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# COMBINED NEWTON-KURCHATOV METHOD UNDER THE GENERALIZED LIPSCHITZ CONDITIONS FOR THE DERIVATIVES AND DIVIDED DIFFERENCES

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Резюме. Доведено локальну збіжність комбінованого ітераційного процесу, побудованого на основі методу Ньютона і методу лінійної інтерполяції Курчатова, для розв'язування нелінійних операторних рівнянь в банаховому просторі за узагальнених умов Ліпшиця для похідних першого порядку і поділених різниць першого та другого порядку. Визначено радіус кулі збіжності і швидкість збіжності методу, знайдено область єдиності розв'язку нелінійного рівняння.

Abstract. The local convergence of combined iterative process, built on the basis of Newton's method and Kurchatov's method of linear interpolation, for solving of nonlinear operator equations in Banach space under the generalized Lipschitz conditions for the derivative of the first order and divided differences of the first and second order is proved. The radius of the convergence ball and convergence order of the method are determined, the ball of uniqueness of the solution of nonlinear equation is found.

## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^* \in D$ of equation

$$F(x) + G(x) = 0, \tag{1}$$

where $F$ is a Fréchet-differentiable nonlinear operator on an open convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$, and $G : D \to Y$ is a continuous nonlinear operator.

Let $x, y$ be two points of $D$. A linear operator from $X$ into $Y$, denoted $\delta G(x, y)$, which satisfies the condition

$$\delta G(x, y)(x - y) = G(x) - G(y) \tag{2}$$

is called a divided difference of $G$ at points $x$ and $y$.

Let $x, y, z$ be three points of $D$. A operator $\delta G(x, y, z)$ will be called a divided difference of the second order of the operator $G$ at the points $x$, $y$ and $z$ , if it satisfies the condition

$$\delta G(x, y, z)(y - z) = \delta G(x, y) - \delta G(x, z). \tag{3}$$

*Key words.* Banach space, Newton's method; Kurchatov's method; Combined iterative method; Divided difference; Local convergence; Convergence order; Generalized Lipschitz condition.

A well-known simple difference method for solving nonlinear equations $F(x) = 0$ is the Secant method

$$x_{n+1} = x_n - (\delta F(x_{n-1}, x_n))^{-1} F(x_n),\ n = 0,1,2,\ldots, \tag{4}$$

where $\delta F(x_{n-1}, x_n)$ is a divided difference of the first order and $x_0, x_{-1}$ are given.

Secant method for solving nonlinear operator equations in a Banach space was explored by the authors [5,14,15,19,30] under the condition that the divided differences of a nonlinear operator $F$ satisfy the Lipschitz (Hölder) condition with constant $L$ of type

$$\|\delta F(x, y) - \delta F(u, v)\| \le L(\|x - u\| + \|y - v\|).$$

In [11] it was proposed one-point iterative Secant-type method with memory. In [29] it was explored the Kurchatov method under the classical Lipschitz conditions for the divided differences of the first and second order and it was determined the quadratic convergence of it. The iterative formula of Kurchatov method has the form [4, 5, 18, 29]

$$x_{n+1} = x_n - (\delta F(2x_n - x_{n-1}, x_{n-1}))^{-1} F(x_n),\ n = 0,1,2,\ldots, \tag{5}$$

where $\delta F(u, v)$ is a divided difference of the first order, $x_0, x_{-1}$ are given.

In paper [20] Potra investigated the three-point difference method with convergence order $1.839\ldots$ for classical Lipschitz conditions for divided differences of the first and second order [20]

$$\begin{aligned} &x_{n+1} = x_n - A_n^{-1} F(x_n), \\ &A_n = \delta F(x_n, x_{n-1}) + \delta F(x_{n-2}, x_n) - \delta F(x_{n-2}, x_{n-1}),\ n = 0,1,2,\ldots, \end{aligned} \tag{6}$$

$x_0, x_{-1}, x_{-2}$ are given. This method first has been proposed for scalar nonlinear equations by Traub in [30].

Regarding the local convergence of Newton method, Traub and Wozniakowski in [31] and Wang in [33] gave the best estimate of the radii of convergence balls when the first derivatives are Lipschitz continuous around a solution.

Besides, there are a lot of the works on the weakness and/or extension of the hypothesis made on the nonlinear operators; see works of Argyros, Ezquerro, Hernandez, Rubio, Gutierrez, Wang, Li [5, 12, 13, 32–35] and references therein. In particular, Wang introduced in [33] the notions of generalized Lipschitz conditions or Lipschitz conditions with $L$ average, where instead of constant $L$ it is used some positive integrable function.

The center Lipschitz condition with $L$ average in the inscribe sphere makes us unify the convergence criteria containing the Kantorovich theorem and the Smale $\alpha$-theory, while the radius Lipschitz conditions with $L$ average unify the estimates of the radii of convergence balls for operators with Lipschitz continuous first derivatives and analytic operators [10, 32, 33].

In our work [27] for the first time we have introduced a similar generalized Lipschitz condition for the operator of the first order divided difference and under this condition the convergence of Secant method was studied and was found that its convergence order is $(1 + \sqrt{5})/2$. In the paper [26] we have introduced a generalized Lipschitz condition also for the divided differences of the second order and we have studied the local convergence of Kurchatov method (5).

Note that in many papers such as [1,2,7,16,21], the authors investigated the Secant and Secant-type methods under the generalized conditions for the first divided differences of the form

$$\|(\delta F(x,y)-\delta F(u,v)))\| \le \omega(\|x-y\|,\|u-v\|) \quad \forall x,y,u,v\in D, \tag{7}$$

where $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is continuous nondecreasing function in their two arguments. Under these same conditions in the work of Argyros [4] it is proven semilocal convergence of Kurchatov method and in [22] of Ren and Argyros the semilocal convergence of combined Kurchatov method and Secant method. In both cases only the linear convergence of the methods is received.

In this paper, we study the local convergence of the combined Newton–Kurchatov method

$$x_{n+1} = x_n - (F'(x_n) + \delta G(2x_n - x_{n-1}, x_{n-1}))^{-1}(F(x_n)+G(x_n)), \quad n = 0,1,2,\dots, \tag{8}$$

where $F'(u)$ is a Fréchet derivative, $\delta G(u,v)$ is a divided difference of the first order, $x_0, x_{-1}$ are given, which is built on the basis of the mentioned Newton and Kurchatov methods under relatively weak, generalized Lipschitz conditions for the derivatives and divided differences of nonlinear operators. Setting $G(x) \equiv 0$, we receive the results for Newton method [33], and when $F(x) \equiv 0$ we get the known results for Kurchatov method [26].

We first proposed the method (8) in the paper [28]. Semilocal convergence of the method (8) under the classical Lipschitz conditions is studied in the mentioned article, but there was determined the convergence only with the order $(1+\sqrt{5})/2$.

In this article we prove the quadratic order of convergence of the method (8), which is higher than the convergence order $(1+\sqrt{5})/2$ for the Newton–Secant method [5,8,9,23]

$$x_{n+1} = x_n - (F'(x_n) + \delta G(x_{n-1}, x_n))^{-1}(F(x_n)+G(x_n)), \quad n = 0,1,2,\dots, \tag{9}$$

Method (9) was proposed in [9] and proved its convergence with the order $(1+\sqrt{5})/2$ under the classical Lipschitz conditions for the first derivative $F'(x)$ and bounded norm of the second-order divided difference $\delta G(x,y,z)$. The same order of convergence was received in [5] with weaker conditions - classical Lipschitz conditions for the first derivative $F'(x)$ and the first-order divided difference $\delta G(x,y)$.

Note that in the work [23] we have considered combined Newton–Secant method (9) and we have proposed a methodology of studying the convergence of combined methods for solving nonlinear equations with nondifferentiable operator under the relatively weak, generalized Lipschitz conditions for the first derivatives and divided differences of nonlinear operators. Under the same conditions in [24] it was studied the convergence of the combined two-step method for the solution of nonlinear equations.

The results of the numerical study of the method (8) and other combined methods on the test problems are provided in our works [25, 28].

## 2. Local convergence of Newton–Kurchatov method

Lets denote $B(x_0, r) = \{x : \|x - x_0\| < r\}$ an open ball of radius $r > 0$ with center at point $x_0 \in D$, $B(x_0, r) \subset D$.

Condition on the divided difference operator $\delta F(x, y)$

$$\|\delta F(x, y) - \delta F(u, v)\| \leq L(\|x - u\| + \|y - v\|) \quad \forall x, y, u, v \in D \tag{10}$$

is called Lipschitz condition in domain $D$ with constant $L > 0$. If the condition is being fulfilled

$$\|\delta F(x, y) - F'(x_0)\| \leq L(\|x - x_0\| + \|y - x_0\|) \quad \forall x, y \in B(x_0, r), \tag{11}$$

then we call it the center Lipschitz condition in the ball $B(x_0, r)$ with constant $L$.

However $L$ in Lipschitz conditions can be not a constant, and can be a positive integrable function. In this case, if for $x^* \in D$ inverse operator $[F'(x^*)]^{-1}$ exists, then the conditions (10 ) and (11) for $x_0 = x^*$ can be replaced respectively for

$$\begin{aligned} &\|F'(x^*)^{-1}(\delta F(x, y) - \delta F(u, v)))\| \leq \\ &\leq \int_0^{\|x-y\|+\|u-v\|} L(z)dz \quad \forall x, y, u, v \in D \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\|F'(x^*)^{-1}(\delta F(x, y) - F'(x^*))\| \leq \\ &\leq \int_0^{\|x-x^*\|+\|y-x^*\|} L(z)dz \quad \forall x, y \in B(x^*, r). \end{aligned} \tag{13}$$

Simultaneously Lipschitz condition (12) – (13) are called generalized Lipschitz conditions or Lipschitz conditions with the $L$ average.

Similarly, we introduce the generalized Lipschitz condition for the divided difference of the second order

$$\begin{aligned} &\|F'(x^*)^{-1}(\delta F(u, x, y) - \delta F(v, x, y))\| \leq \\ &\leq \int_0^{\|u-v\|} N(z)dz \; \forall x, y, u, v \in B(x^*, r), \end{aligned} \tag{14}$$

where $N$ is a positive integrable function.

**Remark 8.** *Note than the operator $F$ is Fréchet differentiable on $D$ when the Lipschitz conditions (10) or (12) are fulfilled $\forall x, y, u, v \in D$ (the divided differences $\delta F(x, y)$ are Lipschitz continuous on $D$) and $\delta F(x, x) = F'(x)$ $\forall x \in D$ [3].*

The radius of the convergence ball and the convergence order of the combined Newton–Kurchatov method (8) are determined in next theorem.

**Theorem 1.** *Let $F$ and $G$ be continuous nonlinear operators defined in open convex domain $D$ of a Banach space $X$ with values in the Banach space $Y$. Lets suppose, that: 1) $H(x) \equiv F(x) + G(x) = 0$ has a solution $x^* \in D$, for which there exists a Fréchet derivative $H'(x^*)$ and it is invertible; 2) $F$ has the Fréchet derivative of the first order, and $G$ has divided differences of the first and second order on $B(x^*, 3r) \subset D$, which are satisfying on $B(x^*, 3r)$ the generalized Lipschitz conditions*

$$\|H'(x^*)^{-1}(F'(x) - F'(x^\tau))\| \le \int_{\tau\rho(x)}^{\rho(x)} L_1(u)du,\ 0 \le \tau \le 1, \tag{15}$$

$$\|H'(x^*)^{-1}(\delta G(x,y) - \delta G(u,v))\| \le \int_0^{\|x-u\|+\|y-v\|} L_2(z)dz, \tag{16}$$

$$\|H'(x^*)^{-1}(\delta G(u,x,y) - \delta G(v,x,y))\| \le \int_0^{\|u-v\|} N(z)dz, \tag{17}$$

*where $x^\tau = x^* + \tau(x - x^*)$, $\varrho(x) = \|x - x^*\|$, $L_1$, $L_2$ and $N$ are positive nondecreasing integrable functions and $r > 0$ satisfies the equation*

$$\frac{\frac{1}{r}\int_0^r L_1(u)udu + \int_0^r L_2(u)du + 2r\int_0^{2r} N(u)du}{1 - \Big(\int_0^r L_1(u)du + \int_0^{2r} L_2(u)du + 2r\int_0^{2r} N(u)du\Big)} = 1. \tag{18}$$

*Then for all $x_0, x_{-1} \in B(x^*, r)$ the iterative method (8) is correctly defined and the generated by it sequence $\{x_n\}_{n\ge 0}$, which belongs to $B(x^*, r)$, converges to $x^*$ and satisfies the inequality*

$$\begin{aligned}
&\|x_{n+1} - x^*\| \le \\
&\le \Big\{\frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu + \int_0^{\rho(x_n)} L_2(u)du + \\
&+ \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|\Big\}\times \\
&\times\Big\{1 - \Big(\int_0^{\rho(x_n)} L_1(u)du + \int_0^{2\rho(x_n)} L_2(u)du + \\
&+ \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|\Big)\Big\}^{-1}\|x_n - x^*\|.
\end{aligned} \tag{19}$$

*Proof.* First we show that

$$f(t) = \frac{1}{t^2}\int_0^t L_1(u)udu, \quad g(t) = \frac{1}{t}\int_0^t L_2(u)du$$

in $h(t) = \frac{1}{t}\int_0^t N(u)du$ monotonically nondecreasing with respect to $t$. Indeed, under the monotony of $L_1, L_2, N$ we have

$$\Big(\frac{1}{t_2^2}\int_0^{t_2} - \frac{1}{t_1^2}\int_0^{t_1}\Big)L_1(u)udu = \Big(\frac{1}{t_2^2}\int_{t_1}^{t_2} + \Big(\frac{1}{t_2^2} - \frac{1}{t_1^2}\Big)\int_0^{t_1}\Big)L_1(u)udu \geq$$

$$\geq L(t_1)\Big(\frac{1}{t_2^2}\int_{t_1}^{t_2} + \Big(\frac{1}{t_2^2} - \frac{1}{t_1^2}\Big)\int_0^{t_1}\Big)udu = L_1(t_1)\Big(\frac{1}{t_2^2}\int_0^{t_2} - \frac{1}{t_1^2}\int_0^{t_1}\Big)udu = 0,$$

$$\Big(\frac{1}{t_2}\int_0^{t_2} - \frac{1}{t_1}\int_0^{t_1}\Big)L_2(u)du = \Big(\frac{1}{t_2}\int_{t_1}^{t_2} + \Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)\int_0^{t_1}\Big)L_2(u)du \geq$$

$$\geq L_2(t_1)\Big(\frac{1}{t_2}\int_{t_1}^{t_2} + \Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)\int_0^{t_1}\Big)du = L_2(t_1)\Big(\frac{t_2 - t_1}{t_2} + t_1\Big(\frac{1}{t_2} - \frac{1}{t_1}\Big)\Big) = 0$$

for $0 < t_1 < t_2$. So, $f(t)$, $g(t)$ are nondecreasing with respect to $t$. Similarly we get for $h(t)$.

We denote by $A_n$ linear operator $A_n = F'(x_n) + \delta G(2x_n - x_{n-1}, x_{n-1})$. Easy to see that if $x_n, x_{n-1} \in B(x^*, r)$, then $2x_n - x_{n-1}, x_{n-1} \in B(x^*, 3r)$. Then $A_n$ is invertible and the inequality holds

$$\begin{aligned}\|A_n^{-1}H'(x^*)\| &= \|[I - (I - H'(x^*)^{-1}A_n)]^{-1}\| \leq \\ &\leq \Big(1 - \Big(\int_0^{\rho(x_n)} L_1(u)du + \int_0^{2\rho(x_n)} L_2(u)du + \\ &+ \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|\Big)\Big)^{-1}.\end{aligned} \tag{20}$$

Indeed from the formulas (15)–(17) we get

$$\begin{gathered}\|I - H'(x^*)^{-1}A_n\| = \|H'(x^*)^{-1}(F'(x^*) - F'(x_n) + \delta G(x^*, x^*) - \\ -\delta G(x_n, x_n) + \delta G(x_n, x_n) - \delta G(2x_n - x_{n-1}, x_{n-1})\|) \leq \\ \leq \int_0^{\rho(x_n)} L_1(u)du + \|H'(x^*)^{-1}(\delta G(x^*, x^*) - \delta G(x_n, x_n) + \\ +\delta G(x_n, x_n) - \delta G(x_n, x_{n-1}) + \delta G(x_n, x_{n-1}) - \delta G(2x_n - x_{n-1}, x_{n-1}))\| \leq \\ \leq \int_0^{\rho(x_n)} L_1(u)du + \int_0^{2\rho(x_n)} L_2(u)du + \\ +\|H'(x^*)^{-1}(\delta G(x_n, x_{n-1}, x_n) - \delta G(2x_n - x_{n-1}, x_{n-1}, x_n))(x_n - x_{n-1})\| \leq \\ \leq \int_0^{\rho(x_n)} L_1(u)du + \int_0^{2\rho(x_n)} L_2(u)du + \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|.\end{gathered}$$

From the definition $r$ (18) we get

$$\begin{aligned}&\int_0^r L_1(u)du + \int_0^{2r} L_2(u)du + 2r\int_0^{2r} N(u)du = \\ &= 1 - \frac{1}{r}\int_0^r L_1(u)du - \int_0^r L_2(u)du - 2r\int_0^{2r} N(u)du < 1.\end{aligned} \tag{21}$$

Using the Banach theorem on inverse operator [17], we get formula (20). Then we can write

$$\begin{aligned}
&\|x_{n+1}-x^*\| = \|x_n - x^* - A_n^{-1}(F(x_n)-F(x^*)+G(x_n)-G(x^*))\| = \\
&= \Big\|-A_n^{-1}(\int_0^1 (F'(x_n^\tau)-F'(x_n))d\tau + \delta G(x_n,x^*) - \\
&- \delta G(2x_n - x_{n-1}, x_{n-1}))(x_n - x^*)\Big\| \le \\
&\le \|A_n^{-1}H'(x^*)\|\Big(\|H'(x^*)^{-1}\int_0^1\int_{\tau\rho(x_n)}^{\rho(x_n)} L_1(u)dud\tau + \\
&+ \|H'(x^*)^{-1}(+\delta G(x_n,x^*) - \delta G(2x_n - x_{n-1}, x_{n-1}))\|\Big)\|x_n - x^*\|.
\end{aligned} \tag{22}$$

According to the condition (15)–(17) of the theorem we get

$$\|H'(x^*)^{-1}(\int_0^1\int_{\tau\rho(x_n)}^{\rho(x_n)} L_1(u)dud\tau + \delta G(x_n,x^*) - A_n)\| =$$

$$= \frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu + \|H'(x^*)^{-1}(\delta G(x_n,x^*) - \delta G(x_n,x_n) +$$

$$+\delta G(x_n,x_n) - \delta G(x_n,x_{n-1}) + \delta G(x_n,x_{n-1}) - \delta G(2x_n - x_{n-1}, x_{n-1}))\| \le$$

$$\le \frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu + \|H'(x^*)^{-1}(\delta G(x_n,x^*) - \delta G(x_n,x_n))\| +$$

$$+\|H'(x^*)^{-1}(\delta G(x_n,x_{n-1},x_n) - \delta G(2x_n - x_{n-1}, x_{n-1}, x_n))(x_n - x_{n-1})\| \le$$

$$\le \frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu + \int_0^{\rho(x_n)} L_2(u)du +$$

$$+ \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|.$$

From (20) and (22) shows that fulfills (19). Then from (19) and (18) we get

$$\|x_{n+1}-x^*\| < \|x_n - x^*\| < ... < \max\{\|x_0 - x^*\|, \|x_{-1} - x^*\|\} < r.$$

Therefore, the iterative process (5) is correctly defined and the sequence that it generates belongs to $B(x^*, r)$. From the last inequality and estimates (19) we get $\lim\limits_{n\to\infty}\|x_n - x^*\| = 0$. Since the sequence $\{x_n\}_{n\ge 0}$ converges to $x^*$, then

$$\|x_n - x_{n-1}\| \le \|x_n - x^*\| + \|x_{n-1} - x^*\| \le 2\|x_{n-1} - x^*\|$$

and $\lim\limits_{n\to\infty}\|x_n - x_{n-1}\| = 0$.

The theorem is proven.

**Corollary 5.** *The order of convergence of the iterative procedure (8) is quadratic.*

*Proof.* Lets denote $\rho_{\max} = \max\{\rho(x_0), \rho(x_{-1})\}$. Since $g(t)$ and $h(t)$ are monotonically nondecreasing, then with taking into account the expressions

$$\frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu = \frac{\int_0^{\rho(x_n)} L_1(u)udu\rho(x_n))}{(\rho(x_n))^2} \leq$$

$$\leq \frac{\int_0^{\rho_{\max}} L_1(u)udu\rho(x_n)}{(\rho_{\max})^2} =: A_1\rho(x_n),$$

$$\int_0^{\rho(x_n)} L_2(u)du = \frac{\int_0^{\rho(x_n)} L_2(u)du\rho(x_n)}{\rho(x_n)} \leq \frac{\int_0^{\rho_{\max}} L_2(u)du\rho(x_n)}{\rho_{\max}} =: A_2\rho(x_n),$$

$$\int_0^{\|x_n-x_{n-1}\|} N(u)du = \frac{\int_0^{\|x_n-x_{n-1}\|} N(u)du\|x_n-x_{n-1}\|}{\|x_n-x_{n-1}\|} <$$

$$< \frac{\int_0^{\|x_0-x_{-1}\|} N(u)du\|x_n-x_{n-1}\|}{\|x_0-x_{-1}\|} =: A_3\|x_n-x_{n-1}\|$$

and

$$\Big(1-\Big(\int_0^{\rho(x_n)} L_1(u)du + 2\int_0^{\rho(x_n)} L_2(u)du +$$

$$+\int_0^{\|x_n-x_{n-1}\|} N(u)du\|x_n-x_{n-1}\|\Big)\Big)^{-1} <$$

$$< \Big(1-\Big(\int_0^{\rho_{\max}} L_1(u)du + 2\int_0^{\rho_{\max}} L_2(u)du +$$

$$+\int_0^{\|x_0-x_{-1}\|} N(u)du\|x_0-x_{-1}\|\Big)\Big)^{-1} =: A_4,$$

from the inequality (19) follows

$$\|x_{n+1}-x^*\| \leq A_4(A_1\rho(x_n) + A_2\rho(x_n) + A_3\|x_n-x_{n-1}\|^2)\|x_n-x^*\|.$$

or

$$\|x_{n+1}-x^*\| \leq C_3\|x_n-x^*\|^2 + C_4\|x_n-x_{n-1}\|^2\|x_n-x^*\|. \tag{23}$$

Here $A_k, k=1,...,4, C_3, C_4$ are some positive constants.

Suppose now that the order of convergence of the iterative process (8) is lower 2, therefore there exist $C_5 \geq 0$ and $N > 0$, that for all $n \geq N$ the inequality holds

$$\|x_n-x^*\| \geq C_5\|x_{n-1}-x^*\|^2.$$

Since

$$\|x_n-x_{n-1}\|^2 \leq (\|x_n-x^*\| + \|x_{n-1}-x^*\|)^2 \leq 4\|x_{n-1}-x^*\|^2,$$

then from (23) we get

$$\begin{aligned} \|x_{n+1}-x^*\| &\le C_3\|x_n-x^*\|^2+4C_4\|x_{n-1}-x^*\|^2\|x_n-x^*\| \\ &\le (C_3+4C_4/C_5)\|x_n-x^*\|^2 = C_6\|x_n-x^*\|^2. \end{aligned} \tag{24}$$

But inequality (24) means that the order of convergence not lower 2. Thus, the convergence rate of sequence $\{x_n\}_{n\ge 0}$ to $x^*$ is quadratic.

Next theorem determines the ball of uniqueness of the solution $x^*$ of (1) in $B(x^*,r)$.

**Theorem 2.** *Lets assume that: 1) $H(x)\equiv F(x)+G(x)=0$ has a solution $x^*\in D$, in which there exists a Fréchet derivative $H'(x^*)$ and it is invertible; 2) $F$ has a continuous Freĉhet derivative in $B(x^*,r)$, $F'$ satisfies the generalized Lipschitz condition*

$$\|H'(x^*)^{-1}(F'(x)-F'(x^*))\| \le \int_0^{\rho(x)} L_1(u)du \quad \forall x\in B(x^*,r),$$

*the divided difference $\delta G(x,y)$ satisfies the generalized Lipschitz condition*

$$\|H'(x^*)^{-1}(\delta G(x,x^*)-G'(x^*))\| \le \int_0^{\rho(x)} L_2(u)du \quad \forall x\in B(x^*,r),$$

*where $L_1$ and $L_2$ are positive integrable functions. Let $r>0$ satisfy*

$$\frac{1}{r}\int_0^r (r-u)L_1(u)du + \int_0^r L_2(u)du \le 1.$$

*Then the equation $H(x)=0$ has a unique solution $x^*$ in $B(x^*,r)$.*

*Proof* analogous to [23, 24].

## 3. Corollaries

In the study of iterative methods the traditional assumption is that the derivatives and/or the divided differences satisfy the classical Lipschitz conditions. Assuming that $L_1$, $L_2$ and $N$ are constants, we get from theorem 2.1 and 3.1 important corollaries, which are of interest on its own.

**Corollary 6.** *Let's assume that: 1) $H(x)\equiv F(x)+G(x)=0$ has a solution $x^*\in D$, in which there exists Fréchet derivative $H'(x^*)$ and it is invertible; 2) $F$ has a continuous Fréchet derivative and $G$ has divided differences of the first and second order $\delta G(x,y)$ and $\delta G(x,y,z)$ in $B(x^*,3r)\subset D$, which satisfy the Lipschitz condition*

$$\|H'(x^*)^{-1}(F'(x)-F'(x^*+\tau(x-x^*))\| \le (1-\tau)L_1\|x-x^*\|,$$

$$\|H'(x^*)^{-1}(\delta G(x,y)-\delta G(u,v))\| \le L_2(\|x-u\|+\|y-v\|),$$

$$\|H'(x^*)^{-1}(\delta G(u,x,y)-\delta G(v,x,y))\| \le N\|u-v\|,$$

*where $x,y,u,v\in B(x^*,r)$, $L_1$, $L_2$, $N$ are positive numbers and $r$ is the positive root of the equation*

$$\frac{L_1r/2+L_2r+4Nr^2}{1-L_1r-2L_2r-4Nr^2}=1.$$

*Then Newton-Kurchatov method (5) converges for all $x_{-1}, x_0 \in B(x^*, r)$ and there fulfills*

$$\|x_{n+1} - x^*\| \le \frac{(L_1/2 + L_2)\|x_n - x^*\| + N\|x_n - x_{n-1}\|^2}{1 - \left(L_1 + 2L_2\|x_n - x^*\| + N\|x_n - x_{n-1}\|^2\right)}.$$

*Moreover, $r$ is the best of all possible.*

Note that the received $r$ coincides with the value of $r = \dfrac{2}{3L_1}$ for Newton method for solving equation $F(x) = 0$ [20, 31, 33] and with $r = 2/(3L_2 + \sqrt{9L_2^2 + 32N})$ for Kurchatov method for solving the equation $G(x) = 0$, as derived in [29].

**Corollary 7.** *Suppose that: 1) $H(x) \equiv F(x) + G(x) = 0$ has a solution $x^* \in D$, in which there exists the Fréchet derivative $H'(x^*)$ and it is invertible; 2) $F$ has continuous derivative and $G$ has divided difference $\delta G(x, x^*)$ in $B(x^*, r) \subset D$, which satisfy the Lipschitz conditions*

$$\|H'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_1\|x - x^*\|,$$
$$\|H'(x^*)^{-1}(\delta G(x, x^*) - G'(x^*))\| \le L_2\|x - x^*\|$$

*for all $x \in B(x^*, r)$, where $L_1$ and $L_2$ are positive numbers and $r = \dfrac{2}{L_1 + 2L_2}$. Then the equation $H(x) = 0$ has a unique solution $x^*$ in the open ball $B(x^*, r)$. Moreover, the given $r$ is the best of all possible and does not depend on $F$ and $G$.*

Note that the resulting radius of the uniqueness ball of the solution coincides with $r = \dfrac{2}{L_1}$ for Newton method for solving the equation $F(x) = 0$ [33] and with $r = \dfrac{1}{L_2}$ for Kurchatov method for solving the equation $G(x) = 0$ [29].

## 4. Conclusions

In the papers [5, 15, 29] it was studied the local convergence of Secant and Kurchatov methods in the case of fulfilment of Lipschitz conditions for the divided differences, which hold some Lipschitz constants. In the work [33] it has been justified the convergence of Newton method for the generalized Lipschitz conditions for the Fréchet derivative of the first order. We explored the local convergence of Newton-Kurchatov method under the generalized Lipschitz conditions for Fréchet derivative of differentiable part of the operator and the divided differences of the nondifferentiable part, in which instead of Lipschitz constants some positive integrable functions are used. Our results contain the known ones as partial cases.

## Bibliography

1. Amat S. On the local convergence of secant-type methods / S. Amat // Intern. J. Comput. Math. – 2004. – Vol. 81. – P. 1153-1161.
2. Amat S. On a higher order Secant method / S. Amat, S. Busquier // Appl. Math. Comput. – 2003. – Vol. 141. – P. 321-329.

3. Argyros I. K. On the secant method / I. K. Argyros // Publ. Math. Debrecen. – 1993. – Vol. 43. – P. 233-238.
4. Argyros I. K. A Kantorovich-type analysis for a fast iterative method for solving nonlinear equations / I. K. Argyros // J. Math. Anal. Appl. – 2007. – Vol. 332. – P. 97-108.
5. Argyros I. K. Convergence and applications of Newton-type iterations / I. K. Argyros. – New-York: Springer–Verlag Publ., 2008.
6. Argyros I. K. Improving the rate of convergence of Newton methods on Banach spaces with a convergence structure and applications / I. K. Argyros // Appl. Math. Lett. – 1997. – Vol. 10 – P. 21-28.
7. Argyros I. K. Chebyshev-Secant type methods for non-differentiable operator / I. K. Argyros, J. A. Ezquerro, J. M. Gutiérrez, M. A. Hernández, S. Hilout // Milan J. Math. – 2013. – Vol. 81. – P. 25-35.
8. Argyros I. K. The theorie and application of iteration methods / I. K. Argyros, F. Szidarovszky. – Bosa Raton, Florida: C.R.C. Press, 1993.
9. Catinas E. On some iterative methods for solving nonlinear equations / E. Catinas // Revue d'Analyse Numérique et de Théorie de l'Approximation. – 199. – Vol. 23. – P. 47-53.
10. Chong C. Convergence behavior of Gauss-Newton's method and extensions of the Smale point estimate theory / C. Chong, N. Hu, J. Wang // Journal of Complexity. – 2010. – Vol. 26. – P. 268-295.
11. Ezquerro J. A. Solving non-differentiable equations by a new one-point iterative method with memory / J. A. Ezquerro, M. Grau-Sánchez, M. Hernández // Journal of Complexity. – 2012. – Vol. 28. – P. 48-58.
12. Ezquerro J. A. Generalised differentiability conditions for Newton's method / J. A. Ezquerro, M. Hernández // IMA J. Numer. Anal. – 2002. – Vol. 22. – P. 187-205.
13. Gutiérrez J. M. Newton's method under weak Kantorovich conditions / J. M. Gutiérrez, M. A. Hernández // IMA J. Numer Anal. – 2000. – Vol. 20. – P. 521-532.
14. Hernandez M. A. The Secant method and divided differences Hölder continuous / M. A. Hernandez, M. J. Rubio // Applied Mathematics and Computation. – 2001. – Vol. 124. – P. 139-149.
15. Hernandez M. A. A uniparametric family of iterative processes for solving nondifferentiable equations / M. A. Hernandez, M. J. Rubio // J. Math. Anal. Appl. – 2002. – Vol. 275. – P. 821-834.
16. Hernandez M. A. The Secant method for nondifferentiable operators / M. A. Hernandez, M. J. Rubio // Appl. Math. Lett. – 2002. – Vol. 15. – P. 395-399.
17. Kantorovich L. V. Functional Analysis / L. V. Kantorovich, G. P. Akilov. – Oxford: Pergamon Press, 1982.
18. Kurchatov V. A. On a method of linear interpolation for the solution of functional equations / V. A. Kurchatov // Dokl. Akad. Nauk SSSR. – 1971. – Vol. 198 (3). – P. 524-526. (in Russian); translation in Soviet Math. Dokl. – 1971. – Vol. 12. – P. 835-838.
19. Ortega J. M. Iterative Solution of Nonlinear Equations in Several Variables / J. M. Ortega, W. C. Rheinboldt. – New-York: Academic Press, 1970.
20. Potra F. A. On an iterative algorithm of order 1.839... for solving nonlinear operator equations / F. A. Potra // Numer. Funct. Anal. Optim. – 1985. – Vol. 7 (1). – P. 75-106.
21. Ren H. New sufficient convergence conditions of the Secant method nondifferentiable operators / H. Ren // Appl. Math. Comput. – 2006. – Vol. 182. – P. 1255-1259.
22. Ren H. A new semilocal convergence theorem with nondifferentiable operators / H. Ren, I. K. Argyros // J. Appl. Math. Comput. – 2010. – Vol. 34. – P. 39-46.
23. Shakhno S. Convergence of combined Newton–Secant method and uniqueness of the solution of nonlinear equations / S. Shakhno // Visnyk Ternopil National Technical University. – 2013. – Vol. 69. – P. 242-252. (in Ukrainian).
24. Shakhno S. M. Convergence of the two-step combined method and uniqueness of the solution of nonlinear operator equations / S. M. Shakhno // J. Comput. Appl. Math. – 2014. – Vol. 261. – P. 378-386.

25. Shakhno S. M. Analysis of convergence of a combined method for the solution of nonlinear equations / S. M. Shakhno, I. V. Melnyk, H. P. Yarmola // Journal of Mathematical Sciences. – 2014. – Vol. 201. – P. 32-43.
26. Shakhno S. M. Method of linear interpolation of Kurchatov under generalized Lipschitz conditions for divided differences of first and second order / S. M. Shakhno // Visnyk Lviv. Univ. Ser. Mech. Math. – 2012. – Vol. 77. – P. 235-242. (in Ukrainian).
27. Shakhno S. M. On the Secant method under generalized Lipschitz conditions for the divided difference operator / S. M. Shakhno // PAMM – Proc. Appl. Math. Mech. – 2007. – Vol. 7. – P. 2060083-2060084.
28. Shakhno S. M. Twopoint method for solving nonlinear equation with nondifferentiable operator / S. M. Shakhno, H. P. Yarmola // Mat. Stud. – 2011.v– Vol. 36. – P. 213-220. (in Ukrainian).
29. Shakhno S. M. On a Kurchatov's method of linear interpolation for solving nonlinear equations / S. M. Shakhno // PAMM – Proc. Appl. Math. Mech. – 2004. – Vol. 4. – P. 650-651.
30. Traub J. F. Iterative Methods for the Solution of Equations / J. F. Traub. – Englewood Cliffs, NY: Prentice-Hall, Inc., 1964.
31. Traub J. F. Convergence and complexity of Newton iteration for operator equations / J. F. Traub, H. Woźniakowski // J. Assoc. Comput. Mach. – 1979. – Vol. 26. – P. 250-258.
32. Wang X. H. Convergence of Newton's method and inverse function theorem in Banach space / X. H. Wang // Math. Comp. – 1999. – Vol. 68. – P. 169-186.
33. Wang X. H. Convergence of Newton's method and uniqueness of the solution of equations in Banach space / X. H. Wang // IMA Journal of Numerical Analysis. – 2000. – Vol. 20. – P. 123-134.
34. Wang X. H. Local and global behavior for algorithms of solving equations / X. H. Wang, C. Li // Chinese Sci. Bull. – 2001. – Vol. 46. – P. 444-451.
35. Wang X. H. Convergence of Newton's method and uniqueness of the solution of equations in Banach space II / X. H. Wang, C. Li // Acta Math. Sin. (Engl. Ser.). – 2003. – Vol. 19. – P. 405-412.
36. Zabrejko P. P. The majorant method in the theory of Newton–Kantorovich approximations and the Pták error estimates / P. P. Zabrejko, D. F. Nguen // Numer. Funct. Anal. Optim. – 1987. – Vol. 9. – P. 671-684.

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