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CONTINUOUS PROBLEMS OF OPTIMAL MULTIPLEX-PARTITIONING OF SETS WITHOUT CONSTRAINTS AND SOLVING METHODS

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Резюме. Розглядається неперервна лінійна задача оптимального мультиплексного розбиття множин у двох варіантах: з фіксованими центрами і з їх розміщенням. Описано методи розв'язання таких задач розбиття. Для задачі з фіксованими центрами оптимальний розв'язок знайдено аналітично у вигляді характеристичних вектор-функцій підмножин вищих порядків, що складають оптимальне мультиплексне розбиття заданої множини. Досліджено деякі властивості оптимальних мультиплексних розбиттів. Розв'язання задачі оптимального мультиплексного розбиття множини з розміщенням центрів зводиться до розв'язування скінченновимірної задачі мінімізації негладкої функції. Наведено результати розв'язання тестових задач. Продемонстрована можливість побудови діаграм Вороного вищих порядків у результаті формулювання та розв'язання неперервних задач мультиплексного розбиття множин з певними критеріями якості розбиття.

Abstract. We consider the continuous linear problem of optimal multiplex-partitioning of sets in two versions: with given coordinates of service centers or with their placing in a given region. The methods of solving such partitioning problems are described. For the problem with fixed centers the optimal solution was found analytically in the form of characteristic vector-functions of subsets of higher-order, which compose the optimal multiplex-partitioning of a given set. Some properties of optimal multiplex-partitions are investigated. The solution of the problem of optimal multiplex-partitioning of set with placing centers is reduced to the finite-dimensional problem of non-smooth function minimization. The results of the test problems are presented. We demonstrate the possibility of construction of higher order Voronoi diagrams via formulating and solving continuous problems of multiplex-partitioning of sets with some criterion of partitioning quality.

1. Introduction

The problems of optimal organization of service or manufacturing networks, including "Optimal set partitioning (OSP) problem", "Facility location problem", "Continuous Location-Allocation Problem", are actively studied over the past 50 years [1–15]. The main problem that can be solved using OSP models and methods is the arrangement of given region into several subregions served by only one service center. The criterion for choosing the optimal partition

Key words. sets partitioning of the k-th order, optimal multiplex-partitioning of set, Voronoi diagrams of higher orders, continuous problems of optimal sets partitioning, non-differentiable optimization.

may be the minimization of costs of service provision or obtainment. The majority of models of partitioning problems are discrete. In [3] it is shown that the discrete models and problems of partitioning - placement on a plane are NP-complete problems. The discrete problems of optimal sets partitioning and their solving methods are studied, in particular, in [3, 11].

The problems, in which the partitioned set is continuous, in scientific literature are called continuous problems of sets partitioning. Such problems are explored in [1, 2, 4–10]. Different formulations of continuous problems of optimal sets partitioning are presented in [14, 15]. There is described an unified approach, which underlies the methods and algorithms for solving such problems.

We present the mathematical formulations of problems of optimal partitioning of a given region into subregions, each of which covers customers that have the same k nearest service centers among N existing (or possible) centers. It is assumed that customers from each subregion can be served by any of the closest k centers.

The first mathematical models of continuous problems of optimal multiplex-partitioning of sets were presented in [16]. There was also substantiated the choice of name for a new class of partitioning problems. It was indicated that the order of partition can be pointed in the name of new partitioning problems. Similarly with computational geometry during the construction of a set of points that have the same set of k nearest centers among N existing (possible) ones the Voronoi cell of k-th order is obtained. The set of all such possible cells associated with N generator points (centers) is called Voronoi diagram of k-th order [17].

The name of a new class of problems takes into consideration the fact that the partitioning of customers (consumers) is carried out so that each subset is served by two, three or more service centers. There is an english term "duplex" (triplex») that in Russian (Ukrainian) translation means "which is designed for two (three) families", "multiplex" is a complex, compound. Thus, the name "problems of optimal multiplex -partitioning of sets" is total for all new OSP problems. Among them the problems of optimal duplex-partitioning of sets (continuous problems of optimal partitioning of sets of the second order), the problems of optimal triplex-partitioning of sets (continuous problems of optimal partitioning of sets of the third order) may be separated. For more detailed specification of multiplex-partitioning problems the words "continuous linear" can be added in its name considering accepted terminology of the theory of OSP problems [14, 15], where the first word means that the partitioned set is continuous, the second one indicates the property of functional and restrictions of the problem.

The difference between "multiplex-partitioning" and "multiple partitioning" is also denoted in [16]. In the first case the partitioning is associated with N homogeneous points called centers and the set is divided into subsets of points, which have the same set of k nearest neighbors among N centers. In the second case a regular partitioning of a given set is carried out for several times. It happens, for example, while solving multistep (multistage) OSP problems, where service centers have different categories and customers must be partitioned for each category separately [18]. We also deal with multiple partitioning while solving multiproduct OSP problems, when each service center can provide multiple services (produce several items) and the partitioning of clients is performed for each service (product) separately [14, 15].

The methods of solvig continuous linear problems of optimal multiplex-partitioning of sets are based on the following general idea (similar to presented in [14, 15]): initial problems of optimal partitioning of sets are mathematically formulated as infinite-dimensional optimization problems and reduced to auxiliary finite-dimensional nonsmooth maximization problems or to nonsmooth maximin problems using the Lagrange functional and after that modern efficient methods of nondifferentiable optimization [19] are used to get their numerical solution. The feature of this approach for linear OSP problems is that the solution of initial infinite-dimensional optimization problems can be obtained analytically in explicit form and, at the same time, the obtained analytical expression can include parameters presented as optimal solutions to the above-mentioned auxiliary finite-dimensional optimization problems with nonsmooth objective functions.

The purpose of the article is to describe solvig methods of optimization problems of partitioning of a given region into subregions that cover customers with the same k nearest service centers among N existing (or possible) centers.

The articles [20,21] describe a unified approach to the construction of Voronoi diagrams that is based on the formulation of continuous problems of optimal partitioning of sets from an n-dimensional Euclidean space into subsets. The development of the theory of continious problems of optimal multiplex-partitioning of sets gives an opportunity to construct the Voronoi diagrams of higher orders and their different generalizations. We will demonstrate it below.

2. The mathematical formulations of continuous linear problems of optimal multiplex-partitioning of sets without constraints

Let Ω be a bounded Lebesgue measurable closed set in the space $E_n; \tau_i = (\tau_i^{(1)}, ..., \tau_i^{(n)})$ from Ω, for all $i = \overline{1, N}$, are some points, called "centers" (they can be fixed or subjected to determination).

We introduce the following notations: $\mathrm{N} = \{1, 2, ..., N\}$ is a set of all centers indeces; M (N, k) is a set of all k-elements subsets of the set N, $|\mathrm{M}\ (\mathrm{N},\ k)| = C_N^k = L$; $\sigma_l = \{j_1^l, j_2^l, ..., j_k^l\}$, $l = \overline{1, L}$ are elements of the set M(N, k). We associate each element σ_l from the set M(N, k) with some subset Ω_{σ_l} of points from Ω, $l = \overline{1, L}$. In its turn, subset Ω_{σ_l} is associated with a set of centers $\{\tau_{j_1^l}, \tau_{j_2^l}, ..., \tau_{j_k^l}\}$.

The collection of Lebesgue measurable subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, ..., \Omega_{\sigma_L}$ from $\Omega \subset E_n$ (among which can be empty) will be called as a partition of the k-th order of the set Ω into disjoint subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, ..., \Omega_{\sigma_L}$, if

$$\bigcup_{i=1}^{L} \Omega_{\sigma_i} = \Omega,$$

$$mes(\Omega_{\sigma_i} \cap \Omega_{\sigma_j}) = 0, \quad \sigma_i, \sigma_j \in \mathrm{M(N},k), \quad i \neq j, \quad i, j = \overline{1, L},$$

where $mes(\cdot)$ means Lebesgue measure.

The subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, ..., \Omega_{\sigma_L}$ of the set Ω we call as subsets of the k-th order of this set. Suppose $\Sigma_\Omega^{N,k}$ is a class of all possible **partitions of the k-th order** of the set Ω into disjoint subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, ..., \Omega_{\sigma_L}$:

$$\Sigma_\Omega^{N,k} = \Big\{ \overline{\omega} = \{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\} : \bigcup_{i=1}^{L} \Omega_{\sigma_i} = \Omega;$$

$$mes(\Omega_{\sigma_i} \cap \Omega_{\sigma_j}) = 0, \ \sigma_i, \sigma_j \in \mathrm{M(N},k), \ i \neq j, \ i, j = \overline{1, L} \Big\}.$$

Problem A1-k: Find

$$F\Big(\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\}\Big) \to \min_{\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\} \in \Sigma_\Omega^{N,k}},$$

$$F\Big(\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\}\Big) = \sum_{l=1}^{L} \int\limits_{\Omega_{\sigma_l}} \sum_{i \in \sigma_l} (c(x, \tau_i)/w_i + a_i)\rho(x)dx$$

where $x = (x^{(1)}, ..., x^{(n)}) \in \Omega$; $\tau^N = (\tau_1, ..., \tau_i, ..., \tau_N) \in \Omega^N$, coordinates $\tau_i^{(1)}, ..., \tau_i^{(n)}$ of a center $\tau_i, i = \overline{1, N}$, are fixed; functions $c(x, \tau_i)$ are bounded defined on $\Omega \times \Omega$ measurable at x for any fixed $\tau_i = (\tau_i^{(1)}, ..., \tau_i^{(n)})$ from Ω for all $i = \overline{1, N}$; $\rho(x)$ is bounded measurable function integral on the set Ω ; $w_i > 0, \ a_i \geq 0, \ i = \overline{1, N}$, are given numbers.

The partition of the k-th order $\overline{\omega}^* = \{\Omega_{\sigma_1}^*, ..., \Omega_{\sigma_L}^*\}$ of $\Omega \subset E_n$ that affords minimum to the functional F, is called **optimal solution of the problem A1-k**.

If the centers $\tau_i, i = \overline{1, N}$ in the problem **A1-k** are not fixed in advance and there are some centers to be placed in a given set $\Omega \subset E_n$ along with finding its partition of the k-th order $\overline{\omega}^* = \{\Omega_{\sigma_1}^*, \Omega_{\sigma_2}^*, ..., \Omega_{\sigma_L}^*\}$, then we will have a new problem of optimal multiplex-partitioning of sets.

Problem A2-k: Find

$$\min_{\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\} \in \Sigma_\Omega^{N,k}, \ \{\tau_1, ..., \tau_N\} \in \Omega^N} F\Big(\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\}, \{\tau_1, ..., \tau_N\}\Big),$$

where

$$F(\overline{\omega}, \tau^N) = F\Big(\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\}, \{\tau_1, ..., \tau_N\}\Big) =$$
$$= \sum_{l=1}^{L} \int\limits_{\Omega_{\sigma_l}} \sum_{i \in \sigma_l} (c(x, \tau_i)/w_i + a_i)\rho(x)dx, \tag{1}$$

all functions and parameters are the same as in the problem **A1-k**; coordinates $\tau_i^{(1)}, ..., \tau_i^{(n)}$ of the centers $\tau_i, i = \overline{1, N}$, are unknown in advance.

An allowable pair $(\overline{\omega}^*, \tau_*^N) = \Big(\{\Omega_{\sigma_1}^*, \Omega_{\sigma_2}^*, ..., \Omega_{\sigma_L}^*\}, \{\tau_1^*, \tau_2^*, ..., \tau_N^*\}\Big)$ that affords minimum to the functional 1 is called **optimal solution of the problem A2-k**.

3. The solving method of the problems of optimal multiplex-partitioning of sets with fixed centers

By analogy with solving method of continuous linear OSP problems [14] first we write the initial problem **A1-k** as a problem of infinite-dimensional mathematical programming with Boolean variables.

Let $\overline{\omega} = \{\Omega_{\sigma_1}, ..., \Omega_{\sigma_l}, ..., \Omega_{\sigma_L}\}$ is some partition of the k-th order of the set Ω. For each point $x \in \Omega_{\sigma_l}, l = \overline{1, L}$, we introduce LN-dimensional vector $\lambda^l(x) = (\lambda_1^l(x), ..., \lambda_N^l(x))$, which coordinates are determined as follows:

$$\lambda_i^l(x) = \begin{cases} 1, & x \in \Omega_{\sigma_l} \ \& \ i \in \sigma_l, \\ 0, & \textit{in the other cases} \end{cases} \qquad i = \overline{1, N}, l = \overline{1, L}, \tag{2}$$

where $\sigma_l \in$ M(N, k), $\sigma_l = \{j_1^l, j_2^l, ..., j_N^l\}$ is the set of centers $\tau_{j_1^l}, \tau_{j_2^l}, ..., \tau_{j_k^l}$ indeces associated with a subset Ω_{σ_l}. Using these functions we introduce characteristic functions of the subsets Ω_{σ_l}, $l = \overline{1, L}$, forming the partition of the k-th order of the set Ω:

$$\chi_l(x) = \begin{cases} 1, & x \in \Omega_{\sigma_l}, \\ 0, & x \in \Omega \setminus \Omega_{\sigma_l}, \end{cases} \qquad \Leftrightarrow \qquad \chi_l(x) = \prod_{i=l,\ i \in \sigma_l}^{N} \lambda_i^l(x), l = \overline{1, L},$$

Therefore, the vector-function $\lambda^l(x) = (\lambda_1^l(x), ..., \lambda_N^l(x))$ defined on the set Ω with coordinates matched to 2 will be called as characteristic vector-function of the subset Ω_{σ_l} included into the partition of the k-th order of Ω (by analogy with the way as characteristic vector for a subset of a finite set in discrete mathematics is given).

Let us rewrite the problem **A1-k** in terms of characteristic functions of subsets that form the partition of the k-th order of the set Ω.

Problem B1-k. Find $\min\limits_{\lambda(\cdot) \in \Gamma_0^k} I(\lambda(\cdot))$,

$$I(\lambda(\cdot)) = \int\limits_{\Omega} \sum_{l=1}^{L} \Big(\sum_{i=1}^{N} (c(x, \tau_i)/w_i + a_i) \lambda_i^l(x) \Big) \rho(x) dx$$

$$\Gamma_0^k = \Big\{ \lambda(x) = (\lambda^1(x), ..., \lambda^l(x), ..., \lambda^L(x)) : \\ \lambda^l(x) = (\lambda_1^l(x), ..., \lambda_N^l(x)); \ \lambda_i^l(x) = 0 \vee 1, \\ i = \overline{1, N}, \ \sum_{i=1}^{N} \lambda_i^l(x) = k, \ l = \overline{1, L} \text{ a. e. for } x \in \Omega \Big\};$$

$\tau^N = (\tau_1, ..., \tau_N) \in \underbrace{\Omega \times ... \times \Omega}_{N} = \Omega^N$ is given vector.

Along with the problem **B1-k** we will consider the corresponding problem with variable values $\lambda_i^l(\cdot)$, $i = \overline{1, N}$, $l = \overline{1, L}$, from segment $[0; 1]$.

Problem C1-k. Find $\min\limits_{\lambda(\cdot)\in\Gamma_1^k} I(\lambda(\cdot))$,

$$I(\lambda(\cdot)) = \int\limits_{\Omega} \sum_{l=1}^{L} \Big(\sum_{i=1}^{N} (c(x,\tau_i)/w_i + a_i)\lambda_i^l(x) \Big) \rho(x)dx$$

where

$$\Gamma_1^k = \Big\{ \lambda(x) = (\lambda^1(x), ..., \lambda^l(x), ..., \lambda^L(x)) : \lambda^l(x) = (\lambda_1^l(x), ..., \lambda_N^l(x)); 0 \le \lambda_i^l(x) \le 1, i = \overline{1,N}, \sum_{i=1}^{N} \lambda_i^l(x) = k, \; l = \overline{1,L} \text{ a. e. for } x \in \Omega \Big\};$$

$\tau^N = (\tau_1, ..., \tau_N) \in \Omega^N$ is given vector.

Obviously, $\Gamma_0^k \subset \Gamma_1^k$. It is easy to show that the set Γ_1^k is bounded closed convex set from the Hilbert space $L_2^{LN}(\Omega)$ with the norm

$$\|\lambda(\cdot)\| = \Big(\int\limits_{\Omega} \sum_{l=1}^{L} \sum_{n=1}^{N} [\lambda_i^l(x)]^2 \Big)^{1/2}.$$

The space $L_2^{LN}(\Omega)$ is reflexive. Fig. 1 depicts elements of the set Γ_1^2 corresponding to one of the points $x \in \Omega$.

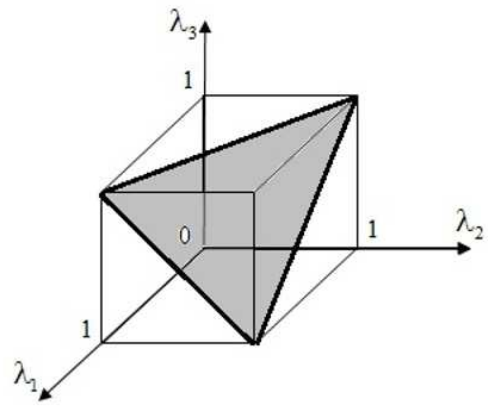


FIG. 1. The element of the set Γ_1^2 corresponding to each point $x \in \Omega$

The functional $I(\lambda(\cdot))$ is linear continuous about $\lambda(\cdot)$ on Γ_1^k at any fixed $\tau^N \in \Omega^N$.

The following statements are true.

Statement 3.1. At any fixed $\tau^N \in \Omega^N$ bounded closed convex set Γ_1^k from the Hilbert space $L_2^{LN}(\Omega)$ is slightly compact and contains at least one extreme point.

Statement 3.2. There is at least one extreme point among the set Γ_1^k of points, in which linear on $\lambda(\cdot)$ functional $I(\lambda(\cdot))$ reaches its minimum about $\lambda(\cdot)$ on the set Γ_1^k at any fixed $\tau^N \in \Omega^N$.

Statement 3.3. The extreme points of the set Γ_1^k are characteristic functions of the subsets of the k-th order $\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}$ that form a partition of the k-th order of the set Ω at any fixed $\tau^N \in \Omega^N$.

The convex (linear) continuous functional $I(\lambda(\cdot))$ reaches its lower bound on a closed bounded convex set Γ_1^k from the Hilbert space $L_2^{LN}(\Omega)$ by the generalized Weierstrass theorem. Consequently, the problem **C1-**k has a solution.

Thus, there is at least one extreme point of Γ_1^k among the set of optimal solutions of the problem **C1-**k, and extreme points of Γ_1^k are characteristic functions of subsets of the k-th order $\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}$ forming a partition of k-th order of the set Ω. A set of optimal solutions of the problem **C1-**k contains optimal solution of the problem **B1-**k. That is the solution of the last one reduces to the solution of the problem **C1-**k.

For the problem **C1-**k we form the Lagrange functional that includes restrictions $\sum\limits_{i=1}^{N} \lambda_i^l(x) = k$, $l = \overline{1, L}$:

$$W(\lambda(\cdot), \psi_0(\cdot)) =$$

$$= \int\limits_{\Omega} \sum_{l=1}^{L} \Big[\sum_{i=1}^{N} [c(x, \tau_i)/w_i + a_i] \rho(x) \lambda_i^l(x) + \psi_0^l(x) \Big(\sum_{i=1}^{N} \lambda_i^l(x) - k \Big) \Big] dx =$$

$$= \int\limits_{\Omega} \sum_{l=1}^{L} \Big\{ \sum_{i=1}^{N} [(c(x, \tau_i)/w_i + a_i) \rho(x) + \psi_0^l(x)] \lambda_i^l(x) - k \psi_0^l(x) \Big\} dx.$$

The functional $W(\lambda(\cdot), \psi_0(\cdot))$ is determined on the Cartesian product $\Lambda \times \Phi$, where

$$\Lambda = \{\lambda(\cdot) \in L_2^{LN}(\Omega) : \ 0 \le \lambda_i^l(x) \le 1, \ \forall x \in \Omega, \ i = \overline{1, N}, \ l = \overline{1, L}\};$$

$$\Phi = \{(\psi_0^1(\cdot), \psi_0^2(\cdot), ..., \psi_0^L(\cdot)) : \ \psi_0^l(\cdot) \in L_2(\Omega), \ l = \overline{1, L}\}.$$

The pair $(\hat{\lambda}(\cdot), \hat{\psi}_0(\cdot))$ is called **Lagrange functional** $W(\lambda(\cdot), \psi_0(\cdot))$ **saddle point** on set $\Lambda \times \Phi$, if $\forall \lambda(\cdot) \in \Lambda, \forall \psi_0(\cdot) \in \Phi$ the following inequality holds

$$W(\hat{\lambda}(\cdot), \psi_0(\cdot)) \le W(\hat{\lambda}(\cdot), \hat{\psi}_0(\cdot)) \le W(\lambda(\cdot), \hat{\psi}_0(\cdot)).$$

For each $x \in \Omega$ we introduce a function about $(LN + L)$ variables:

$$Q(\lambda(x), \psi_0(x)) = \sum_{l=1}^{L} \Big\{ \sum_{i=1}^{N} [(c(x, \tau_i)/w_i + a_i) \rho(x) + \psi_0^l(x)] \lambda_i^l(x) - k \psi_0^l(x) \Big\},$$

determined on the Cartesian product of sections $\Lambda_x \times \Phi_x$ of the sets Λ and Φ at $x \in \Omega$.

It is easy to prove, that in order to the admissible pair $\Big(\hat{\lambda}(\cdot), \hat{\psi}_0(\cdot)\Big) \in \Lambda \times \Phi$ would be a saddle point of Lagrange functional $W(\lambda(\cdot), \psi_0(\cdot))$, it is necessary and sufficient for the following equality to be hold: a. e. for $x \in \Omega$

$$Q(\hat{\lambda}(x), \hat{\psi}_0(x)) = \max_{\psi_0(x) \in \Phi_x} \min_{\lambda(x) \in \Lambda_x} Q(\lambda(x), \psi_0(x)).$$

That means, that for each fixed $x \in \Omega$ the pair $(\hat{\lambda}(x), \hat{\psi}_0(x))$ must form the saddle point of function $Q(\lambda(x), \psi_0(x))$ on the set $\Lambda_x \times \Phi_x$.

Let x is arbitrary fixed point of set Ω. Because of the separability of function $Q(\lambda(x), \psi_0(x))$ about its parameters the following equality holds:

$$\max_{\psi_0(x) \in \Phi_x} \min_{\lambda(x) \in \Lambda_x} Q(\lambda(x), \psi_0(x)) =$$

$$= \max_{\psi_0(x) \in \Phi_x} \min_{\lambda(x) \in \Lambda_x} \sum_{l=1}^{L} \Big\{ \sum_{i=1}^{N} [(c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x)]\lambda_i^l(x) - k\psi_0^l(x) \Big\} =$$

$$= \sum_{l=1}^{L} \max_{\psi_0(x) \in \Phi_x} \Big\{ \sum_{i=1}^{N} \min_{0 \le \lambda_i^l(x) \le 1} [(c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x)]\lambda_i^l(x) - k\psi_0^l(x) \Big\}.$$

The point $(\hat{\lambda}(x), \hat{\psi}_0(x))$ will be a saddle point for function $Q(\lambda(x), \psi_0(x))$ on the set $\Lambda_x \times \Phi_x$ then and only then, when the following conditions are performed:

1) $Q(\hat{\lambda}(x), \hat{\psi}_0(x)) = \min_{\lambda(x) \in \Lambda_x} Q(\lambda(x), \hat{\psi}_0(x))$;

2) $\frac{\partial Q(\hat{\lambda}(x), \hat{\psi}_0(x))}{\partial \psi_0^l} = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{N} \hat{\lambda}_i^l(x) - k = 0 \quad \forall l = \overline{1, L}$.

The function $Q(\lambda(x), \psi_0(x))$ gets a minimum value at arbitrary fixed vector $\psi_0(x)$ in all admissible vectors $\lambda(x) \in \Lambda_x$,

$$\Lambda_x = \{\lambda = (\lambda_1^1, ..., \lambda_N^1, ..., \lambda_1^L, ..., \lambda_N^L) : \ 0 \ \le \ \lambda_i^l \ \le \ 1, i = \overline{1, N}, \ l = \overline{1, L}\},$$

in the point $\hat{\lambda}(x)$, which components are calculated by the formula: for each $i = \overline{1, N}, \ l = \overline{1, L}$

$$\hat{\lambda}_i^l(x) = \begin{cases} 1, & \text{if } (c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x) < 0, \\ 0, & \text{if } (c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x) > 0, \\ \alpha \in [0; 1], & \text{if } (c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x) = 0. \end{cases} \tag{3}$$

Taking into account the fact that among all solutions of the problem **C1-k** we are interested only in those that are extreme points of the feasible set of problem's solutions, then because of an arbitrary choice of value $\alpha \in [0; 1]$ for equalities we can assume that a particular case of 3 is the following formula: for each $i = \overline{1, N}, \ l = \overline{1, L}$

$$\hat{\lambda}_i^l(x) = \begin{cases} 1, & \text{if } (c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x) < 0, \\ 0, & \text{if } (c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x) > 0, \\ 0 \vee 1, & \text{if } (c(x, \tau_i)/w_i + a_i)\rho(x) + \psi_0^l(x) = 0. \end{cases} \tag{4}$$

At $\psi_0(x) = \hat{\psi}_0(x)$ coincidently with constraints 3, including 4, the following equalities are performed:

$$\sum_{i=1}^{N} \hat{\lambda}_i^l(x) - k = 0 \quad \forall l = \overline{1, L}. \tag{5}$$

It follows from 5 that for each fixed $l = \overline{1, L}$ among components of the vector $\hat{\lambda}^l(x) = (\hat{\lambda}_1^l(x), ..., \hat{\lambda}_N^l(x))$ in 4 exactly k components must be equal to 1. Let $\sigma_l = \{j_1^l, j_2^l, ..., j_k^l\}$ is the index set, for which the following inequalities hold:

$$(c(x, \tau_{j_m^l})/w_{j_m^l} + a_{j_m^l})\rho(x) + \hat{\psi}_0^l(x) \le 0,\ j_m^l \in \sigma_l, m = \overline{1, k};$$
$$(c(x, \tau_i)/w_i + a_i)\rho(x) + \hat{\psi}_0^l(x) \ge 0,\ i \in \mathrm{N} \setminus \sigma_l. \tag{6}$$

Then 4 can be written in the next form:

$$\hat{\lambda}_i^l(x) = \begin{cases} 1, & \text{if } i \in \sigma_l, \\ 0, & \text{if } i \in \mathrm{N} \setminus \sigma_l. \end{cases} \tag{7}$$

And thus, because of the arbitrary choice of point $x \in \Omega$ and number $l = \overline{1, L}$ the formula 7 determines the value of the characteristic vector-function of the subset Ω_{σ_l} of the k-th order in the point $x \in \Omega$ associated with a set of centers $\{\tau_{j_1^l}, \tau_{j_2^l}, ..., \tau_{j_k^l}\}$. That means that the formula 6 and 7 indicate the conditions of appurtenance of points x to the subset of the k-th order Ω_{σ_l}, $l = \overline{1, L}$.

We consider the system of inequalities :

$$\begin{cases} (c(x, \tau_{j_1^l})/w_{j_1^l} + a_{j_1^l})\rho(x) + \hat{\psi}_0^l(x) \le 0, \\ (c(x, \tau_{j_2^l})/w_{j_2^l} + a_{j_2^l})\rho(x) + \hat{\psi}_0^l(x) \le 0, \\ ... \\ (c(x, \tau_{j_k^l})/w_{j_k^l} + a_{j_k^l})\rho(x) + \hat{\psi}_0^l(x) \le 0, \\ -(c(x, \tau_i)/w_i + a_i)\rho(x) - \hat{\psi}_0^l(x) \le 0,\ i \in \mathrm{N} \setminus \sigma_l. \end{cases} \tag{8}$$

The system 8 is solvable, since the problem **C1-k** (as well as **B1-k**) has a solution. Summing in 8 each of the first k inequalities with each i-th inequality from the group $\mathrm{N} \setminus \sigma_l$ we obtain the following expressions: for each $l = \overline{1, L}$ and $j \in \sigma_l$

$$(c(x, \tau_j)/w_j + a_j)\rho(x) \le (c(x, \tau_i)/w_i + a_i)\rho(x), \quad \forall i \in \mathrm{N} \setminus \sigma_l.$$

Under the assumption that $\rho(x) \ge 0$ almost everywhere for $x \in \Omega$ we can write the formula for calculation of the characteristic functions of subsets of the k-th order $\Omega^*_{\sigma_l}$, $l = \overline{1, L}$ that form an optimal multiplex-partitioning of Ω as follows:
for each $l = \overline{1, L}$ the point x belongs to $\Omega^*_{\sigma_l}$, if the following inequalities hold

$$c(x, \tau_j)/w_j + a_j \le c(x, \tau_i)/w_i + a_i, \quad \forall j \in \sigma_l \text{ and } \forall i \in \mathrm{N} \setminus \sigma_l.$$

Thus, the following theorem is true.

Theorem 1. *In order a possible partition of the k-th order*

$$\overline{\omega}^* = \{\Omega^*_{\sigma_1}, ..., \Omega^*_{\sigma_L}\} \in \Sigma^{N,k}_{\Omega}$$

*of the set Ω is optimal for problem **A1**-k, it is necessary to fulfill an inequalities a.e. for $x \in \Omega^*_{\sigma_l}$*

$$c(x,\tau_j)/w_j + a_j \leq c(x,\tau_i)/w_i + a_i, \quad \forall j \in \sigma_l \text{ and } \forall i \in N \setminus \sigma_l, \; l = \overline{1,L}. \tag{9}$$

Corollary 1. *Let in the problem **A1**-k function $\rho(x) \geq 0$ a.e. for $x \in \Omega$, $\overline{\omega}^* = \{\Omega^*_{\sigma_1}, ..., \Omega^*_{\sigma_L}\} \in \Sigma^{N,k}_{\Omega}$ is optimal partition, points $x \in \Omega$ belong to the boundary between the non-empty subsets of the k-th order $\Omega^*_{\sigma_m}$ and $\Omega^*_{\sigma_l}$, $(m \neq l; m, l = \overline{1,L})$. Then there is a subset of indices $\zeta = \{j_1, ..., j_r\}$, $1 \leq r < k$ such that $(\zeta \subset \sigma_l) \& (\zeta \subset \sigma_m)$ and for each $j \in \sigma_l \setminus \zeta$ and $i \in \sigma_m \setminus \zeta$ the equality sign in 9 is achieved, i.e.:*

$$c(x,\tau_j)/w_j + a_j = c(x,\tau_i)/w_i + a_i, \quad \forall j \in \sigma_l \setminus \zeta \text{ and } \forall i \in \sigma_m \setminus \zeta. \tag{10}$$

Proof. Let $x \in \Omega$ is arbitrary fixed point, which belongs to the boundary between the non-empty subsets of the k-th order $\Omega^*_{\sigma_m}$ and $\Omega^*_{\sigma_l}$, $(m \neq l; m, l = \overline{1,L})$. Because of $x \in \Omega^*_{\sigma_m}$ the following inequalities system has a solution:

$$c(x,\tau_j)/w_j + a_j \leq c(x,\tau_i)/w_i + a_i, \quad \forall j \in \sigma_m \text{ and } \forall i \in \mathrm{N} \setminus \sigma_m,$$

and by the fact that $x \in \Omega^*_{\sigma_m}$ the following inequalities are true:

$$c(x,\tau_j)/w_j + a_j \leq c(x,\tau_i)/w_i + a_i, \quad \forall j \in \sigma_l \text{ and } \forall i \in \mathrm{N} \setminus \sigma_l.$$

It follows that

$$c(x,\tau_j)/w_j + a_j \leq c(x,\tau_i)/w_i + a_i, \quad \forall j \in \sigma_m \cap \sigma_l, \; \forall i \in \mathrm{N} \setminus \sigma_m;$$

$$c(x,\tau_j)/w_j + a_j \leq c(x,\tau_i)/w_i + a_i, \quad \forall j \in \sigma_m \cap \sigma_l, \; \forall i \in \mathrm{N} \setminus \sigma_l.$$

Let $\zeta = \sigma_m \cap \sigma_l$, $\zeta = \{j_1, ..., j_r\}$, $1 \leq r < k$. Then $\forall p \in \sigma_l \setminus \zeta \quad c(x,\tau_p)/w_p + a_p \leq c(x,\tau_i)/w_i + a_i \quad \forall i \in \mathrm{N} \setminus \sigma_l$.

On the other hand, since $p \in \mathrm{N} \setminus \sigma_m$, then $\forall i \in \sigma_m \quad c(x,\tau_p)/w_p + a_p \geq c(x,\tau_i)/w_i + a_i$, among them all indexes $i \in \sigma_m \setminus \zeta$. And thus, $\forall i \in \sigma_m \setminus \zeta$ and $\forall p \in \sigma_l \setminus \zeta$ the following double inequality is true:

$$c(x,\tau_i)/w_i + a_i \leq c(x,\tau_p)/w_p + a_p \leq c(x,\tau_i)/w_i + a_i.$$

It is possible only when $\forall i \in \sigma_m \setminus \zeta$ and $\forall p \in \sigma_l \setminus \zeta$ and the equality 10 holds, i.e.:

$$c(x,\tau_p)/w_p + a_p = c(x,\tau_i)/w_i + a_i.$$

The corollary 1 is proved.

Corollary 2. *Let in the problem **A1**-k function $\rho(x) \geq 0$ a.e. for $x \in \Omega$, $\overline{\omega}^* = \{\Omega^*_{\sigma_1}, ..., \Omega^*_{\sigma_L}\} \in \Sigma^{N,k}_{\Omega}$ is optimal partition, points $x \in \Omega$ are corner points of the partition, i.e. x belongs to the boundary between several non-empty subsets of the k-th order $\Omega^*_{\sigma_m}$, $m \in \{l_1, l_2, ..., l_s\}$; $1 \leq l_q \leq L$, $q = \overline{1,s}$; $s > 2$. Then there is a subset of indices $\zeta = \{j_1, ..., j_r\}$, $1 \leq r < k$ such* *that* $\zeta \subset \bigcap_{q=\overline{1,s}} \sigma_{l_q}$ *and for each* $i \in \sigma_{l_p} \setminus \zeta, p = 1, 2, ..., s$ *there are indices* j_p *from the set* $\sigma_{l_p} \setminus \zeta,\ \forall q \neq p,\ q = 1, 2, ..., s,$ *at which equal sign in 9 is achieved, i.e.:*

$$c(x, \tau_{j_p})/w_{j_p} + a_{j_p} = c(x, \tau_i)/w_i + a_i, \quad \forall q \neq p,\ q = 1, 2, ..., s. \tag{11}$$

The proof of the Corollary 2 is analogous to the proof of the Corollary 1.

Remark 1. *The necessary condition 9 is a sufficient condition of optimality for the problem **A1**-k because of* $I(\lambda(\cdot))$ *linearity.*

The Figures 2a, 2b are illustrations of the validity of the Corollary 1 and Theorem 1 in the case of optimal duplex and triplex partitioning of a square area with seven centers. Hereinafter, in order not to overload the figures the subsets of the k-th order are denoted as a set of indices $\sigma_l = \{j_1^l, j_2^l, ..., j_k^l\}$ of appropriate centers.

The implementation of the Corollary 2 for optimal triplex-partitioning of a square area with the same centers can be traced on the Fig. 2c. Let us describe this Figure in details. Let the point $x \in \Omega$ is a corner point of the partition, which lies on the border between the following subsets: $\Omega^*_{\{123\}}$, $\Omega^*_{\{237\}}$, $\Omega^*_{\{357\}}$, $\Omega^*_{\{135\}}$. The intersection of all indices sets corresponding to mentioned subsets of the third order is the set $\zeta = \{3\}$. The center τ_3 is really the closest one to a fixed point x among all seven predetermined centers. The remaining centers, which indices make up a set $\bigcup_{q=\overline{1,s}} \sigma_{l_q} \setminus \zeta = \{1, 2, 5, 7\}$, are in the same distance from the point x (the shortest one without taking into account the distance between the center τ_3 and x).

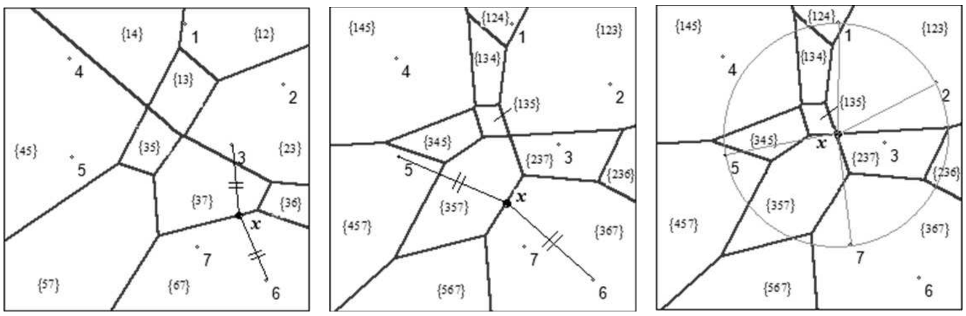


FIG. 2. Illustration of the equalities 10 (a, b) and 11 (c) for points of the boundary between subsets

Thus, from the Theorem 1 we see that the optimal solution of **B1**-k is reached on the vector-function $\lambda^*(x) = (\lambda_*^1(x), ..., \lambda_*^l(x), ..., \lambda_*^L(x))$, each component $\lambda^l(x)$ of which is calculated by the formula: a.e. for $x \in \Omega$

$$\lambda_{*i}^l(x) = \begin{cases} 1, & \text{if } c(x, \tau_i)/w_i + a_i \leq c(x, \tau_j)/w_j + a_j, \\ & \quad \text{simultaneously with } \forall i \in \sigma_l,\ j \in \mathrm{N} \setminus \sigma_l, \\ 0, & \text{in other cases} \qquad l = \overline{1, L},\ i = \overline{1, N}. \end{cases} \tag{12}$$

The functional of the problem **B1-**k at $\lambda(\cdot) = \lambda^*(\cdot)$ is noted as follows:

$$I(\lambda^*(\cdot)) = \int\limits_{\Omega} \min_{l=\overline{1,L}} \Big(\sum_{i \in \sigma_l} (c(x,\tau_i)/w_i + a_i) \Big) \rho(x) dx. \tag{13}$$

Remark 2. *Assume that for a certain point $x \in \Omega$ there are several sets of indices, for example, $\sigma_l = \{j_1^l, j_2^l, ..., j_k^l\}$ and $\sigma_q = \{j_1^q, j_2^q, ..., j_k^q\}$, under which system of inequalities 9 holds. It is possible only when these several indices sets have nonempty intersection and on the set of indices $\sigma_l \triangle \sigma_q$ ($\triangle$ is symmetric difference) an equal sign in the inequality 9 is achieved. Then the solution of the problem **C1-**k will consist of not one extreme point, but at least two ones. It is easy to see that the value of the functional 10 is the same for both extreme points. Since for the visual interpretation of the solution 12 and for the method implementation the selection of the certain extreme point (hence, the set Ω partition) is very important, then the ambiguity can be eliminated using conventional techniques: from several sets of indices $\sigma_l \triangle \sigma_q$, where $c(x,\tau_j)/w_j + a_j = c(x,\tau_i)/w_i + a_i$ is achieved, the smallest index is chosen.*

Usually, while formulating the OSP problems as a function $c(x,\tau_i)$ a particular case of Minkowski power distance $c(x,\tau_i) = \|x - \tau_i\|_p = \sqrt[p]{\sum\limits_{j=1}^{n} (x^j - \tau_i^j)^p}$ is selected: at $p = 2$ – Euclidean, at $p = 1$ – Manhattan (taxicab geometry), at $p = \infty$ – "domination" metrics (Chebyshev metrics).

The partitions of the 1-t, 2-d, 3-d order of the square area $\Omega \subset E_2$ with centers $\tau_i,\ i = 1, 2, ..., 8$ in case, when the function $c(x,\tau_i)$ in the functional 1 is Minkovsky distance at $p = 8$; $w_i = 1,\ a_i = 0,\ i = \overline{1,8}$, are presented on the Fig. 3. For each subset Ω_{σ_l} included in the multiplex partition of Ω (at $k = 2, 3$) it is defined a pair or a trio of indeces of corresponding centers. It is easy to notice that in the duplex partition only 14 (Fig. 3b) from $L = C_8^2 = 28$ subsets $\Omega_{\sigma_1}, \Omega_{\sigma_2}, ..., \Omega_{\sigma_L}$, which compose optimal Ω partition of the 2-d order, are nonempty. In the triplex partition (Fig. 3c) many subsets of the 3-rd order also were empty. The number of empty subsets included in the multiplex partition of the set depends not only on centers' location $\tau_i,\ i = \overline{1,N}$, its number, but also on the selection of metrics [16].

Remark 3. *If in the problem **A1-**k function $c(x,\tau_i)$ is Euclidean metric, $a_i = 0,\ w_i = 1,\ i = \overline{1,N}$; $\rho(x) = 1\ \forall x \in \Omega$, then the optimal solution determined by vector-function $\lambda(\cdot) = \lambda^*(\cdot)$ as 12 turns out Voronoi diagram of the k-th order known in the computational geometry [17], i.e. such partition of set Ω into subsets $\Omega_1, ..., \Omega_L$ that:*

$$\bigcup_{i=1}^{L} \Omega_i = \Omega;\ mes(\Omega_i \cap \Omega_j) = 0,\ \forall i \neq j,\ i,j = \overline{1,L},$$

$$\Omega_m = \Big\{ x \in \Omega :\ \forall j \in T_m \quad c(x,\tau_j) < c(x,\tau_i),\ i \in N \setminus T_m \Big\},$$

where $T_m = \{i_1^m, i_2^m, ..., i_k^m\},\ m = \overline{1,L}$, are all possible k-element subsets of the set N of indeces.

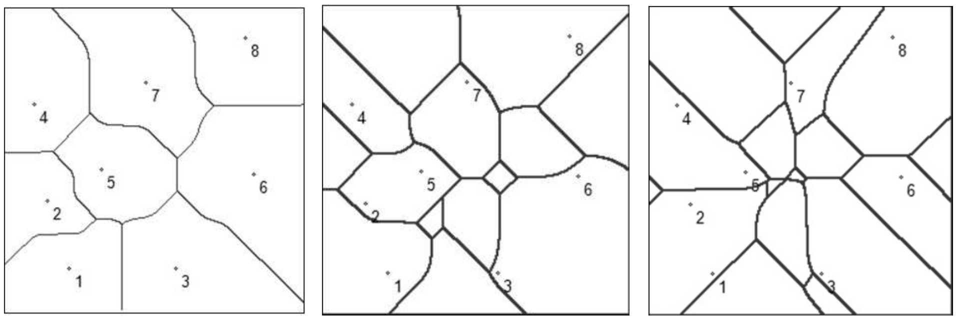


FIG. 3. The optimal partitions of k-th order of a square with 8 centers: $a - k = 1$; $b - k = 2$; $c - k = 3$

Let us examine this fact to the problem **A1-2** (duplex OSP problem without constraints and with fixed centers) under initial data: $c(x, \tau_i)$ is Euclidean metric, $a_i = 0$, $w_i = 1, i = \overline{1, N}$; $\rho(x) = 1\ \forall x \in \Omega$. Under these conditions the formula 12 can be written as follows:

$$\lambda_{*i}^{l}(x) = \begin{cases} 1, & \text{if} c(x, \tau_i) \le c(x, \tau_j), \text{simultaneously with} \forall i \in \sigma_l, j \in \mathrm{N} \setminus \sigma_l, \\ 0, & \text{in other cases.} \end{cases}$$

The optimal partition for this problem is shown on the Fig. 4. Suppose $x \in \Omega$ is arbitrary fixed point. Let us consider, for example, indexes sets $\sigma_q = \{7, 8\}$, $\sigma_r = \{6, 7\}$, $\sigma_m = \{6, 8\}$. Then $\lambda^q(x) = \{0, 0, 0, 0, 0, 0, 1, 1\}$, $\lambda_i^r(x) = 0$, $\lambda_i^m = 0$, $\forall i = \overline{1, N}$, because only for indexes $i \in \sigma_q$ condition is performed:

$$c(x, \tau_i) \le c(x, \tau_j),\ \forall j \in \{1, 2, ..., 8\} \setminus \sigma_q.$$

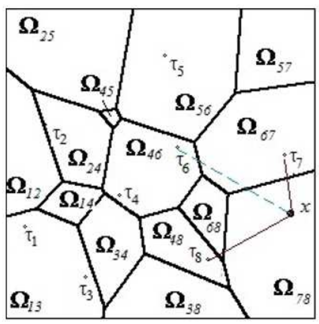


FIG. 4. The Voronoi diagram of second order

Suppose now, $\sigma_s = \{3, 7\}$. On the Fig. 4 we can see, that there is no point $x \in \Omega$, for which at given centers $\tau_i, i = 1, 2, ..., 8$, ratios would be carried out:

$$c(x, \tau_3) \le c(x, \tau_j),\ \forall j \in \{1, 2, ..., 8\} \setminus \sigma_s.$$

$$c(x, \tau_7) \le c(x, \tau_j),\ \forall j \in \{1, 2, ..., 8\} \setminus \sigma_s.$$

Therefore, the subset of the 2-nd order Ω_{37} included in the duplex partition of the set Ω is empty.

Thus, by solving the problems **A1-**k of optimal multiplex - partitioning of sets under different parameter values of the objective functional, we can get a higher order Voronoi diagrams and their generalizations: additively weighted ($\exists$ at least one $i : a_i \neq 0, i = \overline{1,N}$), multiplicatively weighted ($\exists$ at least one $j : w_j \neq 1$, obviously $\forall j : w_j \neq 0, j = \overline{1,N}$), additively and multiplicatively weighted (simultaniously $\exists$ at least one i and at least one $j : a_i \neq 0, w_j \neq 1, i,j = \overline{1,N}$).

For the problem **B2-**k, which is equivalent to **A2-**k but written in terms of characteristic functions of subsets that constitute the partition of the k-th order of a given set Ω, the following theorem holds.

Theorem 2. *The optimal solution of the problem* ***B2-****k has the following form: for* $i = \overline{1,N}, l = \overline{1,L}$ *and almost all* $x \in \Omega$

$$\lambda^l_{*i}(x) = \begin{cases} 1, & \text{if } c(x,\tau_{*i})/w_{*i} + a_i \leq c(x,\tau_{*j})/w_{*j} + a_j,\ i \in \sigma_l,\ j \in N \setminus \sigma_l, \\ 0, & \text{in other cases,} \end{cases}$$

in the capacity of $\tau_{*1}, ..., \tau_{*N}$ *the optimal solution of the problem*

$$G(\tau) \to \min_{\tau^N \in \Omega^N}, \tag{14}$$

is chosen, where

$$G(\tau) = \int\limits_{\Omega} \min_{\sigma_l \in M(N,,k)} \sum_{i \in \sigma_l} [c(x,\tau_i)/w_i + a_i] \rho(x) dx. \tag{15}$$

Hence, with a help of the Theorem 2 solving the continuous problem of optimal multiplex-partitioning of sets is reduced to a finite-dimensional minimization problem 14 solving with non-differentiable function 15 by any known method of non-smooth optimization [19].

In article we present only the results of solving some problems of multiplex-partitioning of sets with centers placing. Fig. 4, 5, respectively, demonstrate the results of solving the optimal duplex and triplex partitioning of square area with centers placing under parameters: $c(x,\tau_i)$ is Euclidean metric, $a_i = 0,\ w_i = 1, i = \overline{1,N};\ \rho(x) = 1\ \forall x \in \Omega$. To solve finite-dimensional problem 14 with non-differentiable function 15, the algorithm of pseudo-gradients was used with space dilatation in the direction of the difference between two successive gradients; this algorithm is close to Shor's r-algorithm [19].

Due to the fact that the Shor's r-algorithm provides a search of non-differentiable function local minimum, and the problem 14 is multiextremal, then under different initial approximations various local solutions of the problem **A2-**k can be obtained. For example, in the case of solving this problem for $N = 15,\ k = 2,3$ except of the optimal solutions depicted in the Table 1, can be obtained the solutions presented in Fig. 5.

The identification of the properties of optimal solutions of the problem **A2-**k under certain initial data is the direction of further research.

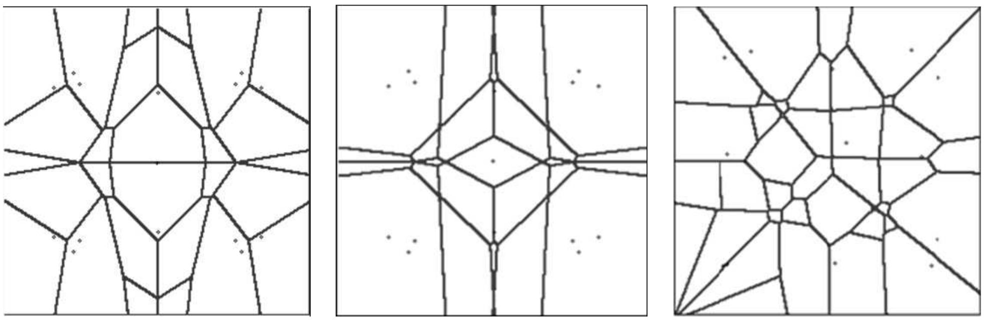


FIG. 5. The local solutions of the problem A2-k at $N = 15$: $a - k = 2$; $b, c - k = 3$

4. Conclusion

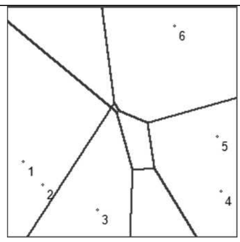
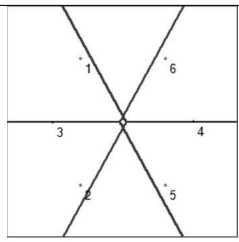
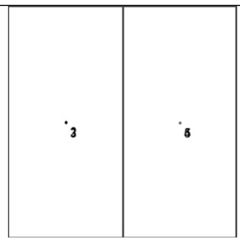
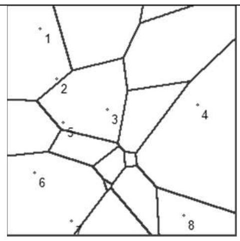
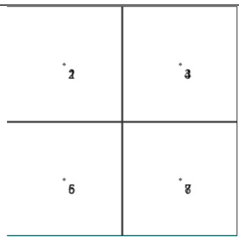
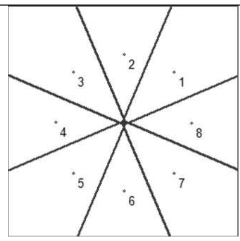
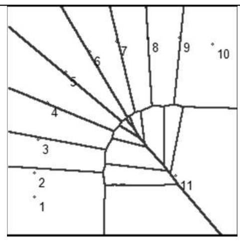
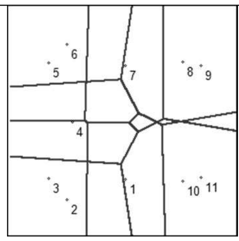
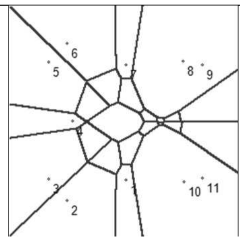
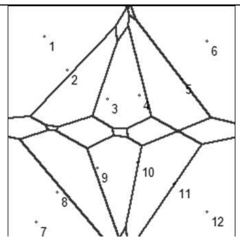
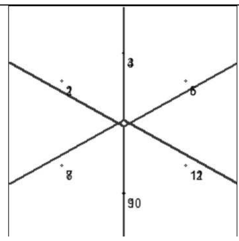
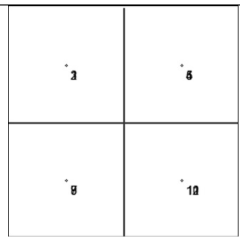
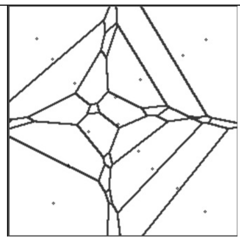
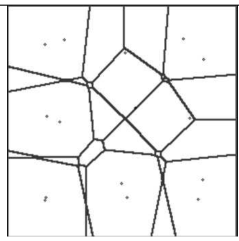
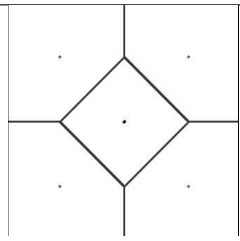
Thus, the solutions of continuous linear problems of optimal multiplex-partitioning of sets without restrictions with fixed centers and with their placing are obtained. In the latter case, the optimal solution of the multiplex partitioning problem contains unknown parameters that are obtained in the process of solving the finite-dimensional nonsmooth function minimization. The results of computational experiments are presented. The considered mathematical models can be attributed to the so-called minisum problems of partitioning-placement in terms of a quality criterion of multiplex-partitioning by analogy with the objectives of location-allocation problems of the graph theory [11, 22].

We can consider a different form of the functional of multiplex-partitioning problem, for example:

$$F_1\Big(\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\}\Big) = \sum_{l=1}^{L} \int\limits_{\Omega_{\sigma_l}} \max_{i \in \sigma_l}(c(x, \tau_i)/w_i + a_i)\rho(x)dx.$$

In this case, the multiplex-partitioning problem is not linear and refers to the so-called minimax problems of partitioning-placement [11, 22]. The development and substantiation of methods of solving these problems is one of the directions for further research in the theory of multiplex-partitioning of sets. We only note that even with this criterion the problems of optimal multiplex-partitioning of sets include as a particular case the continuous OSP problems studied in details in [14]. It is interesting to compare the solutions of the problems of optimal multiplex-partitioning of sets with different quality critera. It can be assumed that while solving the problem with placement of centers $\tau^N = (\tau_1, ..., \tau_i, ..., \tau_N) \in \Omega^N$ the functional $F_1\Big(\{\Omega_{\sigma_1}, ..., \Omega_{\sigma_L}\}\Big)$ will provide such their optimal location,that will be the solution of the optimal multiple covering of set $\Omega \subset E_n$ by circles with these centers [23].

TABL. 1. The optimal solutions of problems **A1**-k and **A2**-k

Centers number N	The partition of the k-th order of the set Ω		
	With fixed coordinates of the center	With the optimal location of centers in the set Ω	
	$k=2$	$k=2$	$k=3$
6			
8			
11			
12			
15			

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