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ON THE NON-LINEAR INTEGRAL EQUATION APPROACHES FOR THE BOUNDARY RECONSTRUCTION IN DOUBLE-CONNECTED PLANAR DOMAINS

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Резюме. Розглядається задача реконструкції внутрішньої кривої за заданими даними Коші гармонійної функції на зовнішній кривій плоскої області. За допомогою функції Гріна і теорії потенціалу нелінійна обернена задача редукована до системи нелінійних граничних інтегральних рівнянь. Розроблено три ітераційні алгоритми для її чисельного розв'язування. Знайдено похідні Фреше відповідних операторів і показано єдиність розв'язку лінеаризованих систем. Повна дискретизація здійснена методом тригонометричних квадратур. Через некоректність вихідної задачі до отриманих систем лінійних рівнянь застосовано регуляризацію Тіхонова. Чисельні результати показують, що пропоновані методи дають достатньо добру точність реконструкції при економних обчислювальних затратах.

Abstract. We consider the reconstruction of an interior curve from the given Cauchy data of a harmonic function on the exterior boundary of the planar domain. With the help of Green's function and potential theory the non-linear boundary reconstruction problem is reduced to the system of non-linear boundary integral equations. The three iterative algorithms are developed for its numerical solution. We find the Fréchet derivatives for the corresponding operators and show unique solviability of the linearized systems. Full discretization of the systems is realized by a trigonometric quadrature method. Due to the inherited ill-possedness in the obtained system of linear equations we apply the Tikhonov regularization.

The numerical results show that the proposed methods give a good accuracy of reconstructions with an economical computational cost.

1. Introduction

The mathematical modeling of electrostatic or thermal imaging methods in nondestructive testing and evaluation leads to inverse boundary value problems for the Laplace equation. In principle, in these applications an unknown inclusion within a conducting host medium with a constant conductivity is resolved from the overdetermined Cauchy data on the accessible part of the boundary of the medium.

The idea to reduce the problem of the boundary reconstruction to the system of non-linear equations and to employ a regularized iterative procedure was firstly suggested in [11]. This approach was successfully extended in [1, 3, 6, 11,

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12] for the case of the Laplace equation and in [4,5,7–9,13–15] for the Helmholtz equation.

As an alternative to the reciprocity gap approach based on Green's integral theorem we propose iterative solution methods based on the Green's function. Although the proposed methods are restricted to the class of domains for which the Green's function can be easily found the methods have several advantages over the reciprocity gap approach. In particular, the corresponding single layer potential is bounded at infinity and hence its modification is not needed. Moreover, for the complicated boundary conditions such as generalized impedance the proposed methods will be easier to adopt.

We assume that D is a doubly connected bounded domain in $\mathbb{R}^2$ with the boundary ∂D consisting of two disjoint closed C^2 curves Γ and Λ such that Γ is contained in the interior of Λ.

The corresponding direct problem is: Given a function f on Λ consider the Dirichlet problem for $u \in C^2(D) \cap C(\bar{D})$ satisfying the Laplace equation

$$\Delta u = 0 \quad \text{in } D \tag{1}$$

and the boundary conditions

$$u = 0 \quad \text{on } \Gamma, \tag{2}$$

$$u = f \quad \text{on } \Lambda. \tag{3}$$

The inverse problem we are concerned with is: Given the Dirichlet data f on Λ with $f \neq 0$ and the Neumann data

$$g := \frac{\partial u}{\partial \nu} \quad \text{on } \Lambda, \tag{4}$$

determine the shape of the interior boundary Γ. Here, and in the sequel, by ν we denote the outward unit normal to Γ or to Λ. We tacitly assume that f has enough smoothness, for example $f \in C^{1,\alpha}(\Lambda)$ for classical solutions or $f \in H^{1/2}(\Lambda)$ for weak solutions, to ensure the existence of the normal derivative on Λ. As opposed to the forward boundary value problem, the inverse problem is nonlinear and ill-posed.

The issue of uniqueness, i.e., identifiability of the unknown curve Γ from the Cauchy data on Λ, is settled by the following theorem (see [10]).

Theorem 1. *Let Γ and $\widetilde{\Gamma}$ be two closed curves contained in the interior of Λ and denote by u and $\widetilde{u}$ the solutions to the Dirichlet problem* (1)–(3) *for the interior boundaries Γ and $\widetilde{\Gamma}$, respectively. Assume that $f \neq 0$ and*

$$\frac{\partial u}{\partial \nu} = \frac{\partial \widetilde{u}}{\partial \nu}$$

on an open subset of Λ. Then $\Gamma = \widetilde{\Gamma}$.

The plan of the paper is as follows. In Section 2 we reduce the inverse boundary value problem (1)–(4) to two boundary integral equations using Green's function. Section 3 contains three iterative schemes for the numerical solution of the non-linear integral equations. We show the injectivity of the corresponding linearized operators. The practical realization of suggested algorithms is discussed in Section 4. Section 5 concludes the paper with some numerical examples illustrating the feasibility of the non-linear integral equation method for approximate solution of the inverse boundary value problem.

2. Reduction to boundary integral equations

To this end, we denote the interior of Λ by B. Then, by G we denote the Green's function for B, that is, G is defined for all $x \neq y$ in $\overline{B}$ and of the form

$$G(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} + \tilde{G}(x,y),$$

where, for a fixed $y \in B$, the function $\tilde{G}$ is harmonic in B with respect to x such that $G(\cdot\,, y) = 0$ on Λ. We note that for Λ a circle of radius R centered at the origin $\tilde{G}$ is explicitly given by

$$\tilde{G}(x,y) = \frac{1}{4\pi} \ln \frac{R^4 + |x|^2|y|^2 - 2R^2\, x \cdot y}{R^2}\,.$$

The solution w to the Dirichlet problem in B with boundary values $w = f$ on Λ can be represented in the form

$$w(x) = -\int_\Lambda \frac{\partial G(x,y)}{\partial \nu(y)}\, f(y)\, ds(y), \quad x \in B. \tag{5}$$

In the case of Λ a circle the representation (5) reduces to the Poisson integral. In a more abstract sense, we may interpret (5) as solution operator that maps the boundary value f into the solution w of the Dirichlet problem in B. Seeking the unique solution of (1)–(3) in the form

$$u(x) = \int_\Gamma G(x,y)\varphi(y)\, ds(y) + w(x), \quad x \in D, \tag{6}$$

now leads to the integral equation of the first kind

$$\int_\Gamma G(x,y)\varphi(y)\, ds(y) = -w(x), \quad x \in \Gamma, \tag{7}$$

for the unknown density φ. We name the integral equation (7) as a field equation. The given flux g on Λ leads to the integral equation

$$\int_\Gamma \varphi(y) \frac{\partial G(x,y)}{\partial \nu(x)} ds(y) = g(x) - \frac{\partial w}{\partial \nu}(x), \quad x \in \Lambda, \tag{8}$$

which is named a data equation.
Let introduce the single-layer potential

$$(S\varphi)(x) := \int_\Gamma G(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma, \tag{9}$$

and the operator

$$(A\varphi)(x) := \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)}\, \varphi(y)\, ds(y), \quad x \in \Lambda, \tag{10}$$

for the normal derivative of the single-layer potential on Λ.

Theorem 2. *The inverse boundary value problem* (1)–(4) *is equivalent to the system of integral equations*

$$S\varphi = -w \quad on \quad \Gamma, \tag{11}$$

$$A\varphi = g - \frac{\partial w}{\partial \nu} \quad on \quad \Lambda. \tag{12}$$

Proof. Analogously to [11]. □

Theorem 3. *The operator* $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is bijective and has bounded inverse. The operator* $A : L^2(\Gamma) \to L^2(\Lambda)$ *is injective and has dense range.*

Proof. The bijectivity of S is the classical result and can be found in [10]. The injectivity of A is proved in [2]. □

To describe the algorithms conveniently a parametrization of boundary curves is required. Let

$$\lambda(s) = \{(x_1(s), x_2(s)) : s \in [0, 2\pi]\}$$

is the parametrization for the exterior curve Λ. For simplicity we consider only starlike interior curves, i.e., we choose a parametrization in polar coordinates of the form

$$\gamma_r(s) = \{r(s)c(s) : s \in [0, 2\pi]\}, \tag{13}$$

where $c(s) = (\cos s, \sin s)$ and $r : \mathbb{R} \to (0, \infty)$ is a 2π periodic function representing the radial distance from the origin. However, we wish to emphasize that the concepts described below, in principle, are not confined to starlike boundaries only. We introduce the parametrized density as $\varphi(t) := \varphi(\gamma_r(t))$ or $\phi(t) := \varphi(\gamma_r(t))|\gamma_r'(t)|$. We indicate the dependence on r by denoting the curve with parametrization (13) by Γ_r. The corresponding operators defined through (9) and (10) for $\Gamma = \Gamma_r$ are given by

$$(S_r\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)G(\gamma_r(t), \gamma_r(\tau))d\tau,$$

$$(\tilde{S}_r\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)G(\gamma_r(t), \gamma_r(\tau))|\gamma_r'(\tau)|d\tau,$$

$$(A_r\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)\frac{\partial G}{\partial \nu(\lambda(t))}(\lambda(t), \gamma_r(\tau))d\tau$$

and

$$(\tilde{A}_r\varphi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\frac{\partial G}{\partial \nu(\lambda(t))}(\lambda(t), \gamma_r(\tau))|\gamma_r'(\tau)|d\tau.$$

3. Iterative schemes

Operators S_r, A_r and $\tilde{A}_r$ have the following Freéchet derivatives with respect to the radial function r

$$(S'[r,\phi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)[q(\tau)L_r^{(1)}(t,\tau) + q(t)L_r^{(2)}(t,\tau)]d\tau,$$

$$(A'[r,\phi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\tau)q(\tau)H_r^{(1)}(t,\tau)d\tau.$$

and

$$(\tilde{A}'[r,\varphi]q)(t) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)\Big[q(\tau)H_r^{(1)}(t,\tau)|\gamma_r'(\tau)| + \\ + \frac{r(\tau)q(\tau)+r'(\tau)q'(\tau)}{|\gamma_r'(t)|}H_r^{(2)}(t,\tau)\Big]d\tau. \tag{14}$$

Here we introduced the kernels

$$L_r^{(1)}(t,\tau) := -\frac{r(\tau)-r(t)\cos(t-\tau)}{|\gamma_r(t)-\gamma_r(\tau)|^2} + \operatorname{grad}_{\gamma_r(\tau)} \tilde{G}(\gamma_r(t),\gamma_r(\tau))\cdot c(\tau),$$

$$L_r^{(2)}(t,\tau) := -\frac{r(t)-r(\tau)\cos(t-\tau)}{|\gamma_r(t)-\gamma_r(\tau)|^2} + \operatorname{grad}_{\gamma_r(t)} \tilde{G}(\gamma_r(t),\gamma_r(\tau))\cdot c(t),$$

$$H_r^{(1)}(t,\tau) := \operatorname{grad}_{\gamma_r(\tau)} \frac{\partial G(\lambda(t),\gamma_r(\tau))}{\partial\nu(\lambda(t))}\cdot c(\tau)$$

and

$$H_r^{(2)}(t,\tau) := \frac{\partial G(\lambda(t),\gamma_r(\tau))}{\partial\nu(\lambda(t))}.$$

Note that

$$\lim_{\tau\to t}(q(\tau)L_r^{(1)}(t,\tau)+q(t)L_r^{(2)}(t,\tau)) = \frac{r(t)q(t)+r'(t)q'(t)}{|\gamma_r'(t)|^2} + \\ + 2q(t)\operatorname{grad}_{\gamma_r(t)} \tilde{G}(\gamma_r(t),\gamma_r(t))\cdot c(t).$$

These representation were obtained by standard differentiation procedure in (9) and (10). Also we will need the Freéchet derivative for the function w

$$(w'[r]q)(t) = -\frac{1}{2\pi}\int_0^{2\pi} f(\tau)q(\tau)W_r(t,\tau)d\tau$$

with

$$W_r(t,\tau) := |\lambda'(\tau)|\operatorname{grad}_{\gamma_r(t)} \frac{\partial G(\gamma_r(t),\lambda(\tau))}{\partial\nu(\lambda(\tau))}\cdot c(t).$$

The linear operators $S'[r,\varphi]$ and $A'[r,\varphi]$ have the following properties.

Theorem 4. *Let r be the radial function of the interior boundary Γ_r and let ϕ be a solution to the integral equation* (11)*, i.e. $S_r\phi = -w$ on Γ_r. Assume that $q \in C^2[0,2\pi]$ and $\psi \in L^2[0,2\pi]$ solve the homogeneous system*

$$S_r\psi + S'[r,\phi]q + w'[r]q = 0, \tag{15}$$

$$A_r\psi + A'[r,\phi]q = 0. \tag{16}$$

Then $q = 0$ and $\psi = 0$.

Proof. As it is shown in [6], for sufficiently small q, the perturbed interior curve as given in polar coordinates by

$$\Gamma_{r+q} = \{(r(t)+q(t))c(t) : t\in[0,2\pi]\}$$

can be represented in the form

$$\Gamma_{r+q} = \{r(t)c(t)+\widetilde{q}(t)\nu(t) : t\in[0,2\pi]\}$$

in terms of the normal vector

$$\nu(t) = r'(t)(-\sin t, \cos t) - r(t)(\cos t, \sin t)$$

to the unperturbed curve Γ_r and a function $\widetilde{q}$. Now in the Fréchet derivatives S', A' and w' we may replace the perturbation vector $\zeta(t) = q(t)c(t)$ by $\widetilde{\zeta} = \widetilde{q}\,\nu$. We introduce the function

$$V(x) := \int_0^{2\pi} \psi(\tau) G(x, \gamma_r(\tau)) d\tau - \\ - \int_0^{2\pi} \operatorname{grad}_x G(x, \gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\phi(\tau)\, d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

Then (16) implies that $\frac{\partial V}{\partial \nu} = 0$ on Λ. The function V satisfies the Laplace equation in the exterior of Λ, it decays at infinity, therefore by the uniqueness for the exterior Neumann problem we conclude that $V \equiv 0$ in the exterior of Λ. By analyticity we obtain $V \equiv 0$ in the exterior of Γ_r. Approaching Γ_r from the exterior by the jump relations we obtain

$$0 = \int_0^{2\pi} \psi(\tau) G(\gamma_r(t), \gamma_r(\tau) d\tau \\ - \int_0^{2\pi} \operatorname{grad}_{\gamma_r(t)} G(\gamma_r(t), \gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\phi(\tau)\, d\tau + \frac{1}{2}\widetilde{q}(t)\phi(t), \quad t \in [0, 2\pi].$$

Employing the above equality and recalling the definition (6) of u we can rewrite (15) as follows

$$\widetilde{\zeta} \cdot \operatorname{grad} u \circ \gamma_r = 0.$$

Due to the definition of u and the condition on φ we have $u = 0$ on Γ_r, which is equivalent to

$$\widetilde{\zeta} \cdot \nu \circ \gamma_r \left(\frac{\partial u}{\partial \nu} \right) \circ \gamma_r = 0.$$

Since by Holmgren's theorem $\frac{\partial u}{\partial \nu}$ cannot vanish on open subsets of Γ_r we obtain $\widetilde{\zeta} \cdot \nu \circ \gamma_r = \widetilde{q} = 0$ and hence $q = 0$. Analogously to [11] by continuity of a single-layer potential and the uniqueness of the interior Dirichlet problem we obtain $V = 0$ in $\mathbb{R}^2$ and therefore the density $\psi = 0$. □

Theorem 5. *Let r be the radial function of the interior boundary Γ_r and let ϕ be a solution to the integral equation (12), i.e. $A_r\phi = g - \frac{\partial w}{\partial \nu}$ on Λ. Assume that $q \in C^2[0, 2\pi]$ solves the homogeneous equation*

$$S'[r, \phi]q + w'[r]q = 0. \tag{17}$$

Then $q = 0$.

Proof. Since ϕ is a solution to $A_r\phi = g - \frac{\partial w}{\partial \nu}$ on Λ it also satisfies $S_r\phi = -w$ on Γ_r. We represent the perturbed interior curve again as

$$\Gamma_{r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t \in [0, 2\pi]\}$$

and introduce the function

$$V(x) := \int_0^{2\pi} \phi(\tau) G(x, \gamma_r(\tau)) d\tau - \int_\Lambda \frac{\partial G(x, y)}{\partial \nu(y)} f(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

The function V is a solution to the interior Dirichlet boundary value problem with the homogeneous condition. In view of the unique solution we obtain $V \equiv 0$ in the interior of Γ_r and therefore $\left.\dfrac{\partial V}{\partial \nu}\right|_{\Gamma_r} = 0$, i.e.

$$0 = \widetilde{q}(t)\nu(t) \cdot \operatorname{grad}_{\gamma_r(t)} \int_0^{2\pi} G(\gamma_r(t), \gamma_r(\tau))\phi(\tau)\,d\tau + \frac{1}{2}\widetilde{q}(t)\phi(t)$$
$$-\widetilde{q}(t)\nu(t) \cdot \operatorname{grad}_{\gamma_r(t)} \int_\Lambda \frac{\partial G(\gamma_r(t), y)}{\partial \nu(y)} f(y)\,ds(y), \quad t \in [0, 2\pi].$$

From (17) we find

$$0 = -\frac{1}{2}\widetilde{q}(t)\phi(t) - \int_0^{2\pi} \operatorname{grad}_{\gamma_r(t)} G(\gamma_r(t), \gamma_r(\tau)) \cdot \widetilde{\zeta}(\tau)\,\phi(\tau)\,d\tau, \quad t \in [0, 2\pi]. \tag{18}$$

We define a double layer potential

$$W(x) := -\int_0^{2\pi} \operatorname{grad}_x G(x, \gamma_r(\tau)) \cdot \nu(\tau)\widetilde{q}(\tau)\phi(\tau)\,d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r.$$

Since the function W is harmonic in the interior of Γ_r and satisfies the homogeneous Dirichlet boundary condition, (18), it implies $W \equiv 0$ in the interior of Γ_r. One can show, similarly to [10, Theorem 6.21], that the operator $-I + K$ is injective, where

$$(K\psi)(t) = \int_0^{2\pi} \operatorname{grad}_{\gamma_r(t)} G(\gamma_r(t), \gamma_r(\tau)) \cdot \nu(\tau)\psi(\tau)\,d\tau, \quad t \in [0, 2\pi]$$

Hence from (18) we obtain

$$\widetilde{q}(t)\phi(t) = 0, \quad t \in [0, 2\pi]$$

By the jump relations for the function V we have

$$\frac{1}{|\gamma_r'|}\phi = \left.\frac{\partial V^-}{\partial \nu}\right|_{\Gamma_r} - \left.\frac{\partial V^+}{\partial \nu}\right|_{\Gamma_r} = -\left.\frac{\partial V^+}{\partial \nu}\right|_{\Gamma_r}.$$

Since by Holmgren's theorem $\frac{\partial V^+}{\partial \nu}$ cannot vanish on open subsets of Γ_r and $|\gamma_r| \neq 0$ we obtain $\widetilde{q} = 0$ and hence $q = 0$. □

Remark (about the Algorithm 2).
If the interior boundary is a circle, then exists a nontrivial solution $q = const$ to the homogeneous equation $A'[r, \varphi]q = 0$. Indeed, introducing the function

$$V(x) = -q \operatorname{grad}_x \int_0^{2\pi} G(x, \gamma_r(\tau)) \cdot \nu(\tau)\varphi(\tau)d\tau, \quad x \in \mathbb{R}^2 \setminus \Gamma_r$$

we obtain that V is a unique solution to the Neumann boundary value problem with the homogeneous condition in the exterior of Λ, and hence $V^+|_{\Gamma_r} = 0$. Since the null-space of the operator of the integral equation

$$\frac{1}{2}\varphi(t) - \operatorname{grad}_x \int_0^{2\pi} G(t, \gamma_r(\tau)) \cdot \nu(\tau)\varphi(\tau)d\tau = 0, \quad t \in [0, 2\pi]$$

is not empty, one can find $q \neq 0$ which solves $A'[r, \varphi]q = 0$.

In view of this remark we introduced the modified version $\widetilde{A}'[r,\varphi]$, (14), instead of the operator $A'[r,\varphi]$.

Now we describe three iterative algorithms for the numerical solution of (11)-(12).

Algorithm 1.

1. Choose some starting value r. Solve the well-posed integral equation

$$S_r\phi = -w_r. \tag{19}$$

2. For the given r and φ solve the system of linearized ill-posed integral equations

$$S_r\psi + S'[r,\phi]q + w'[r]q = -S_r\phi - w_r, \tag{20}$$

$$A_r\psi + A'[r,\phi]q = g - \frac{\partial w}{\partial \nu} - A_r\phi \tag{21}$$

with respect to functions ψ and q.

3. Calculate new approximations for the radial function $r = r + q$ and for the density $\phi = \phi + \psi$.

4. Repeat steps 2-3 until some stopping criterion is satisfied.

Algorithm 2.

1. Choose some starting value r.

2. Solve the well-posed integral equation

$$\tilde{S}_r\varphi = -w_r. \tag{22}$$

3. For the given r and φ solve the linearized ill-posed integral equation

$$\tilde{A}'[r,\varphi]q = g - \frac{\partial w}{\partial \nu} - \tilde{A}_r\varphi \tag{23}$$

with respect to function q.

4. Calculate new approximations for the radial function $r = r + q$.

5. Repeat steps 2-4 until some stopping criterion is satisfied.

Algorithm 3.

1. Choose some starting value r.

2. Solve the ill-posed integral equation

$$A_r\phi = g - \frac{\partial w}{\partial \nu}. \tag{24}$$

3. For given r and φ solve the linearized ill-posed integral equation

$$S'[r,\phi]q + w'[r]q = -S_r\phi - w_r, \tag{25}$$

with respect to function q.

4. Calculate new approximations for the radial function $r = r + q$.

5. Repeat steps 2-4 until some stopping criteria is satisfied. Note here that we need to use some regularization method in the case of ill-posed integral equations. According to properties of the corresponding integral operators an application of the Tikhonov regularization is justified for the algorithms $1, 3$.

4. Implementation

Algorithm 1.

Step1. On the first step of this algorithm we need to solve the well posed integral equation of the first kind (19) with a logarithmic singularity for a current approximation of r. Since all functions in this equation are 2π periodic we implement the trigonometric quadrature method. To do this we rewrite the equation (19) in the following equivalent form

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(\tau)\left[-\frac{1}{2}\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+K_r(t,\tau)\right]d\tau=-w_r(t),\quad t\in[0,2\pi],$$

where

$$K_r(t,\tau):=\frac{1}{2}\ln\frac{\frac{4}{e}\sin^2\frac{t-\tau}{2}}{|\gamma_r(t)-\gamma_r(\tau)|^2}+\tilde{G}(\gamma_r(t),\gamma_r(\tau)),\ \ t\neq\tau$$

with the diagonal term

$$K_r(t,t)=\frac{1}{2}\ln\frac{1}{e|\gamma_r'(t)|^2}+\tilde{G}(\gamma_r(t),\gamma_r(t)).$$

The following trigonometric quadratures with equidistant points $t_j=\frac{j\pi}{n}$, $j=0,\dots,2n-1$ are used

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau\approx\sum_{k=0}^{2n-1}R_k(t)\,f(t_k)\tag{26}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\,d\tau\approx\frac{1}{2n}\sum_{k=0}^{2n-1}f(t_k)\tag{27}$$

with explicit expressions for the weight functions given in [10]. It leads to the following system of linear equations with respect to $\phi_{ni}\approx\phi(t_i)$

$$\sum_{i=0}^{2n-1}\phi_{ni}[-\frac{1}{2}R_i(t_k)+\frac{1}{2n}K(t_k,t_i)]=-\tilde{w}_r(t_k),\ \ k=0,\dots,2n-1$$

with

$$\tilde{w_r}(t)=-\frac{1}{2n}\sum_{i=0}^{2n-1}f(t_i)H(t,t_i),$$

where

$$H(t,\tau):=\frac{\partial G(\gamma_r(t),\lambda(\tau))}{\partial\nu(\lambda(\tau))}|\lambda'(t)|.$$

The convergence and error analysis for this method can be found in [10].

Step2. We search the unknown corrections in the system (20)-(21) as

$$\psi_n=\sum_{i=0}^{2n-1}\psi_{ni}l_i^1,\quad q_m=\sum_{i=0}^{2m}q_{mi}l_i^2,$$

where l_i^1, $i = 0, \dots, 2n-1$ are basic Lagrangian trigonometric polynomials and l_i^2, $i = 0, \dots, 2m$ are known basic functions. The quadrature method applied to (20)-(21) give us the linear system

$$\sum_{i=0}^{2n-1} \psi_{ni} \mathcal{A}_{ki}^{(11)} + \sum_{i=0}^{2m} q_{mi} \mathcal{A}_{ki}^{(12)} = b_k^{(1)}, \quad k = 0, \dots, 2n-1,$$

$$\sum_{i=0}^{2n-1} \psi_{ni} \mathcal{A}_{ki}^{(21)} + \sum_{i=0}^{2m} q_{mi} \mathcal{A}_{ki}^{(22)} = b_k^{(2)}, \quad k = 0, \dots, 2n-1$$

with matrix coefficients

$$\mathcal{A}_{ki}^{(11)} = -\frac{1}{2} R_i(t_k) + \frac{1}{2n} K_r(t_k, t_i), \quad \mathcal{A}_{ki}^{(21)} = \frac{1}{2n} H_r^{(2)}(t_k, t_i),$$

$$\mathcal{A}_{ki}^{(12)} = \frac{1}{2n} \sum_{j=0}^{2n-1} \{\phi_{nj}[l_i^2(t_j) L_r^{(1)}(t_k, t_j) + l_i^2(t_k) L_r^{(2)}(t_k, t_j)] + l_i^2(t_j) f(t_i) W_r(t_k, t_j)\},$$

$$\mathcal{A}_{ki}^{(22)} = \frac{1}{2n} \sum_{j=0}^{2n-1} \phi_{nj} l_i^2(t_j) H_r^{(1)}(t_k, t_j)$$

and right hand side

$$b_k^{(1)} = \sum_{i=0}^{2n-1} \phi_{ni}[-\frac{1}{2} R_i(t_k) - \frac{1}{2n} K_r(t_k, t_i)] - \tilde{w}_r(t_k),$$

$$b_k^{(2)} = g(t_k) - \frac{\partial \tilde{w}_r}{\partial \nu}(t_k) - \frac{1}{2n} \sum_{i=0}^{2n-1} \phi_{ni} H_r^{(2)}(t_k, t_i).$$

Here $2n \geq 2m + 1$.
Thus the received ill-posed linear system is overdetermined and therefore we reduce it to the least-squares problem. The following cost functional needs to be minimized

$$F(\psi_{n0}, \dots, \psi_{n,2n-1}, q_{m0}, \dots, q_{m,2m}) =$$

$$= \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj} \mathcal{A}_{ij}^{(11)} + \sum_{j=0}^{2m} q_{mj} \mathcal{A}_{ij}^{(12)} - b_i^{(1)} \right|^2 +$$

$$\sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj} \mathcal{A}_{ij}^{(21)} + \sum_{j=0}^{2m} q_{mj} \mathcal{A}_{ij}^{(22)} - b_i^{(2)} \right|^2 +$$

$$\alpha \sum_{j=0}^{2n-1} \omega_{1j} \psi_{nj}^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2$$

with the regularization parameters $\alpha > 0$ and $\beta > 0$ and weight coefficients ω_{1j} and ω_{2j}. Clearly, the final linear system has the form

$$\alpha\omega_{1i}\psi_{ni} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(11)} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(12)} = \mathbf{b}_i^{(1)}, \quad i = 0, \ldots, 2n-1,$$

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(21)} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(22)} = \mathbf{b}_i^{(2)}, \quad i = 0, \ldots, 2m,$$

where

$$\mathbf{a}_{ij}^{(\ell p)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(\ell 1)}\mathcal{A}_{kj}^{(p1)} + \sum_{k=0}^{2m} \mathcal{A}_{ki}^{(\ell 2)}\mathcal{A}_{kj}^{(p2)}$$

and

$$\mathbf{b}_i^{(\ell)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(\ell 1)} b_k^{(1)} + \sum_{k=0}^{2m} \mathcal{A}_{ki}^{(\ell 2)} b_k^{(2)}.$$

Step 3. Now we can evaluate the new values for the radial function $r_m = r_m + q_m$ and for the density $\phi_n = \phi_n + \psi_n$.
The following stopping criterion can be used

$$\frac{\|q_m\|_{L^2[0,2\pi]\|}}{\|r_m\|_{L^2[0,2\pi]}} < \epsilon$$

with sufficiently small $\epsilon > 0$, or a discrepancy principle, as well.
Algorithm 2.
Step2. It is analogous to the *Step 1* from the Algorithm 1.
Step3. To find the correction q from (23) we make the discretization by the quadrature method and due to its ill-posednes we minimize the following Tikhonov functional

$$F(q_{m0}, \ldots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(22)} - b_i^{(2)} \right|^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2, \; 2n \geq 2m+1.$$

The corresponding linear system has the form

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij} = \mathbf{b}_i, \quad i = 0, \ldots, 2m$$

with

$$\mathbf{a}_{ij} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(22)}\mathcal{A}_{kj}^{(22)}, \quad \mathbf{b}_i = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(22)} b_k^{(2)}.$$

Algorithm 3.
Step2. The discretization in (24) and ill-posednes of the received linear system lead to the minimization of the following Tikhonov functional

$$F(\psi_{n0}, \ldots, \psi_{n,2n-1}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2n-1} \psi_{nj}\mathcal{A}_{ij}^{(21)} - \tilde{b}_i^{(2)} \right|^2 + \alpha \sum_{j=0}^{2n-1} \omega_{1j}\psi_{nj}^2$$

with

$$\tilde{b}_i^{(2)} = g(t_k) - \frac{\partial \tilde{w}_r}{\partial \nu}(t_k).$$

which is equivalent to solving the linear system

$$\alpha\omega_{1i}\psi_{ni} + \sum_{j=0}^{2n-1} \psi_{nj}\mathbf{a}_{ij}^{(1)} = \mathbf{b}_i^{(1)}, \quad i = 0, \dots, 2n-1,$$

where

$$\mathbf{a}_{ij}^{(1)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(21)}\mathcal{A}_{kj}^{(21)}, \quad \mathbf{b}_i^{(1)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(21)}\tilde{b}_k^{(2)}.$$

Step3. To find the correction q from (25) we make the discretization by quadrature method and due to its ill-posednes we minimize the following Tikhonov functional

$$F(q_{m0}, \dots, q_{m,2m}) = \sum_{i=0}^{2n-1} \left| \sum_{j=0}^{2m} q_{mj}\mathcal{A}_{ij}^{(12)} - b_i^{(1)} \right|^2 + \beta \sum_{j=0}^{2m} \omega_{2j} q_{mj}^2, \; 2n \geq 2m+1.$$

Thus the corresponding linear system has the form

$$\beta\omega_{2i}q_{mi} + \sum_{j=0}^{2m} q_{mj}\mathbf{a}_{ij}^{(2)} = \mathbf{b}_i^{(2)}, \quad i = 0, \dots, 2m,$$

where

$$\mathbf{a}_{ij}^{(2)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(12)}\mathcal{A}_{kj}^{(12)}, \quad \mathbf{b}_i^{(2)} = \sum_{k=0}^{2n-1} \mathcal{A}_{ki}^{(12)} b_k^{(1)}.$$

5. Numerical examples

We demonstrate the feasibility of the proposed methods for the inverse problem (1)-(4) with the following boundaries $\lambda(t) = \{Rc(t), t \in [0, 2\pi]\}$ with $R = 2$, and

$$\gamma_r(t) = \left\{ \sqrt{\cos^2 t + 0.25\sin^2 t}\, c(t), t \in [0, 2\pi] \right\}.$$

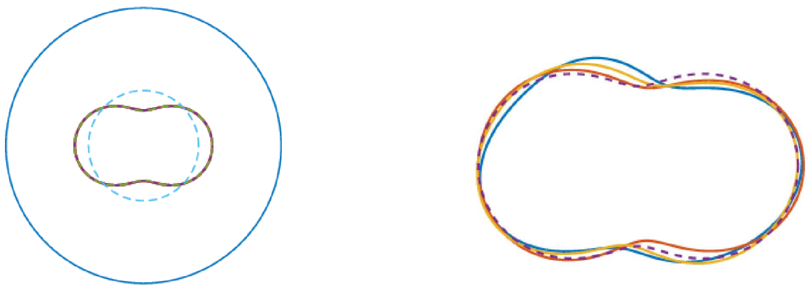
The Cauchy data on Λ were generated by solving the direct problem (1)-(3) for $f = 1$ on Λ and calculating g as the normal derivative on Λ. The noisy data were formed as

$$g^\delta = g + \delta(2\eta - 1)\|g\|_{L_2(\Lambda)}$$

with the noise level δ and the random value $\eta \in (0, 1)$. The results of the numerical experiments for exact and noisy data with $\delta = 5\%$ are reflected on Fig. 1. Here we used the following discretization parameters $n = 16$, $m = 4$ and $\epsilon = 0.0001$. The values of regularization parameters, numbers of iterations and L_2-errors are given in Tabl. 1.

	δ	It.	E	α	β
Algorithm 1	0%	7	0.00561	10^{-13}	10^{-5}
	5%	8	0.07367	10^{-10}	10^{-3}
Algorithm 2	0%	21	0.00614		10^{-2}
	5%	17	0.03843		10^{-1}
Algorithm 3	0%	21	0.00322	10^{-14}	10^{-7}
	5%	15	0.04714	10^{-5}	10^{-1}

TABL. 1. Errors and regularization parameters



a). Reconstruction for the exact data b). Reconstruction for 5% noise in the data

FIG. 1. Reconstruction of the boundary Γ

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