Журнал обчислювальної та прикладної математики | 2016 № 2 (122) | Journal of Numerical & Applied Mathematics

UDC 517.9

ON THE GENERALIZED SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM WITH NEUMANN CONDITION FOR THE WAVE EQUATION BY THE USE OF THE RETARDED DOUBLE LAYER POTENTIAL AND THE LAGUERRE TRANSFORM

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Резюме. Описано і обгрунтовано підхід до розв'язування мішаної задачі Неймана для однорідного хвильового рівняння, який базується на інтегральному перетворенні Лагера за часовою змінною і граничних інтегральних рівняннях. Для подання узагальненого розв'язку такої задачі використано запізнюючий потенціал подвійного шару, густину якого шукають у вигляді ряду Фур'є-Лагера. Для коефіцієнтів розвинення отримано аналітичні формули. В результаті вихідну нестаціонарну задачу зведено до еквівалентної послідовності граничних інтегральних рівнянь.

Abstract. Approach for solving of the initial-boundary value problem for the homogeneous wave equation with the Neumann condition is described and proved. It is based on the Laguerre transform in the time domain and the boundary integral equations. The retarded double layer potential is used for representation of generalized solution of such problem in some weighted Sobolev spaces. The density of retarded potential is expanded in Fourier-Laguerre series, coefficients of which have special convolution form. As a result, the initial-boundary value problem is reduced to an equivalent sequence of boundary integral equations.

1. Introduction

Retarded surface potentials are useful tools for the integral representation of generalized solutions of initial-boundary value problems for the wave equation with homogeneous initial conditions [1, 2, 6]. Their advantages in applications are, first of all, caused by the generality of domain form. In addition, they allow to reduce initial-boundary value problems to equivalent time-dependent boundary integral equations (TDBIEs, also known as retarded potential boundary integral equations), with unknown densities of potentials that are determined at each moment of time only on the domain's boundary [7, 12, 17]. Further, they implicitly impose radiation conditions at infinity.

However, practical usage of retarded potentials has some computational complexity, caused by the presence of dependency of potential density on the time and the spatial coordinates (so-called delay, see for example [7]). To overcome

Key words. Boundary integral equation method; wave equation; Sobolev spaces; generalized solution; retarded surface potentials; Laguerre transform; time domain boundary integral equations.

such problems, the following approaches have been used: one of traditional discretisations on spatial variables is applied to unknown values and auxiliary problems are used for calculation of the time dependency. In particular, a convolution quadrature [17] method has been utilized in many applications. It is based on the use of sustainable methods for ordinary differential equations. Using this method in the time is more stable than using Galerkin or collocation time approximations.

Another way to take account of dependence in the time domain is the Fourier-Laplace integral transform over the time variable [1, 6, 7]. This method is well suitable for theoretical investigations, however, it is complex (except for some cases) to perform corresponding inverse transform in applications. In this respect the Laguerre transform, for which the inverse transform is to find the sum of corresponding Fourier-Laguerre series, proved to be more constructive. In combination with the method of boundary integral equations (BIEs) such transform was used in [3, 8, 10, 13, 15, 18, 21] for numerical solution of various evolution problems.

In [16] we considered the generalized solution of the Dirichlet initial-boundary value problem for the wave equation with homogeneous initial conditions. Its representation was built by using the retarded single layer potential in some weighted Sobolev spaces, in which the desired solution and the potential density allow the Fourier-Laguerre expansion over the time. In this case the Fourier-Laguerre coefficients for the potential density are defined as solutions of the BIEs. This work is concerned with applying the same method to the analogical problem for the wave equation but with the Neumann boundary condition. In this case we deal with the retarded double layer potential.

We begin in Section 2 with a brief description of the proposed method. Section 3 contains the basic definitions of proper functional spaces, followed by a formulation of the main theorem about conditions under which the generalized solution of the problem belongs to the desired weighted Sobolev spaces and can be obtained by the proposed method. In Section 4 we investigate the regularity of the retarded double layer potential depending on the smoothness of its density. Definitions of the Laguerre transform and a q-convolution of sequences are introduced in Section 5, as well as the Fourier-Laguerre expansion is given for the potential's density and the representation formula for the corresponding Fourier-Laguerre coefficients are obtained. In Section 6 we explain how this approach leads to a sequence of BIE, solutions of which are Fourier-Laguerre coefficients of the unknown potential's density. At the end we prove a theorem that has been referred to above.

2. Reduction of the Neumann problem to a sequence of BIE

Let Ω be a domain in $\mathbb{R}^3$ with Lipschitz boundary Γ, $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$, $\mathbb{R}_+ := (0, \infty)$, $Q := \Omega \times \mathbb{R}_+$, $\Sigma := \Gamma \times \mathbb{R}_+$, and $\boldsymbol{\nu}(x)$ be a unit vector in the direction of the outward normal to the surface Γ at a point $x \in \Gamma$.

Let us consider the initial-boundary value problem: find a function $u(x, t)$, $(x, t) \in \overline{Q}$, that satisfies (in some sense) the homogeneous wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = 0, \quad (x,t) \in Q, \tag{1}$$

homogeneous initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in \Omega, \tag{2}$$

and the Neumann boundary condition

$$\partial_{\boldsymbol{\nu}(x)} u(x,t) = g(x,t), \quad (x,t) \in \Sigma. \tag{3}$$

Here $\Delta = \sum_{i=1}^{3} \partial^2/\partial x_i^2$ is the Laplace operator and $\partial_{\boldsymbol{\nu}}$ denotes the normal derivative operator. Note that for a sufficiently smooth function u and the surface Γ operator $\partial_{\boldsymbol{\nu}}$ can be expressed as

$$\partial_{\boldsymbol{\nu}(x)} u(x,\cdot) = \boldsymbol{\nu}(x) \cdot \nabla_x u(x,\cdot),$$

where ∇_x is the gradient operator.

We use the retarded double layer potential to solve the problem (1)-(3)

$$(\mathcal{D}\lambda)(x,t) := \frac{1}{4\pi} \int_{\Gamma} \boldsymbol{\nu}(y) \cdot \nabla_y \left(\frac{\lambda(z, t - |x-y|)}{|x-y|} \right) \bigg|_{z=y} d\Gamma_y, \quad (x,t) \in Q, \tag{4}$$

where $\lambda : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a density. It is known (see, e.g., [1], [21]) that if an arbitrary function $\lambda(y,\tau)$, $(y,\tau) \in \Gamma \times \mathbb{R}$, is smooth enough and $\lambda(y,\tau) = 0$ when $y \in \Gamma, \tau \leq 0$, then function

$$u(x,t) := (\mathcal{D}\lambda)(x,t), \quad (x,t) \in Q, \tag{5}$$

satisfies (in classic sense) the wave equation and initial conditions. In order for the function u to satisfy the boundary conditions (3) we will consider the following limit

$$(\mathcal{W}\lambda)(x,t) := \frac{1}{4\pi} \boldsymbol{\nu}(x) \cdot \lim_{x' \to x} \nabla_{x'} \int_{\Gamma} \boldsymbol{\nu}(y) \cdot \nabla_y \left(\frac{\lambda(z, t - |x'-y|)}{|x'-y|} \right) \bigg|_{z=y} d\Gamma_y, \tag{6}$$

where $x' := x - \epsilon \boldsymbol{\nu}(x) \in \Omega$, $\epsilon > 0$ notes a point close to the points $x \in \Gamma$, understanding approach of $x' \to x$ by $\epsilon \to 0$. The function u satisfies the boundary condition (3), if the function λ is a solution of the TDBIE

$$(\mathcal{W}\lambda)(x,t) = g(x,t), \quad (x,t) \in \Sigma. \tag{7}$$

To find the solution of the equation (7) we use the Laguerre transform, namely the expansion of function in the Fourier-Laguerre series by Laguerre polynomials $\{L_j(\sigma\cdot)\}_{j\in\mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}$ is a set of natural numbers and $\sigma > 0$ is a parameter. It is known (see, e.g., [11]) that the system of Laguerre polynomials forms an orthogonal basis in the space $L^2_\sigma(\mathbb{R}_+)$ $v : \mathbb{R}_+ \to \mathbb{R}$ of functions such that $\int_{\mathbb{R}_+} |v(\tau)|^2 e^{-\sigma\tau} d\tau < \infty$, therefore, $v(\tau) = \sum_{j=0}^{\infty} v_j \, L_j(\sigma\tau)$, $\tau \in \mathbb{R}_+$, where $v_j := \sigma \int_{\mathbb{R}_+} v(\tau) \, L_j(\sigma\tau) \, e^{-\sigma\tau} d\tau$ $(j \in \mathbb{N}_0)$ are the Laguerre-Fourier coefficients of function v.

Therefore, the solution of the TDBIE (7) can be expressed as:

$$\lambda(y,\tau) = \begin{cases} \sum\limits_{j=0}^{\infty} \lambda_j(y)\, L_j(\sigma\tau), & y \in \Gamma,\ \tau \in \mathbb{R}_+, \\ 0, & y \in \Gamma,\ \tau \in \mathbb{R} \setminus \mathbb{R}_+, \end{cases} \tag{8}$$

where λ_j ($j \in \mathbb{N}_0$) are the corresponding Laguerre-Fourier coefficients of the unknown function λ. In the case of the retarded argument with arbitrary value $a > 0$ we have an expansion

$$\lambda(y, t-a) = \sum_{j=0}^{\infty} \tilde{\lambda}_j(y,a)\, L_j(\sigma t), \tag{9}$$

where coefficients $\tilde{\lambda}_j(y,a)$ have the representation formula, that was obtained in [16]

$$\tilde{\lambda}_j(y,a) = e^{-\sigma a} \sum_{i=0}^{j} \zeta_{j-i}(\sigma a)\, \lambda_i(y), \quad j \in \mathbb{N}_0, \tag{10}$$

and where

$$\zeta_0(s) := 1,\ \ \zeta_k(s) := L_k(s) - L_{k-1}(s),\ \ s \in \overline{\mathbb{R}_+} = [0,\infty), \quad k \in \mathbb{N}. \tag{11}$$

Then, taking into account (9) and (10), we will have

$$\begin{aligned} \lambda(y, t-|x-y|) =& e^{-\sigma|x-y|} \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \zeta_{j-i}(\sigma|x-y|)\, \lambda_i(y) \right) L_j(\sigma t), \\ & x, y \in \Gamma,\ t \in \mathbb{R}_+, \end{aligned} \tag{12}$$

and then introducing notation similar to (6)

$$(W_k\xi)(x) := \frac{1}{4\pi} \boldsymbol{\nu}(x) \cdot \lim_{x' \to x} \nabla_{x'} \int\limits_{\Gamma} \xi(y) \boldsymbol{\nu}(y) \cdot \nabla_y e_k(x'-y) d\Gamma_y, \tag{13}$$

where

$$e_k(z) := (4\pi|z|)^{-1} \zeta_k(\sigma|z|) e^{-\sigma|z|} \ \text{ at } \ z \in \mathbb{R}^3 \setminus \{0\},\ \ k \in \mathbb{N}_0, \tag{14}$$

for the normal derivative of the retarded double layer potential (6) we obtain an expansion

$$(\mathcal{W}\lambda)(x,t) = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} (W_{j-i}\lambda_i)(x) \right) L_j(\sigma t), \quad x, y \in \Gamma,\ t \in \mathbb{R}_+. \tag{15}$$

Now lets write the Fourier-Laguerre expansion of the function g

$$g(x,t) = \sum_{j=0}^{\infty} g_j(x)\, L_j(\sigma t), \quad (x,t) \in \Sigma, \tag{16}$$

where $g_j(x) = \sigma \int\limits_{\mathbb{R}_+} g(x,\tau)\, L_j(\sigma\tau)\ e^{-\sigma\tau} d\tau,\ x \in \Gamma,\ j \in \mathbb{N}_0$. Taking into account (15) and (16) along with (7) and equating expressions near the Laguerre polynomials with the same indexes, we get an infinite triangular system of BIE for finding the Laguerre-Fourier coefficients $\lambda_0,\ \lambda_1, ..., \lambda_j, ...$ of the density λ

$$\sum_{i=0}^{j}(W_{j-i}\lambda_i)(x) = g_j(x), \quad x \in \Gamma,\ j \in \mathbb{N}_0. \tag{17}$$

It is easy to see that system (17) can be rewritten as a recursive sequence of equations

$$\begin{cases} (W_0\lambda_0)(x) = g_0(x), \\ (W_0\lambda_1)(x) = \tilde{g}_1(x), \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ (W_0\lambda_j)(x) = \tilde{g}_j(x), \quad j \in \mathbb{N}, \quad x \in \Gamma, \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{cases} \tag{18}$$

where

$$\tilde{g}_j(x) := g_j(x) - \sum_{i=0}^{j-1}(W_{j-i}\lambda_i)(x), \quad j \in \mathbb{N}. \tag{19}$$

For every $j \in \mathbb{N}_0$ the corresponding j-th equation (18) is hypersingular equation that has the form

$$(W_0\xi)(x) = h(x), \quad x \in \Gamma. \tag{20}$$

It is known [4,9] that the equation (20) has a unique solution ξ for an arbitrary function h within a fairly broad class. To find the solution of this equation one can use numerical methods (see for example [24] and references there).

After finding the solution $\lambda_0,\ \lambda_1, ...$ of the BIE system (17) (same as a solution of the sequence (18)), the generalized solution of the problem (1)-(3) can be presented using (4), (5) and (12) as a sum of the series

$$u(x,t) = \frac{1}{4\pi}\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\int_{\Gamma}\lambda_i(y)\,\boldsymbol{\nu}(y)\cdot\nabla_y e_{j-i}(x-y)d\Gamma_y\right)L_j(t), \ \ (x,t) \in Q. \tag{21}$$

If we introduce a notation

$$(D_k\xi)(x) := \frac{1}{4\pi}\int_{\Gamma}\xi(y)\boldsymbol{\nu}(y)\cdot\nabla_y e_k(x'-y)d\Gamma_y, \tag{22}$$

the formula (21) can be rewritten as:

$$u(x,t) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}D_{j-i}\lambda_i(x)\right)L_j(\sigma t), \ \ (x,t) \in Q. \tag{23}$$

If there exists a sum of the series (23) we can consider its partial sum as an approximate solution for the problem (1)-(3). In this case one can choose (by some criteria) value N and find from the system (18) the first components $\lambda_0,\ \lambda_1, ..., \lambda_N$ of its solution. Then the approximate solution of the problem (1)-(3) is the partial sum

$$\widetilde{u}_N(x,t) = \sum_{j=0}^{N}\left(\sum_{i=0}^{j}D_{j-i}\lambda_i(x)\right)L_j(\sigma t), \ \ (x,t) \in Q. \tag{24}$$

We can use the representation (24) for the numerical solution of the problem (1)-(3).

3. Variational formulation of the problem (1)-(3)

First, we need to introduce some additional notations. Let $L^2(\Omega)$ be the Lebesgue space of square integrable functions $v : \Omega \to \mathbb{R}$ with inner product

$$(v, w)_{L^2(\Omega)} := \int_\Omega vw dx, \;\; v, w \in L^2(\Omega),$$

and norm $\|v\|_{L^2(\Omega)} := \sqrt{(v, v)_{L^2(\Omega)}}$, and $H^1(\Omega)$ be the Sobolev space of functions $v \in L^2(\Omega)$, having generalized derivatives of $v_{x_1}, v_{x_2}, v_{x_3}$ in $L^2(\Omega)$, with inner product

$$(v, w)_{H^1(\Omega)} := \int_\Omega (\nabla v \nabla w + vw)\, dx, \;\; v, w \in H^1(\Omega),$$

and norm $\|v\|_{H^1(\Omega)} := \sqrt{(v, v)_{H^1(\Omega)}}$, $v \in H^1(\Omega)$. Let us denote $H^{1/2}(\Gamma)$ a space of traces of elements of $H^1(\Omega)$ on the surface Γ, $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ a trace operator, $H^{-1/2}(\Gamma) := \left(H^{1/2}(\Gamma)\right)'$ a conjugate to $H^{1/2}(\Gamma)$ space, and $< \cdot, \cdot >_\Gamma$ a duality relation for $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Also let $H^1_0(\Omega)$ be a closure of the space $C^\infty_0(\Omega)$ with norm $\|\cdot\|_{H^1(\Omega)}$ and $H^{-1}(\Omega) := (H^1_0(\Omega))'$ be the conjugate to $H^1_0(\Omega)$ space. In the space $H^1(\Omega)$ we also consider a subspace $H^1(\Omega, \Delta) := \left\{\, v \in H^1(\Omega) \,|\, \Delta v \in L^2(\Omega)\right\}$ with the norm

$$\|v\|_{H^1(\Omega,\Delta)} := \left(\|v\|^2_{H^1(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Let X be a Hilbert space with inner product $(\cdot, \cdot)_X$ and inducted norm $||\cdot||_X$. For some parameter $\sigma > 0$ we consider a weighted Lebesgue space $L^2_\sigma(\mathbb{R}_+; X)$ [5] with weight $\rho_\sigma(t) = e^{-\sigma t}$, $t \in \mathbb{R}_+$, elements of which are measurable functions $v : \mathbb{R}_+ \to X$ such that $\int_{\mathbb{R}_+} ||v(t)||^2_X\, e^{-\sigma t} dt < \infty$. This space is equipped with inner product

$$(v, w)_{L^2_\sigma(\mathbb{R}_+;X)} := \int_{\mathbb{R}_+} \big(v(t), w(t)\big)_X\, e^{-\sigma t} dt, \;\; v, w \in L^2_\sigma(\mathbb{R}_+; X), \tag{25}$$

and the norm

$$\|v\|_{L^2_\sigma(\mathbb{R}_+;X)} := \sqrt{(v, v)_{L^2_\sigma(\mathbb{R}_+;X)}}, \;\; v \in L^2_\sigma(\mathbb{R}_+; X). \tag{26}$$

Note that the space $L^2_\sigma(\mathbb{R}_+; X)$ is complete [22, section II.1]. We will assume that the elements of space $L^2_\sigma(\mathbb{R}_+; X)$ are extended with zero for non-positive arguments.

For any $m \in \mathbb{N}$ let us denote the weighted Sobolev space as

$$H^m_\sigma(\mathbb{R}_+; X) := \{\;\, v \in L^2_\sigma(\mathbb{R}_+; X) \,|\, v^{(k)} \in L^2_\sigma(\mathbb{R}_+; X), \; k = \overline{1, m}\} \tag{27}$$

with norm

$$\|v\|_{H^m_\sigma(\mathbb{R}_+;X)} := \left(\sum_{k=0}^{m} \left\|v^{(k)}\right\|^2_{L^2_\sigma(\mathbb{R}_+;X)}\right)^{1/2}. \tag{28}$$

Here derivatives $v^k (k \in \mathbb{N})$ are understood in terms of the space $\mathcal{D}'(\mathbb{R}_+;X)$, elements of which are distributions with values in the space X. Note that $H^1_\sigma(\mathbb{R}_+;X) \subset C(\overline{\mathbb{R}_+};X)$ [5, theorem 7, section XVIII].

Let us also denote following spaces:

$$L^2_{\text{loc}}(\overline{\mathbb{R}_+};X) := \{v : \mathbb{R}_+ \to X - \text{ measurable} \mid \ ||v(\cdot)||_X \in L^2(0,\tau) \ \ \forall \tau > 0\},$$

$$H^1_{\text{loc}}(\overline{\mathbb{R}_+};X) := \{v \in L^2_{\text{loc}}(\overline{\mathbb{R}_+};X) \mid \ v' \in L^2_{\text{loc}}(\overline{\mathbb{R}_+};X)\}.$$

Definition 1. Let $g \in L^2_{\text{loc}}(\overline{\mathbb{R}_+};H^{-1/2}(\Gamma))$. A generalized solution of the problem (1)-(3) is a function $u \in H^1_{\text{loc}}(\overline{\mathbb{R}_+};L^2(\Omega)) \cap L^2_{\text{loc}}(\overline{\mathbb{R}_+};H^1(\Omega))$, which satisfies the first of the initial conditions (2) and the integral identity

$$\iint_Q (\nabla u \nabla v - u'v')dxdt = \iint_\Sigma g\gamma_0 v d\Gamma dt \tag{29}$$

for any $v \in H^1(\mathbb{R}_+;L^2(\Omega)) \cap L^2(\mathbb{R}_+;H^1(\Omega))$ such that $\operatorname{supp} v$ is a bounded set.

Note that there exists at most one generalized solution of the problem (1)-(3) [19, Theorem 1, Ch. V, §2].

We introduce a couple more notations. As the sequence of elements of set X we understand mapping $\mathbf{V} : \mathbb{N}_0 \to X$ (denoted by **bold** letter) and write it as a vector-column $\mathbf{v} := (v_0, v_1, ...)^\top$. All possible sequences of elements of the set X are denoted by X^∞. It is clear that when X is a linear space, then X^∞ is also a linear space. Recall that

$$l^2 := \left\{\mathbf{v} \in \mathbb{R}^\infty \mid \quad \sum_{j=0}^{\infty} |v_j|^2 < +\infty\right\}$$

with the inner product $(\mathbf{v}, \mathbf{w}) = \sum\limits_{j=0}^{\infty} v_j w_j$, $\mathbf{v}, \mathbf{w} \in l^2$ and the norm $\|\ \mathbf{v}\|_{l^2} := \left(\sum\limits_{j=0}^{\infty} |v_j|^2\right)^{1/2}$, $\mathbf{v} \in l^2$.

Let X be a Hilbert space with inner product $(\cdot,\cdot)_X$ and inducted norm $||\cdot||_X$. We consider the Hilbert space

$$l^2(X) := \left\{\mathbf{v} \in X^\infty \mid \quad \sum_{j=0}^{\infty} \|v_j\|^2_X < +\infty\right\}$$

with the inner product $(\mathbf{v}, \mathbf{w}) = \sum\limits_{j=0}^{\infty} (v_j, w_j)_X$, $\mathbf{v}, \mathbf{w} \in l^2(X)$ and the norm $||\mathbf{v}||_{l^2(X)} := \left(\sum\limits_{j=0}^{\infty} \|v_j\|^2_X\right)^{1/2}$, $\mathbf{v} \in l^2(X)$. It is obvious that $l^2 = l^2(\mathbb{R})$.

Definition 2 ([14]). Let X, Y, Z be arbitrary sets and $q : X \times Y \to Z$ be some mapping. By a *q-convolution* of sequences $\mathbf{u} \in X^\infty$ and $\mathbf{v} \in Y^\infty$ we understand the sequence $\mathbf{w} := (w_0, w_1, ..., w_j, ...)^\top \in Z^\infty$, whose elements are obtained by the rule

$$w_j := \sum_{i=0}^{j} q\left(u_{j-i}, v_i\right) \equiv \sum_{i=0}^{j} q\left(u_i, v_{j-i}\right), \quad j \in \mathbb{N}_0; \tag{30}$$

the q-convolution of $\mathbf{u}$ and $\mathbf{v}$ is shortly written in the form $\mathbf{w} = \mathbf{u} \underset{q}{\circ} \mathbf{v}$.

Let $X = \mathbb{R}$ and $Y = Z$ be linear spaces and $q(u, v) := uv,\ u \in \mathbb{R},\ v \in Y$. Then the components of q-convolution of arbitrary $\mathbf{u} \in \mathbb{R}^\infty$ and $\mathbf{v} \in Y^\infty$ will be denoted as

$$w_j = \sum_{i=0}^{j} u_{j-i} v_i, \quad j \in \mathbb{N}_0, \tag{31}$$

and the q-convolution would be denoted as $\mathbf{w} := \mathbf{u} \underset{\mathbb{R}\times Y}{\circ} \mathbf{v}$.

If $X = H^{-1/2}(\Gamma),\ Y = H^{1/2}(\Gamma),\ Z = \mathbb{R}$ and $q(u, v) :=< u, v >_\Gamma,\ u \in H^{-1/2}(\Gamma),\ v \in H^{1/2}(\Gamma)$, for components of the q-convolution of arbitrary sequences $\mathbf{u} \in \left(H^{-1/2}(\Gamma)\right)^\infty$ and $\mathbf{v} \in \left(H^{1/2}(\Gamma)\right)^\infty$ we will have

$$w_j = \sum_{i=0}^{j} < u_{j-i}, v_i >_\Gamma, \quad j \in \mathbb{N}_0, \tag{32}$$

and will write $\mathbf{w} := \mathbf{u} \underset{\Gamma}{\circ} \mathbf{v}$.

Another example concerns the q-convolutions of linear operators when $X = \mathcal{L}(Y, Z)$ is the space of linear operators acting from the space Y into the space Z and $q(A, v) := Av$, $A \in \mathcal{L}(Y, Z),\ v \in Y$, for components of the q-convolution of arbitrary sequences $\mathbf{A} \in \left(\mathcal{L}(Y, Z)\right)^\infty$ and $\mathbf{v} \in Y^\infty$ we will have the following formula

$$w_j = \sum_{i=0}^{j} A_{j-i} v_i, \quad j \in \mathbb{N}_0, \tag{33}$$

and will write $\mathbf{w} := \mathbf{A} \underset{Z}{\circ} \mathbf{v}$.

Based on the above, we define the sequence

$$\mathbf{u}(x) = \big(\mathbf{D} \underset{H^1(\Omega)}{\circ} \boldsymbol{\lambda}\big)(x), \quad x \in \Omega, \tag{34}$$

which is the q-convolution of the sequence $\mathbf{D}$ composed of operators $D_k : H^{1/2}(\Gamma) \to H^1(\Omega))$, $k \in \mathbb{N}_0$, given by the formula (22), and the sequence $\boldsymbol{\lambda}$ of Fourier-Laguerre coefficients of the function λ. Similarly, BIE system (17) can be rewritten as

$$\mathbf{W} \underset{H^{-1/2}(\Gamma)}{\circ} \boldsymbol{\lambda} = \mathbf{g} \quad \text{in} \quad l^2(H^{-1/2}(\Gamma)), \tag{35}$$

where $\mathbf{W} : l^2(H^{1/2}(\Gamma)) \to l^2(H^{-1/2}(\Gamma))$ is a boundary operator whose components act in accordance with (13), and $\mathbf{g}$ is the sequence of Fourier-Laguerre coefficients of the function g.

Now we can formulate the main result of this paper as the following statement.

Theorem 1. *Let $g \in H^{m+4}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then there exists a unique generalized solution of the problem (1)-(3), it belongs to the space $H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$ and for any $\sigma \geq \sigma_0$ such an inequality holds*

$$||u||_{H^{m+1}_\sigma(\mathbb{R}_+;H^1(\Omega))} \leq C||g||_{H^{m+4}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))}, \tag{36}$$

where $C > 0$ is a constant that is not dependent on g.

In addition, the generalized solution of the problem (1)-(3) can be represented as the sum of a serie (23), that is convergent in the space $L^2_{\sigma_0}(\mathbb{R}_+; H^1(\Omega, \Delta))$, where $u_j \in H^1(\Omega, \Delta)$ $(j \in \mathbb{N}_0)$ are the corresponding components of the q-convolution (34), and elements of the sequence $\boldsymbol{\lambda} \in l^2(H^{1/2}(\Gamma))$ are solutions of BIE system (35), in which $\mathbf{g} \in l^2(H^{-1/2}(\Gamma))$ is the sequence of Laguerre-Fourier coefficients for the function g.

Proof of Theorem 1 will be presented further on.

4. Some properties of the retarded double layer potential

For examination of the generalized solution of the problem (1)-(3) we need some results of the work [1].

Proposition 1 ([1], Theorem 1)**.** *Let $g \in H^1_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for some $\sigma_0 > 0$. Then unique generalized solution of the problem space (1)-(3) exists, it belongs to space*

$$H^1_{\sigma_0}(\mathbb{R}_+; L^2(\Omega)) \cap L^2_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$$

and the following inequality holds:

$$||u||_{L^2_\sigma(\mathbb{R}_+;H^1(\Omega))} + ||u'||_{L^2_\sigma(\mathbb{R}_+;L^2(\Omega))} \leq C_1||g||_{H^1_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))} \ \forall\sigma \geq \sigma_0, \tag{37}$$

where $C_1 > 0$ is a constant.

In addition, the generalized solution of the problem (1)-(3) can be represented as a retarded double layer potential $\mathcal{D}\lambda$ with density $\lambda \in L^2_\sigma(\mathbb{R}_+; H^{1/2}(\Gamma))$,

$$||\lambda||_{L^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))} \leq C_2||g||_{H^1_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))} \ \forall\sigma \geq \sigma_0, \tag{38}$$

where $C_2 > 0$ is a constant.

Let us outline the proof of the statement 1, received results will be exploited further for the proof of 1.

First, consider some auxiliary spaces. Let X be arbitrary Banach space with a norm $||\cdot||_X$. By $\mathcal{D}'(\mathbb{R}; X)$ we denote the space of distributions with values in the space X and by $\mathcal{D}'_+(\mathbb{R}; X)$ we denote the space of so-called causal distributions, consisting of distributions $v \in \mathcal{D}'(\mathbb{R}; X)$, for which the condition $\langle v, \phi \rangle = 0$ holds for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\phi \subset (-\infty, 0)$. For any $\sigma_0 > 0$ let us define a space

$$\mathcal{L}'_{+,\sigma_0}(\mathbb{R}; X) := \{\ f \in \mathcal{D}'_+(\mathbb{R}; X) \mid e^{-\sigma_0 \cdot} f(\cdot) \in \mathcal{S}'_+(\mathbb{R}; X)\ \},$$

where $\mathcal{S}'_+(\mathbb{R}; X)$ denotes the space of slow casual distributions.

Note that for slow casual distributions one can define the Fourier transform over the time variable (See, e.g., [5, section XVI, §2, definition 7])

$$\mathcal{F} : \mathcal{S}'_+(\mathbb{R}; X) \to \mathcal{S}'_+(\mathbb{R}; X). \tag{39}$$

It is an isomorphic mapping from $\mathcal{S}'_+(\mathbb{R}; X)$ onto $\mathcal{S}'_+(\mathbb{R}; X)$ and enables us to define the Fourier-Laplace transform for any element $f \in \mathcal{L}'_{+,\sigma_0}(\mathbb{R}; X)$ [5, section XVI, §2, definition 8]:

$$\widehat{F}(\omega) := \mathcal{F}(e^{-\sigma \cdot} f(\cdot))(\eta), \;\; \omega = \eta + i\sigma \in \mathbb{R} \times (\sigma_0, +\infty). \tag{40}$$

In case of $f \in \mathcal{L}'_{+,\sigma_0}(\mathbb{R}; X) \cap L^1_{\text{loc}}(\mathbb{R}_+; X)$ this transform has an integral representation

$$\widehat{f}(\omega) := \int_{\mathbb{R}} e^{i\eta t} e^{-\sigma t} f(t) dt = \int_{\mathbb{R}} e^{i\omega t} f(t) dt, \;\; \omega = \eta + i\sigma \in \mathbb{R} \times (\sigma_0, +\infty). \tag{41}$$

As we can see the Fourier-Laplace transform is applicable to the elements of functional spaces that appear in the definition of the generalized solution u of the problem (1)-(3). So with its help the initial-boundary value problem (1)-(3) can be reduced to following boundary value problem regarding a function $\widehat{u}(\cdot, \omega) \in H^1(\Omega, \Delta)$:

$$\Delta \widehat{u} + \omega^2 \widehat{u} = 0 \;\text{ in } \Omega, \tag{42}$$

$$\gamma_1 \widehat{u} = \widehat{g} \text{ on } \Gamma, \tag{43}$$

where $\widehat{g}(\cdot, \omega) \in H^{-1/2}(\Gamma)$ is a known function and $\omega \in \mathbb{R} \times (\sigma_0, +\infty)$ is a parameter.

Solution of the problem (42), (43) can be represented as a double layer potential

$$\widehat{u}(x, \omega) = \big(\widehat{D}_\omega \widehat{\lambda}\big)(x) := \frac{1}{4\pi} \int_\Gamma \widehat{\lambda}(y, \omega)\, \boldsymbol{\nu}(y) \cdot \nabla_y \frac{e^{i\omega|x-y|}}{|x-y|}\, d\Gamma_y, \;\; x \in \Omega, \tag{44}$$

whose density $\widehat{\lambda}(\cdot, \omega) \in H^{1/2}(\Gamma)$ is a solution of BIE

$$\widehat{W}_\omega \widehat{\lambda} = \widehat{g} \text{ in } H^{-1/2}(\Gamma), \tag{45}$$

where $\widehat{W}_\omega := \gamma_1 \circ \widehat{D}_\omega$. A boundary operator $\widehat{W}_\omega$ is $H^{1/2}$-elliptical on Γ, that implies the existence and uniqueness of the solution for BIE(45).

The integral (44) exists because of $\widehat{\lambda}(\cdot, \omega) \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and $\frac{e^{i\omega|x-y|}}{|x-y|}$ is an infinitely differentiable function for an arbitrary fixed point $x \in \Omega$. In addition, according to the [4, Theorem 1], the double layer potential and its normal derivative are bounded operators, respectively, $\widehat{D}_\omega : H^{1/2}(\Gamma) \to H^1(\Omega, \Delta)$ and $\widehat{W}_\omega : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$.

As we see, the boundary value problem (42), (43) and BIE (45) depend on parameter ω, consequently, their solutions, accordingly, $\widehat{u}(\cdot, \omega)$ and $\widehat{\lambda}(\cdot, \omega)$, and the double layer potential $\widehat{D}_\omega$ and the boundary operator $\widehat{W}_\omega$ can be considered as functions of parameter ω. They are proved to be holomorphic in half-space $\mathbb{R} \times (\sigma_0, +\infty)$ and satisfy following estimates [1, inequality (2.6),(2.7) and (2.11)], [23, inequality (3.17) and (3.18)]:

$$||\widehat{u}(\cdot, \omega)||_{H^1(\Omega)} \le \tilde{C}_1 |\omega| ||\widehat{g}(\cdot, \omega)||_{H^{-1/2}(\Gamma)}, \tag{46}$$

$$||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)} \le \tilde{C}_2|\omega|||\widehat{g}(\cdot,\omega)||_{H^{-1/2}(\Gamma)}, \tag{47}$$

$$||\widehat{W}_\omega\widehat{\lambda}||_{H^{-1/2}(\Gamma)} \le \tilde{C}_3|\omega|^2\,||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)}, \tag{48}$$

$$||\widehat{D}_\omega\widehat{\lambda}||_{H^1(\Omega)} \le \tilde{C}_4|\omega|^{3/2}\,||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)}, \tag{49}$$

$$||\widehat{D}_\omega\widehat{\lambda}||_{H^1(\Omega,\Delta)} \le \tilde{C}_5|\omega|^{5/2}\,||\widehat{\lambda}(\cdot,\omega)||_{H^{1/2}(\Gamma)}, \tag{50}$$

where $\tilde{C}_i > 0$ are some constants.

Proposition 2 ([5], section XVI, §2, Theorem 1)**.** *Let X be a Banach space over the field $\mathbb{C}$ of complex numbers with norm $||\cdot||_X$, and $\omega \mapsto \widehat{f}(\omega)$ be a function defined in $\mathbb{C}$ with values in the space X. For the function $\widehat{f}(\omega)$ to be the Fourier-Laplace transform of the distribution $f \in \mathcal{D}'(\mathbb{R};X)$ with support $\operatorname{supp} f \subset [\alpha,+\infty)$ it is necessary and sufficient that $\widehat{f}(\omega)$ is holomorphic in the half-space $\mathbb{R}\times(\sigma_0,+\infty)$ with values in X and satisfies inequality*

$$||\widehat{f}(\omega)||_X \le e^{-\sigma\alpha}\mathrm{Pol}(|\omega|),\ \ \omega = \eta + i\sigma \in \mathbb{R}\times(\sigma_0,+\infty), \tag{51}$$

where $\mathrm{Pol}(|\omega|)$ is a polynom of the variable $|\omega|$.

By the statement 2 one can prove from inequalities (46)-(50) the existence of distributions that match the generalized solution of the problem (1)-(3), retarded double layer potential and its density. They are elements of spaces $\mathcal{L}'_{+,\sigma_0}(\mathbb{R};X)$ with values in the appropriate space X (see e.g. [1, Theorem 1], and [6, section 2]) such that

$$\widehat{\mathcal{D}\lambda} = \widehat{D}_\omega\widehat{\lambda}\ \text{ and }\ \widehat{\mathcal{W}\lambda} = \widehat{W}_\omega\widehat{\lambda}.$$

Using inequalities (46)-(50) we can easily get estimates of the generalized solution of the problem (1)-(3), and the retarded double layer potential. To do this, let us consider in the set $\mathcal{L}'_{+,\sigma_0}(\mathbb{R};X)$ for arbitrary values $\sigma \ge \sigma_0$ and $p \in \mathbb{R}$ a space

$$\mathcal{H}^p_\sigma(\mathbb{R}_+;X) := \{\ f \in \mathcal{L}'_{+,\sigma_0}(\mathbb{R};X)\,|\int\limits_{\mathbb{R}+i\sigma}|\omega|^{2p}\,||\widehat{f}(\omega)||^2_X d\omega < +\infty\} \tag{52}$$

with the norm

$$||f||_{\mathcal{H}^p_\sigma(\mathbb{R}_+;X)} := \left(\frac{1}{2\pi}\int\limits_{\mathbb{R}+i\sigma}|\omega|^{2p}\,||\widehat{f}(\omega)||^2_X d\omega\right)^{1/2}. \tag{53}$$

Proposition 3 ([2], section 3.1)**.** *Let $\sigma > 0$, $m \in \mathbb{N}_0$. A function v belongs to the space $H^m_\sigma(\mathbb{R}_+;X)$ if and only if it belongs to the space $\mathcal{H}^m_{\sigma/2}(\mathbb{R}_+;X)$.*

Note that statement 3 is the consequence of Parseval-Plancherel identity:

$$\int\limits_{\mathbb{R}} e^{-2\sigma t}\left(f(t),g(t)\right)_X dt = \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\sigma}\left(\hat{f}(\omega),\hat{g}(\omega)\right)_X d\omega. \tag{54}$$

Lemma 1. *Let $\sigma > 0$, $m \in \mathbb{N}_0$. If an arbitrary function λ is an element of the space $H^{m+2}_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then $\mathcal{D}\lambda \in H^m_\sigma(\mathbb{R}_+;H^1(\Omega))$. If $\lambda \in H^{m+3}_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then $\mathcal{D}\lambda \in H^m_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$ and $\mathcal{W}\lambda \in H^m_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$.*

Proof. Let us show that for any fixed values of $p \in \mathbb{R}$ and $\alpha > 0$ the operator

$$\mathcal{D} : \mathcal{H}_\alpha^{p+3/2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}_\alpha^p(\mathbb{R}_+; H^1(\Omega)) \tag{55}$$

is bounded. To achieve this, for an arbitrary function $\lambda \in \mathcal{H}_\alpha^{p+3/2}(\mathbb{R}_+; H^{1/2}(\Gamma))$, $\alpha \geq \alpha_0$, taking into account norm definition (53) and inequality (49), following estimate can be performed:

$$\begin{aligned}
||\mathcal{D}\lambda||^2_{\mathcal{H}_\alpha^p(\mathbb{R}_+;H^1(\Omega))} &= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}||\widehat{\mathcal{D}\lambda}||^2_{H^1(\Omega)}d\omega = \\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}||\widehat{D}(\cdot,\omega)\widehat{\lambda}(\cdot,\omega)||^2_{H^1(\Omega)}d\omega \leq \\
&\leq \frac{\tilde{C}_4^2}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p+3}||\widehat{\lambda}(\cdot,\omega)||^2_{H^{1/2}(\Gamma)}d\omega = \\
&= \tilde{C}_4^2||\lambda||^2_{\mathcal{H}_\alpha^{p+3/2}(\mathbb{R}_+;H^{1/2}(\Gamma))} \leq \tilde{C}_4^2||\lambda||^2_{\mathcal{H}_\alpha^{p+2}(\mathbb{R}_+;H^{1/2}(\Gamma))}.
\end{aligned} \tag{56}$$

Hence, the operator (55) is bounded, and, in particular, for the values $p = m$ and $\alpha = \sigma/2$ the following operator is also bounded

$$\mathcal{D} : H_\sigma^{m+2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to H_\sigma^m(\mathbb{R}_+; H^1(\Omega)). \tag{57}$$

Similarly to the previous case, but using inequality (50), for arbitrary $p \in \mathbb{R}$ and $\alpha > 0$ it can be shown that the operator

$$\mathcal{D} : \mathcal{H}_\alpha^{p+5/2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}_\alpha^p(\mathbb{R}_+; H^1(\Omega,\Delta)) \tag{58}$$

is also bounded, and when $p = m$ and $\alpha = \sigma/2$ the same will apply to the operator

$$\mathcal{D} : H_\sigma^{m+3}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to H_\sigma^m(\mathbb{R}_+; H^1(\Omega,\Delta)), \quad m \in \mathbb{N}_0, \tag{59}$$

which means $\mathcal{D}\lambda \in H_\sigma^m(\mathbb{R}_+; H^1(\Omega,\Delta))$. It is known [4, theorem Lemma 3.2, 1] that for elements of space $H^1(\Omega,\Delta)$ we can define linear continuous operator of normal derivative $\gamma_1 : H^1(\Omega,\Delta) \to H^{-1/2}(\Gamma)$. Therefore, in this case it is legitimate to define the composition of operators $\gamma_1 \circ \mathcal{D} =: \mathcal{W}$, for which for any $p \in \mathbb{R}$ and $\alpha > 0$ using inequality (48) following estimate can be applied:

$$\begin{aligned}
||\mathcal{W}\lambda||^2_{\mathcal{H}_\alpha^p(\mathbb{R}_+;H^{-1/2}(\Gamma))} &= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}||\widehat{W}(\cdot,\omega)\widehat{\lambda}(\cdot,\omega)||^2_{H^{-1/2}(\Gamma)}d\omega \leq \\
\leq \frac{\tilde{C}_3^2}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2p}|\omega|^4||\widehat{\lambda}(\cdot,\omega)||^2_{H^{1/2}(\Gamma)}d\omega &= \tilde{C}_3^2||\lambda||^2_{\mathcal{H}_\alpha^{p+2}(\mathbb{R}_+;H^{1/2}(\Gamma))}.
\end{aligned} \tag{60}$$

This means that the operator

$$\mathcal{W} : \mathcal{H}_\alpha^{p+2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}_\alpha^p(\mathbb{R}_+; H^{-1/2}(\Gamma)) \tag{61}$$

is bounded, and when $p = m$ and $\alpha = \sigma/2$ following operator is also bounded:

$$\mathcal{W} : H_\sigma^{m+2}(\mathbb{R}_+; H^{1/2}(\Gamma)) \to H_\sigma^m(\mathbb{R}_+; H^{-1/2}(\Gamma)). \tag{62}$$

□

5. Application of the Laguerre transform to retarded potentials

Now let us give the definition of the Laguerre transform and outline some of its properties which we have obtained in [16]. Consider a mapping $\mathcal{L}: L^2_\sigma(\mathbb{R}_+;X) \to X^\infty$, where X is Hilbert space with inner product $(\cdot,\cdot)_X$ and inducted norm $||\cdot||_X$, which operates according to the rule

$$f_k := \sigma \int\limits_{\mathbb{R}_+} f(t)\, L_k(\sigma t)\, e^{-\sigma t} dt, \quad k \in \mathbb{N}_0, \tag{63}$$

where $\{L_k(\sigma\cdot)\}_{k\in\mathbb{N}_0}$ are Laguerre polynomials, which form orthogonal basis in the space $L^2_\sigma(\mathbb{R}_+)$. We will also use the notation

$$\mathcal{L}_k f \equiv (\mathcal{L}f)(k) := f_k \;\; \forall k \in \mathbb{N}_0.$$

Note that since the function $t \mapsto ||f(t)||_X |L_k(\sigma t)| e^{-\sigma t} \in L^1(\mathbb{R}_+)$, the Bochner integral in formula (63) is convergent and its value is an element of space X.

Also consider the mapping $\mathcal{L}^{-1}: l^2(X) \to L^2_\sigma(\mathbb{R}_+;X)$, which maps an arbitrary sequence $\mathbf{h} = (h_0,\, h_1, ...,\, h_k, ...)^\top$ to a function

$$h(t) := (\mathcal{L}^{-1}\mathbf{h})(t) = \sum_{k=0}^{\infty} h_k\, L_k(\sigma t),\; t \in \mathbb{R}_+. \tag{64}$$

Proposition 4 ([16], Theorem 2)**.** *The mapping $\mathcal{L}: L^2_\sigma(\mathbb{R}_+;X) \to X^\infty$ that maps the arbitrary function f to the sequence $\mathbf{f} = (f_0,\, f_1, ...,\, f_k, ...)^\top$ according to the formula (63), is injective and its image is the space $l^2(X)$, and*

$$\|f\|^2_{L^2_\sigma(\mathbb{R}_+;X)} = \frac{1}{\sigma}\sum_{k=0}^{\infty} ||f_k||^2_X. \tag{65}$$

In addition, for the arbitrary function $f \in L^2_\sigma(\mathbb{R}_+;X)$ we have an equality

$$\mathcal{L}^{-1}\mathcal{L}f = f, \tag{66}$$

where the mapping $\mathcal{L}^{-1}: l^2(X) \to L^2_\sigma(\mathbb{R}_+;X)$ is the inverse to $\mathcal{L}$ and maps the arbitrary sequence $\mathbf{h} = (h_0,\, h_1, ...,\, h_k, ...)^\top$ to the function h according to the formula (64).

Definition 3. Let $\sigma > 0$ and X be a Hilbert space. Mappings

$$\mathcal{L}: L^2_\sigma(\mathbb{R}_+;X) \to l^2(X) \quad \text{and} \quad \mathcal{L}^{-1}: l^2(X) \to L^2_\sigma(\mathbb{R}_+;X),$$

mentioned in theorem 4, are called, respectively, direct and inverse Laguerre transforms, and the formula (65) is an analog of the Parseval equality.

Proposition 5 ([16], Lemma 1)**.** *Let $\sigma > 0$, $a > 0$ and X be a Hilbert space with inner product $(\cdot,\cdot)_X$ and the norm $||\cdot||_X$. Then for an arbitrary function $f \in L^2_\sigma(\mathbb{R}_+;X)$ function $f(\cdot - a)$ belongs to space $L^2_\sigma(\mathbb{R}_+;X)$ too and the following equalities hold:*

$$||f(\cdot - a)||_{L^2_\sigma(\mathbb{R}_+;X)} = e^{-\frac{\sigma a}{2}} ||f(\cdot)||_{L^2_\sigma(\mathbb{R}_+;X)}, \tag{67}$$

$$\tilde{\mathbf{f}}_a = e^{-\sigma a} \boldsymbol{\zeta}(\sigma a) \underset{\mathbb{R}\times X}{\circ} \mathbf{f}, \tag{68}$$

$$f(\,\cdot\, - a) = e^{-\sigma a}\sum_{j=0}^{\infty}\Big(\sum_{i=0}^{j}\zeta_{j-i}(\sigma a) f_i\Big) L_j(\sigma\cdot) \;\; in \;\; L^2_\sigma(\mathbb{R}_+;X), \qquad (69)$$

where $\boldsymbol{f} = \mathcal{L}f(\cdot)$ *and* $\tilde{\mathbf{f}}_a := \mathcal{L}f(\cdot - a)$.

Using statements 4 and 5 we can outline conditions for the density λ of the retarded double layer potential $\mathcal{D}\lambda$, which guarantees that the Fourier-Laguerre expansions for this potential

$$(\mathcal{D}\lambda)(t) = \sum_{j=0}^{\infty} u_j\, L_j(\sigma t), \;\; x \in \Omega,\, t \in \mathbb{R}_+, \qquad (70)$$

and its normal derivative

$$(\mathcal{W}\lambda)(x,t) = \sum_{j=0}^{\infty} \tilde{u}_j(x)\, L_j(\sigma t), \;\; x \in \Gamma,\, t \in \mathbb{R}_+, \qquad (71)$$

where $u_j := (\mathcal{L}_j\mathcal{D}\lambda)$ and $\tilde{u}_j := (\mathcal{L}_j\mathcal{W}\lambda)$, are convergent in the corresponding Sobolev spaces.

Lemma 2. *Let $\sigma > 0$ be an arbitrary constant.*
(i) *If an arbitrary function λ belongs to space $H^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then expansion (70) is convergent in the space $L^2_\sigma(\mathbb{R}_+;H^1(\Omega))$. If $\lambda \in H^3_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then expansions (70) and (71) are convergent in spaces $L^2_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$ and $L^2_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$, correspondingly.*
(ii) *Coefficients $u_j,\ \tilde{u}_j,\ j \in \mathbb{N}_0$, are components of q-convolutions (34) and*

$$\tilde{\mathbf{u}}(x) = \mathbf{W} \underset{H^{-1/2}(\Gamma)}{\circ} \boldsymbol{\lambda}, \;\; x \in \Gamma, \qquad (72)$$

correspondingly, where $\boldsymbol{\lambda} = \mathcal{L}\lambda \in l^2(H^{1/2}(\Gamma))$.

Proof. The first statement of this lemma follows from the fact that by Lemma 1 the retarded double layer potential with a density that is an element of the space $H^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, belongs to space $L^2_\sigma(\mathbb{R}_+;H^1(\Omega))$. If $\lambda \in H^3_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$, then $\mathcal{D}\lambda \in L^2_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$, and $\mathcal{W}\lambda \in L^2_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$. Then by Theorem 4 the Laguerre transform can be applied to both the potential and its normal derivative, and expansions (70) and (71) with obtained coefficients are convergent in the appropriate spaces.

Let us consider the retarded potential (4) with density $\lambda \in H^2_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))$ at an arbitrary point $x \in \Omega$, and apply formula (63) to it as to an element of the space $L^2_\sigma(\mathbb{R}_+;H^1(\Omega))$:

$$\begin{aligned} u_j(x) :=& \mathcal{L}_j\, \mathcal{D}\lambda(x) = \\ =& \frac{\sigma}{4\pi}\int_{\mathbb{R}_+} e^{-\sigma t} L_j(\sigma t)\int_{\Gamma} \boldsymbol{\nu}(y)\cdot\nabla_y\Big(\frac{\lambda(z,t-|x-y|)}{|x-y|}\Big)\Big|_{z=y} d\Gamma_y dt, \qquad (73)\\ & j \in \mathbb{N}_0. \end{aligned}$$

As points x and y do not coincide (i.e. partial derivatives in inner integral are bounded) and $||u_j||_{H^1(\Omega)} < +\infty$, then we can change the order of integration according to the Fubini theorem

$$u_j(x) = \frac{1}{4\pi}\int\limits_{\Gamma} \partial_{\vec{\nu}(y)}\left(\frac{\sigma}{|x-y|}\int\limits_{\mathbb{R}_+}\lambda(z,t-|x-y|)e^{-\sigma t}L_j(\sigma t)dt\right)\Big|_{z=y} d\Gamma_y, \quad x \in \Omega. \tag{74}$$

Note that in the obtained expression, the inner integral is expressing the j-th Fourier-Laguerre coefficient of "retarded" function λ. Therefore, according to Lemma 5 and formulas (68),(14) we can write the following:

$$\begin{aligned} u_j(x) =& \frac{1}{4\pi}\int\limits_{\Gamma} \partial_{\vec{\nu}(y)}\left(\frac{e^{-\sigma|x-y|}}{|x-y|}\sum_{i=0}^{j}\zeta_{j-i}(x-y)\lambda_i(z)\right)\Big|_{z=y} d\Gamma_y = \\ =& \sum_{i=0}^{j}\int\limits_{\Gamma}\lambda_i(y)\partial_{\vec{\nu}(y)}e_{j-i}(x-y)\, d\Gamma_y, \;\; j\in\mathbb{N}_0, \;\; x\in\Omega, \end{aligned} \tag{75}$$

where $\lambda_j := \mathcal{L}_j\lambda, \ j \in \mathbb{N}_0$.

For an arbitrary fixed point $x \in \Omega$ all components of sequence $\mathbf{e}(x-\cdot)$ are continuously differentiable functions on Γ. Since $\lambda_j \in H^{1/2}(\Gamma), \ j \in \mathbb{N}_0$, then for the Lipschitz surface Γ integrals in (75) can be interpreted as the inner product of elements in $L^2(\Gamma)$ and can be extended to the duality relation on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$:

$$u_j(x) = \sum_{i=0}^{j}\left\langle \partial_{\vec{\nu}(\cdot)}e_{j-i}(x-\cdot), \lambda_i(\cdot)\right\rangle_{\Gamma}, \ x\in\Omega, \ j\in\mathbb{N}_0. \tag{76}$$

So we received coefficients of the q-convolution (34).

If $\lambda \in H^3_\sigma(\mathbb{R}_+; H^{1/2}(\Gamma))$ we have $||u_j||_{H^1(\Omega,\Delta)} < +\infty$ and, obviously, for any point $x \in \Omega$ previous considerations regarding functions in integrals in formulas (73)-(76) hold. Therefore the form of coefficients $u_j, \ j \in \mathbb{N}_0$, is the same. Besides, for these coefficients as elements of the space $H^1(\Omega, \Delta)$, we can define linear continuous operator of normal derivative [4, Lemma 3.2, Theorem 1]. Let us show that $\tilde{u}_j = \gamma_1 u_j, \ j \in \mathbb{N}_0$.

Consider an arbitrary point $x \in \Gamma$ and apply the Laguerre transform to $\mathcal{W}\lambda$:

$$\begin{aligned} &\tilde{u}_j(x) := \mathcal{L}_j\,\mathcal{W}\lambda(x) = \frac{\sigma}{4\pi}\int\limits_{\mathbb{R}_+} e^{-\sigma t}L_j(\sigma t)\times \\ &\boldsymbol{\nu}(x)\cdot\lim_{x'\to x}\nabla_{x'}\int\limits_{\Gamma}\boldsymbol{\nu}(y)\cdot\nabla_y\left(\frac{\lambda(z,t-|x'-y|)}{|x'-y|}\right)\Big|_{z=y} d\Gamma_y dt < +\infty. \end{aligned} \tag{77}$$

If we move differentiation by normal at the point x out of the integral over the time variable, we receive $\tilde{u}_j(x) = \gamma_1 u_j(x)$. □

Note that we do not move outer differentiation inside the integral over the boundary Γ in order to avoid a high order of the singularity in a kernel. The definition of normal derivative operator γ_1 in case if $\mathbf{u} \in (H^1(\Omega,\Delta))^\infty$ was presented in [20]. In applications when calculating the respective singular integrals it is possible to replace normal derivatives with corresponding derivatives in the tangent plane (See, e.g. [1, formula (2.16)]).

6. Finding a generalized solution of the problem (1)-(3)

Consider operator

$$\mathcal{G} : \mathcal{H}^1_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma)) \to \mathcal{H}^0_\alpha(\mathbb{R}_+; H^1(\Omega)),\ \alpha = \sigma_0/2, \tag{78}$$

which maps the boundary value g to the generalized solution $u = \mathcal{G}g$ of the problem (1)-(3) according to the proposition 1. Taking into account the obvious inclusion

$$H^1_\sigma(\mathbb{R}_+; H^1(\Omega)) \subset \big(H^1_\sigma(\mathbb{R}_+; L^2(\Omega)) \cap L^2_\sigma(\mathbb{R}_+; H^1(\Omega))\big),$$

let us define a restriction of the operator $\mathcal{G}$ on elements from weighted Sobolev spaces.

Lemma 3. *Let $g \in H^{m+2}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ with some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then for arbitrary values $\sigma \geq \sigma_0$ operator*

$$\mathcal{G} : H^{m+2}_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma)) \to H^m_\sigma(\mathbb{R}_+; H^1(\Omega)) \tag{79}$$

is bounded.

Proof. Let g be an arbitrary function from the space $H^{m+2}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$. Considering it as an element of the space $\mathcal{H}^{m+2}_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ with $\alpha = \sigma_0/2$, we will have the solution $u = \mathcal{G}g$. Let us estimate it using the inequality (46):

$$\begin{aligned} ||u||^2_{\mathcal{H}^m_\alpha(\mathbb{R}_+;H^1(\Omega))} &= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m}||\hat{u}(\cdot,\omega)||^2_{H^1(\Omega)}d\omega \leq \\ &\leq \frac{\tilde{C}_1^2}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m}|\omega|^2||\hat{g}(\cdot,\omega)||^2_{H^{-1/2}(\Gamma)}d\omega = \\ &= \tilde{C}_1^2||g||^2_{\mathcal{H}^{m+2}_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))} < \infty. \end{aligned} \tag{80}$$

Since $u \in \mathcal{H}^m_\alpha(\mathbb{R}_+; H^1(\Omega))$, we get $u \in H^m_\sigma(\mathbb{R}_+; H^1(\Omega))$. □

Similarly, it is possible to examine the dependence of TDBIE solution on the smoothness (7) of the function g.

Lemma 4. *Let $g \in H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ with some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then there exists a unique solution of TDBIE (7) in the space $H^m_\sigma(\mathbb{R}_+; H^{1/2}(\Gamma))$, and it satisfies the following condition with an arbitrary $\sigma \geq \sigma_0$:*

$$||\lambda||_{H^m_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))} \leq C||g||_{H^{m+1}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))}, \tag{81}$$

where $C > 0$ is a constant.

Proof. According to the proposition 1 consider operator

$$\mathcal{V}^{-1} : \mathcal{H}^1_\alpha(\mathbb{R}_+; H^{1/2}(\Gamma)) \to \mathcal{H}^0_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$$

with the value $\alpha = \sigma_0/2$, that maps arbitrary function g to a unique solution of TDBIE $\lambda = \mathcal{V}^{-1}g$. With respect to the inequality (47), we get the following estimate for density λ:

$$\begin{aligned} ||\lambda||^2_{\mathcal{H}^m_\alpha(\mathbb{R}_+;H^{1/2}(\Gamma))} &= \frac{1}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m}||\hat{\lambda}(\cdot,\omega)||^2_{H^{1/2}(\Gamma)}d\omega \leq \\ &\leq \frac{\tilde{C}_2^2}{2\pi}\int\limits_{\mathbb{R}+i\alpha} |\omega|^{2m}|\omega|^2||\hat{g}(\cdot,\omega)||^2_{H^{-1/2}(\Gamma)}d\omega = \\ &= \tilde{C}_2^2||g||^2_{\mathcal{H}^{m+1}_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))} < \infty, \end{aligned} \tag{82}$$

and inequality (81) implies here. □

Thus, Lemmas 3 and 4 specify the conditions regarding the function g, that cause the required smoothness of both the retarded potential density and the generalized solution of the problem (1)-(3) in weighted Sobolev spaces.

Proof of Theorem 1. Let boundary data in the boundary condition (3) be defined with function $g \in H^{m+3}_{\sigma_0}(\mathbb{R}_+;H^{-1/2}(\Gamma))$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then, based on proposition 1, there exists a unique generalized solution of the problem (1)-(3) as element of the space $H^1_{\sigma_0}(\mathbb{R}_+;L^2(\Omega)) \cap L^2_{\sigma_0}(\mathbb{R}_+;H^1(\Omega))$. In addition, we can conclude according with Lemma 3 that with boundary data specified below this solution belongs to the space $H^{m+2}_{\sigma_0}(\mathbb{R}_+;H^1(\Omega)) \subset H^{m+1}_{\sigma_0}(\mathbb{R}_+;H^1(\Omega))$, and for arbitrary $\sigma \geq \sigma_0$ following inequality holds:

$$||u||_{H^{m+2}_\sigma(\mathbb{R}_+;H^1(\Omega))} \leq C||g||_{H^{m+3}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))}, \tag{83}$$

where $C > 0$ is a constant that does not depend on g. Obviously, in that case estimate (36) is correct.

Consider now the TDBIE (7), having $g \in H^{m+3}_{\sigma_0}(\mathbb{R}_+;H^{-1/2}(\Gamma))$. Then by Lemma 4 its solution λ belongs to space $H^{m+2}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$. Based on this, the Laguerre transform is applicable to density λ (by Theorem 4) and $\boldsymbol{\lambda} := \mathcal{L}\lambda \in l^2(H^{1/2}(\Gamma))$. Furthermore, with such density the potential $\mathcal{D}\lambda$ belongs to the space of solutions of the problem (1)-(3), because $\mathcal{D}\lambda \in H^{m+1}_\sigma(\mathbb{R}_+;H^1(\Omega))$ by Lemma 1.

If $g \in H^{m+4}_{\sigma_0}(\mathbb{R}_+;H^{-1/2}(\Gamma))$, then, according to Lemma 4 the density λ has to be element of the space $H^{m+3}_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$ and, by Lemma 1, we have $\mathcal{D}\lambda \in H^m_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$ and $\mathcal{W}\lambda \in H^m_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$. This means (by Lemma 4) that beginning from $m = 0$ the expansions (70) and (71) are convergent in spaces $L^2_\sigma(\mathbb{R}_+;H^1(\Omega,\Delta))$ and $L^2_\sigma(\mathbb{R}_+;H^{-1/2}(\Gamma))$, correspondingly, and the coefficients of these expansions have form of (34) and (72), correspondingly.

Let us build a sequence $\mathbf{g} := \mathcal{L}g \in l^2(H^{-1/2}(\Gamma))$ and substitute the Fourier-Laguerre expansion of the boundary function g in the right hand side of TDBIE (7). If we substitute the expansion (71) in its left hand side, we can equated the expressions beside Laguerre polynomials with the same index. As a result, we get an infinite triangular system of BIEs (35). It is known [20], that this system has a unique solution $\boldsymbol{\lambda}$. □

Consequently, the proposed method enables us to find the generalized solution of the Neumann problem for the homogeneous wave equation with homogeneous initial conditions using the Fourier-Laguerre expansion of the retarded double layer potential. Note that this approach can be adapted for finding the Cauchy datum of generalized solution using a Kirchhoff formula instead of retarded potential.

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Received 08.06.2016