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NUMERICAL SOLUTION OF LORD-SHULMAN THERMOPIEZOELECTRICITY FORCED VIBRATIONS PROBLEM

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Резюме. Ми розглядаємо модель термоп'єзоелектрики Лорда-Шульмана (LS). Для початково-крайової задачі LS-термоп'єзоелектрики формулюється відповідна варіаційна задача. Далі розглядаються вимушені коливання піроелектрика і варіаційна задача переписується у спеціальному вигляді для цього окремого випадку. Доводиться коректність останньої варіаційної задачі. З використанням дискретизації Гальоркіна будується чисельна схема для розв'язування цієї варіаційної задачі. Питання збіжності цієї схеми також розглянуті в цій статті. Зрештою, проводиться чисельний експеримент, який добре ілюструє вплив параметра "часу релаксації" на отримані розв'язки.

Abstract. We consider the Lord-Shulman (LS) model of thermopiezoelectricity. Variational formulation is constructed for the initial boundary value problem of LS-thermopiezoelectricity. Then forced vibrations of pyroelectric specimen are considered and the variational problem is rewritten in the special form for that particular case. Well-posedness of the latter variational problem is proved. Then using Galerkin semidiscretization a numerical scheme for solving this variational problem is built. The questions of convergence of this scheme are also covered in this article. Finally, a numerical experiment is performed, which perfectly illustrates the influence of "relaxation time" parameter on the obtained solutions.

1. Introduction

Nowadays piezoelectric and pyroelectric materials are widely utilized in various modern devices such as sensors, actuators, transducers, etc [14]. The classic theory of linear thermopiezoelectricity was introduced by Mindlin [12]. The further study of the theory was performed by Nowacki [13]. The main drawback of the classic theory is the assumption of infinite speed of propagation of thermal signals in the piezoelectric specimen. To overcome this, Lord and Shulman [10] proposed a modified theory of thermoelasticity (LS-theory), where the classic Fourier' law of heat conduction is replaced by Maxwell-Cattaneo equation with introduction of so-called "relaxation time". Chandrasekharaiah was the first researcher to apply the LS-theory to thermopiezoelectricity [5]. Later a set of generalization theories for thermoelasticity and thermopiezoelectricity

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was developed, for example Green-Lindsay, Chandrasekharaiah-Tzou, Green-Naghdi, etc. A good review of the existing generalization theories can be found in [1], [6], [8], [9]. Different methods were used by researchers to obtain the solutions of the generalized thermopiezoelectricity problem, see [2], [3], [7], [15], [20].

Forced vibrations of pyroelectrics is the special case of the thermopiezoelectricity problem and was studied under the classic (Mindlin's) theory in [11], [21] and [22]. In our previous work [19], we utilized our finite-element-based numerical scheme for solving forced vibrations problem under classic thermopiezoelectricity theory and developed an adaptive algorithm for obtaining solution with a preset level of accuracy. The goal of the present research is to construct a similar FEM-based numerical scheme for forced vibrations problem under LS-thermopiezoelectricity theory.

2. Problem statement

The theory of thermopiezoelectricity describes the coupled interaction of mechanical, electrical and thermal fields in pyroelectric material.

Suppose the piezoelectric specimen occupies a bounded domain Ω in Euclidean space $R^d, d = 1, 2,$ or 3 with continuous by Lipschitz boundary Γ with unit external normal vector $n = \{n_i\}_{i=1}^d$, where $n_i = cos(n, x_i)$. According to the classic theory (see [12, 13, 16, 17]), we need to find elastic displacement vector $\boldsymbol{u} = \boldsymbol{u}(x,t)$, electric potential $p = p(x,t)$ and temperature increment $\theta = \theta(x,t)$, which satisfy the following equations:

$$\rho u_i'' - \sigma_{ij,j} = \rho f_i, \tag{1}$$

$$D'_{k,k} + J_{k,k} = 0, \tag{2}$$

$$\rho(T_0 S' - w) + q_{i,i} = 0, \tag{3}$$

namely, equation of motion, differentiated Maxwell's equation and generalized heat equation respectively, where f_i is a vector of volume mechanical forces and w represents volume heat forces. Here the constitutive equations for stress tensor

$$\sigma_{ij} = c_{ijkm}[\varepsilon_{km} - \alpha_{km}\theta] - e_{kij}E_k, \tag{4}$$

electric displacement vector

$$D_k = e_{kij}\varepsilon_{ij} + \chi_{km}E_m + \pi_k\theta, \tag{5}$$

and entropy density

$$\rho S = c_{ijkm}\alpha_{km}\varepsilon_{ij} + \pi_k E_k + \tfrac{\rho c_v}{T_0}\theta \tag{6}$$

are used.

Vector J_k is the electrical current density, generated by a free electrical charge density. We assume that pyroelectric material is not an ideal dielectric, and the electric current runs through the pyroelectric specimen and satisfies standard Ohm's law, i.e.

$$J_k = z_{km}E_m(p). \tag{7}$$

Heat flux vector $\boldsymbol{q} = \boldsymbol{q}(x,t)$ is assumed to satisfy the standard Fourier's law:

$$q_i = -\lambda_{ij}\theta_{,j}. \tag{8}$$

Strain tensor ε_{km} and electrical field vector E_k are assumed to satisfy the relations

$$\begin{aligned} &\varepsilon_{km} = \varepsilon_{km}(\boldsymbol{u}) = \tfrac{1}{2}(u_{k,m} + u_{m,k}), \\ &E_k = E_k(p) = -p_{,k}, \end{aligned} \tag{9}$$

where comma in the subscript stands for the partial derivative by the spatial variable, i. e. $g_{,k} = -\partial g/\partial x_k$.

The other symbols in the above equations represent the material properties of pyroelectric medium: c_{ijkm} is an elasticity coefficients tensor with common properties of symmetry and ellipticity, that is:

$$\begin{aligned} &c_{ijkm} = c_{jikm} = c_{kmij}, \\ &c_{ijkm}\kappa_{ij}\kappa_{km} \geq c_0\kappa_{ij}\kappa_{km}, c_0 = const > 0, \quad \forall \kappa_{ij} = \kappa_{ji} \in R, \end{aligned} \tag{10}$$

α_{ij} is a thermal expansion tensor with similar properties

$$\begin{aligned} &\alpha_{ij} = \alpha_{ji}, \\ &\alpha_{ij}\xi_i\xi_j \geq \alpha_0\xi_i\xi_j, \alpha_0 = const > 0, \quad \forall \xi_i \in R, \end{aligned} \tag{11}$$

e_{kij} ia a piezoelectricity tensor with properties:

$$e_{kij} = e_{kji}, \tag{12}$$

χ_{ij} is a dielectric permittivity tensor with properties

$$\begin{aligned} &\chi_{ij} = \chi_{ji}, \\ &\chi_{ij}\xi_i\xi_j \geq \chi_0\xi_i\xi_i, \chi_0 = const > 0, \quad \forall \xi_i \in R, \end{aligned} \tag{13}$$

π_k are the pyroelectric coefficients, which are assumed to satisfy the following inequality, mentioned in [13]

$$\chi_{km}y_k y_m + 2\pi_k y_k \xi + \rho c_v \xi^2 \geq 0, \quad \forall \xi, y_k \in R, \tag{14}$$

z_{km} is the electrical conductivity tensor with common properties of symmetry and ellipticity, λ_{ij} is a symmetrical elliptic heat conductivity tensor, ρ, c_v and T_0 represent a mass density, specific heat and a fixed uniform reference temperature of a piezoelectric specimen, respectively. Here and everywhere below the ordinary summation by repetitive indices is expected.

To take into account a viscosity effect in pyroelectric materials, we modify the constitutive equation (4) for stress σ_{ij} by adding the term proportional to strain velocity. Therefore, the stress-relation now looks in the following way:

$$\sigma_{ij} = c_{ijkm}[\varepsilon_{km} - \alpha_{km}\theta] - e_{kij}E_k + a_{ijkm}\varepsilon'_{km}, \tag{15}$$

where a_{ijkm} is a viscosity coefficients tensor with common properties of symmetry and ellipticity.

To characterize the interaction of piezoelectric specimen with the environment, we must consider the boundary conditions. The boundary conditions for mechanical and heat fields are:

$$\begin{cases} u_i = 0 \quad on \quad \Gamma_u \times [0,T], \; \Gamma_u \subset \Gamma, mes(\Gamma_u) > 0, \\ \sigma_{ij}n_j = \hat{\sigma}_i \quad on \quad \Gamma_\sigma \times [0,T], \Gamma_\sigma := \Gamma \setminus \Gamma_u, \end{cases} \tag{16}$$

$$\begin{cases} \theta = 0 \quad on \quad \Gamma_\theta \times [0,T], \; \Gamma_\theta \subset \Gamma, mes(\Gamma_\theta) > 0, \\ q_i n_i = \hat{q} \quad on \quad \Gamma_q \times [0,T], \Gamma_q := \Gamma \setminus \Gamma_\theta. \end{cases} \tag{17}$$

Note that nonuniform boundary conditions on parts Γ_u and Γ_θ can be always transformed into uniform ones.

Similarly, the boundary conditions at the interface between the pyroelectric specimen and an ideal dielectric can be described in the following way:

$$\left[D_k' + J_k\right] n_k = 0 \quad \text{on} \ \ \Gamma_d, \ \Gamma_d \subset \Gamma. \tag{18}$$

Many pyroelectric materials and devices are operated under high electric field, which is applied through surface electrodes. We suppose that the electrode has a constant electric potential p_e on its surface, and is soft enough, so that it does not transfer any mechanical loadings. In this case we consider the following boundary conditions

$$p = 0 \quad on \quad \Gamma_p \times [0,T], \ \Gamma_p \subset \Gamma, mes(\Gamma_p) > 0 \quad \text{(grounded electrode)}, \tag{19}$$

and

$$\begin{cases} \int\limits_{\Gamma_e} \left[D_k' + J_k\right] n_k d\gamma = I, \\ p = const \quad \text{on} \ \ \Gamma_e, \quad \Gamma_e = \Gamma \setminus (\Gamma_d \cap \Gamma_p), \end{cases} \tag{20}$$

where I defines the external electrical current.

In order to terminate the formulation of initial boundary value problem of classic piezothermoelectricity, we consider the initial conditions

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u_0}, \quad \boldsymbol{u'}|_{t=0} = \boldsymbol{v_0}, p|_{t=0} = p_0, \quad \theta|_{t=0} = \theta_0 \quad in \ \ \Omega. \tag{21}$$

The aforementioned mathematical model of thermopiezoelectricity was considered in [16, 17], where its well-posedness is proved. Also a finite element based numerical scheme for solving this problem was constructed and the results of numerical experiments are described in [4, 18].

In present work, instead of (8), we use modified Fourier‘s law (also known as Maxwell-Cattaneo equation):

$$\tau q_i' + q_i = -\lambda_{ij}\theta_{,j}. \tag{22}$$

Here the parameter $\tau > 0$ is so-called "relaxation time". This assumption ensures finite speeds of heat wave propagation and was firstly introduced by Lord and Shulman in [10] and was firstly applied to thermopiezoelectricity theory by Chandrasekharaiah in [5]. Also, for convenience, similar to how Chandrasekharaiah did in [5], we introduce artificial coefficients b_{ij} in the way that the following condition is held:

$$T_0 b_{ij}\lambda_{jm} = \delta_{im}, \ \text{where } \delta_{im} \text{ are the elements of the unit matrix,} \tag{23}$$

and they satisfy ellipticity conditions:

$$b_{ij} y_i y_j \geq 0 \quad \forall y_i, y_j \in R. \tag{24}$$

Then the modified Fourier‘s law can be rewritten in the following form:

$$\tau b_{ij} q_i' + b_{ij} q_i = -T_0^{-1}\theta_{,j}. \tag{25}$$

Using Maxwell-Cattaneo equation (22) implies, that for Lord-Shulman theory a heat flux $\boldsymbol{q}$ is an additional independent variable. Therefore, the initial conditions (21) must be rewritten into:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u_0}, \quad \boldsymbol{u'}|_{t=0} = \boldsymbol{v_0}, p|_{t=0} = p_0, \quad \theta|_{t=0} = \theta_0, \quad \boldsymbol{q}|_{t=0} = \boldsymbol{q_0} \quad in \ \ \Omega. \tag{26}$$

Thus, the equations (1)-(3), (5)-(7), (9), (15) and (25) together with boundary conditions (16)-(20) and initial conditions (26) define the Lord-Shulman mathematical model of thermopiezoelectricity (initial boundary value problem of LS-thermopiezo-electricity).

3. Variational problem

Let us introduce the spaces of admissible elactic displacements, electric potentials, temperature increments and heat fluxes respectively:

$$\begin{array}{l} V = \left\{ \boldsymbol{v} \in [H^1(\Omega)]^d | \boldsymbol{v} = 0 \ \text{ on } \ \Gamma_u \right\}, \\ X = \left\{ \xi \in H^1(\Omega) | \xi = 0 \ \text{ on } \ \Gamma_p, \ \ \xi = const \ \text{ on } \ \Gamma_e \right\} \\ Y = \left\{ \eta \in H^1(\Omega) | \eta = 0 \ \text{ on } \ \Gamma_\theta \right\}, \\ Z = \left\{ \boldsymbol{\zeta} \in [L^2(\Omega)]^d \right\}, \end{array} \tag{27}$$

and notations

$$\Phi = V \times X \times Y \times Z, \ \Phi_1 = V \times X \times Y, \ G = L^2(\Omega), \ H = G^d. \tag{28}$$

Here symbol $H^m(\Omega)$ means a standard Sobolev space.

After applying the principle of virtual works to initial boundary value problem of LS-thermopiezoelectricity, we obtain the following variational problem:

$$\left\{ \begin{array}{l} \textit{given } \psi_0 = (\boldsymbol{u_0}, p_0, \theta_0, \boldsymbol{q_0}) \in \Phi, \ \boldsymbol{v_0} \in H \textit{ and } (l, r, \mu) \in L^2(0,T;\Phi'); \\ \textit{find } \ \psi = (\boldsymbol{u}, p, \theta, \boldsymbol{q}) \in L^2(0,T;\Phi) \textit{ such that} \\ m(\boldsymbol{u}''(t), \boldsymbol{v}) + a(\boldsymbol{u}'(t), \boldsymbol{v}) + c(\boldsymbol{u}(t), \boldsymbol{v}) - e(p(t), \boldsymbol{v}) - \\ \qquad -\gamma(\theta(t), \boldsymbol{v}) = < l(t), \boldsymbol{v} >, \\ \chi(p'(t), \xi) + e(\xi, \boldsymbol{u}'(t)) + z(p(t), \xi) + \pi(\theta'(t), \xi) = < r(t), \xi >, \\ s(\theta'(t), \eta) + \pi(\eta, p'(t)) + \gamma(\eta, \boldsymbol{u}'(t)) - g(\boldsymbol{q}(t), \eta) = < \mu(t), \eta >, \\ \tau b(\boldsymbol{q}'(t), \boldsymbol{\zeta}) + b(\boldsymbol{q}(t), \boldsymbol{\zeta}) + g(\boldsymbol{\zeta}, \theta(t)) = 0 \ \ \forall \ t \in (0,T], \\ m(\boldsymbol{u}'(0) - \boldsymbol{v_0}, \boldsymbol{v}) = 0, \qquad c(\boldsymbol{u}(0) - \boldsymbol{u_0}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in V, \\ \chi(p(0) - p_0, \xi) = 0 \qquad \forall \xi \in X, \\ s(\theta(0) - \theta_0, \eta) = 0 \qquad \forall \eta \in Y, \\ b(\boldsymbol{q}(0) - \boldsymbol{q_0}, \boldsymbol{\zeta}) = 0 \qquad \forall \boldsymbol{\zeta} \in Z \end{array} \right. \tag{29}$$

The introduced bilinear and linear forms are as follows:

$$\begin{array}{l} m(\boldsymbol{u}, \boldsymbol{v}) := \int\limits_\Omega \rho u_i v_i dx = \int\limits_\Omega \rho \boldsymbol{u}.\boldsymbol{v} dx, \quad a(\boldsymbol{u}, \boldsymbol{v}) := \int\limits_\Omega a_{ijkm} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{km}(\boldsymbol{v}) dx, \\ c(\boldsymbol{u}, \boldsymbol{v}) := \int\limits_\Omega c_{ijkm} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{km}(\boldsymbol{v}) dx, \quad < l, \boldsymbol{v} > := \int\limits_\Omega \rho f_i v_i dx + \int\limits_{\Gamma_\sigma} \hat{\sigma}_i v_i d\gamma, \\ \gamma(\xi, \boldsymbol{v}) := \int\limits_\Omega \xi c_{ijkm} \alpha_{km} \varepsilon_{ij} \boldsymbol{v}) dx, \\ e(\xi, \boldsymbol{v}) := \int\limits_\Omega e_{kij} E_k(\xi) \varepsilon_{ij}(\boldsymbol{v}) dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \\ \chi(p, \xi) := \int\limits_\Omega \chi_{km} E_k(p) E_m(\xi) dx, \quad z(p, \xi) := \int\limits_\Omega z_{km} E_k(p) E_m(\xi) dx, \\ < r, \xi > := I\xi|_{\Gamma_e} \quad \forall p, \xi \in X, \\ \pi(\eta, \xi) = \int\limits_\Omega \eta \pi_k E_k(\xi) dx, \quad s(\theta, \eta) = \int\limits_\Omega \rho c_v T_0^{-1} \theta \eta dx, \\ < \mu, \eta > := \int\limits_\Omega T_0^{-1} \rho w \eta dx - \int\limits_{\Gamma_h} T_0^{-1} \hat{h} \eta d\gamma \quad \forall \eta, \theta \in Y, \\ b(\boldsymbol{q}, \boldsymbol{\zeta}) = \int\limits_\Omega b_{ij} q_i \zeta_j dx, \quad g(\boldsymbol{\zeta}, \eta) = \int\limits_\Omega T_0^{-1} \zeta_k \eta_{,k} dx \quad \forall \boldsymbol{q}, \boldsymbol{\zeta} \in Z. \end{array} \tag{30}$$

Now suppose the harmonic loadings with angular frequency $\omega > 0$ are applied to the piezoelectric specimen:

$$\begin{array}{l} l(t) = (l_1 + il_2)e^{-i\omega t}, \\ r(t) = (r_1 + ir_2)e^{-i\omega t}, \\ \mu(t) = (\mu_1 + i\mu_2)e^{-i\omega t}, \quad \forall\, t \in (0, T]. \end{array} \tag{31}$$

Then we can look for approximate solutions of problem (29) in the form of the following expansions:

$$\begin{array}{l} \boldsymbol{u}(x,t) \cong (\boldsymbol{u_1}(x) + i\boldsymbol{u_2}(x))e^{-i\omega t}, \\ p(x,t) \cong (p_1(x) + ip_2(x))e^{-i\omega t}, \\ \theta(x,t) \cong (\theta_1(x) + i\theta_2(x))e^{-i\omega t}, \\ \boldsymbol{q}(x,t) \cong (\boldsymbol{q_1}(x) + i\boldsymbol{q_2}(x))e^{-i\omega t}, \end{array} \tag{32}$$

where $\boldsymbol{u_1}(x)$, $\boldsymbol{u_2}(x)$, $p_1(x)$, $p_2(x)$, $\theta_1(x)$, $\theta_2(x)$ and $\boldsymbol{q_1}(x)$, $\boldsymbol{q_2}(x)$ are the unknown amplitudes of mechanical displacement, electric potential, temperature increment and heat flux respectively.

After substitution of (31) and (32) into (29) and neglection of its initial conditions, we obtain the variational problem for forced harmonic vibrations of piezoelectric specimen:

$$\left\{ \begin{array}{l} \textit{given}\ \omega > 0,\ (l_1, l_2, r_1, r_2, \mu_1, \mu_2, 0, 0) \in W' = \Phi' \times \Phi'; \\ \textit{find}\ \ \psi = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W = \Phi \times \Phi\ \textit{such that} \\ -\omega^2 m(\boldsymbol{u_1}, \boldsymbol{v_2}) + \omega a(\boldsymbol{u_2}, \boldsymbol{v_2}) + c(\boldsymbol{u_1}, \boldsymbol{v_2}) - e(p_1, \boldsymbol{v_2}) - \\ \qquad -\gamma(\theta_1, \boldsymbol{v_2}) =< l_1, \boldsymbol{v_2} >, \\ -\omega^2 m(\boldsymbol{u_2}, \boldsymbol{v_1}) - \omega a(\boldsymbol{u_1}, \boldsymbol{v_1}) + c(\boldsymbol{u_2}, \boldsymbol{v_1}) - e(p_2, \boldsymbol{v_1}) - \\ \qquad -\gamma(\theta_2, \boldsymbol{v_1}) =< l_2, \boldsymbol{v_1} >, \\ \omega\chi(p_2, \xi_1) + \omega e(\xi_1, \boldsymbol{u_2}) + z(p_1, \xi_1) + \omega\pi(\theta_2, \xi_1) =< r_1, \xi_1 >, \\ -\omega\chi(p_1, \xi_2) - \omega e(\xi_2, \boldsymbol{u_1}) + z(p_2, \xi_2) - \omega\pi(\theta_1, \xi_2) =< r_2, \xi_2 >, \\ \omega s(\theta_2, \eta_1) + \omega\pi(\eta_1, p_2) + \omega\gamma(\eta_1, \boldsymbol{u_2}) - g(\boldsymbol{q_1}, \eta_1) =< \mu_1, \eta_1 >, \\ -\omega s(\theta_1, \eta_2) - \omega\pi(\eta_2, p_1) - \omega\gamma(\eta_2, \boldsymbol{u_1}) - g(\boldsymbol{q_2}, \eta_2) =< \mu_2, \eta_2 >, \\ \omega\tau b(\boldsymbol{q_2}, \boldsymbol{\zeta_1}) + b(\boldsymbol{q_1}, \boldsymbol{\zeta_1}) + g(\boldsymbol{\zeta_1}, \theta_1) = 0, \\ -\omega\tau b(\boldsymbol{q_1}, \boldsymbol{\zeta_2}) + b(\boldsymbol{q_2}, \boldsymbol{\zeta_2}) + g(\boldsymbol{\zeta_2}, \theta_2) = 0 \\ \qquad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{array} \right. \tag{33}$$

Having added all the equations of the problem (33), we introduce the bilinear form $\Pi_\omega : W \times W \to R$ and linear form $\chi_\omega : W \to R$ in the following way:

$$\begin{array}{l} \Pi_\omega(\psi, w) = -\omega^2[m(\boldsymbol{u_1}, \boldsymbol{v_2}) - m(\boldsymbol{u_2}, \boldsymbol{v_1})] + \\ +\omega[a(\boldsymbol{u_1}, \boldsymbol{v_1}) + a(\boldsymbol{u_2}, \boldsymbol{v_2})] + [c(\boldsymbol{u_1}, \boldsymbol{v_2}) - c(\boldsymbol{u_2}, \boldsymbol{v_1})] + \\ +[e(p_2, \boldsymbol{v_1}) - e(p_1, \boldsymbol{v_2}) + e(\xi_1, \boldsymbol{u_2}) - e(\xi_2, \boldsymbol{u_1})] + \\ +[\gamma(\theta_2, \boldsymbol{v_1}) - \gamma(\theta_1, \boldsymbol{v_2}) + \gamma(\eta_1, \boldsymbol{u_2}) - \gamma(\eta_2, \boldsymbol{u_1})] + \\ +[\pi(\theta_2, \xi_1) - \pi(\theta_1, \xi_2) + \pi(\eta_1, p_2) - \pi(\eta_2, p_1)] + \\ +[\chi(p_2, \xi_1) - \chi(p_1, \xi_2)] + \omega^{-1}[z(p_1, \xi_1) + z(p_2, \xi_2)] + \\ +[s(\theta_2, \eta_1) - s(\theta_1, \eta_2)] + \\ +\omega^{-1}[g(\boldsymbol{\zeta_1}, \theta_1) + g(\boldsymbol{\zeta_2}, \theta_2) - g(\boldsymbol{q_1}, \eta_1) - g(\boldsymbol{q_2}, \eta_2)] + \\ +\tau[b(\boldsymbol{q_2}, \boldsymbol{\zeta_1}) - b(\boldsymbol{q_1}, \boldsymbol{\zeta_2})] + \omega^{-1}[b(\boldsymbol{q_1}, \boldsymbol{\zeta_1}) + b(\boldsymbol{q_2}, \boldsymbol{\zeta_2})] \\ \qquad \forall\ \psi = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W, \\ \qquad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{array} \tag{34}$$

$$\begin{gathered}< \chi_\omega, w >= - < l_2, \boldsymbol{v_1} > + \omega^{-1}[< r_1, \xi_1 > + < \mu_1, \eta_1 >]+ \\ + < l_1, \boldsymbol{v_2} > + \omega^{-1}[< r_2, \xi_2 > + < \mu_2, \eta_2 >] \\ \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W.\end{gathered} \tag{35}$$

Then variational problem for forced harmonic vibrations of pyroelectric can be rewritten as follows:

$$\left\{\begin{array}{l} \textit{given } \omega > 0,\ \chi_\omega \in W' = \Phi' \times \Phi'; \\ \textit{find } \psi = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W = \Phi \times \Phi \textit{ such that} \\ \Pi_\omega(\psi, w) =< \chi_\omega, w > \quad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2}) \in W. \end{array}\right. \tag{36}$$

4. Well-posedness of the variational problem

Theorem 1. *Let us define the bilinear form $k(\cdot,\cdot)$ as follows:*

$$k(\theta, \eta) = \int\limits_\Omega T_0^{-1} \boldsymbol{\Lambda} \nabla\theta \nabla\eta dx, \tag{37}$$

where $\boldsymbol{\Lambda} = \{\lambda_{ij}\}$ is matrix of thermal conductivity coefficients. Then the below equality is held:

$$(1 + \omega^2\tau^2)[b(\boldsymbol{q_1}, \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2})] = k(\theta_1, \theta_1) + k(\theta_2, \theta_2), \tag{38}$$

where $\boldsymbol{q_1}, \boldsymbol{q_2}, \theta_1, \theta_2$ are the solutions of variational problems (33) and (36), defining amplitudes of heat flux and temperature increment correspondingly.

Proof.

The modified Fourier law

$$\tau \boldsymbol{q}' + \boldsymbol{q} = -\boldsymbol{\Lambda}\nabla\theta \tag{39}$$

is rewritten for the case of harmonic vibrations:

$$-i\omega\tau(\boldsymbol{q_1} + i\boldsymbol{q_2})e^{-iwt} + (\boldsymbol{q_1} + i\boldsymbol{q_2})e^{-iwt} = -\boldsymbol{\Lambda}(\nabla\theta_1 + i\nabla\theta_2)e^{-iwt}. \tag{40}$$

The expression (40) is then splitted into real and imaginary parts. As a result, we obtain:

$$\begin{aligned} \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2} &= -\boldsymbol{\Lambda}\nabla\theta_1, \\ \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1} &= -\boldsymbol{\Lambda}\nabla\theta_2. \end{aligned} \tag{41}$$

After multiplying equations of (41) by $T_0^{-1}\nabla\theta_1$ and $T_0^{-1}\nabla\theta_2$ respectively and integration over the domain Ω we get:

$$\begin{aligned} g(\boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \theta_1) &= -k(\theta_1, \theta_1), \\ g(\boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}, \theta_2) &= -k(\theta_2, \theta_2). \end{aligned} \tag{42}$$

Then two last equations of the variational problem (33) are considered and a substitution of admissible functions $\boldsymbol{\zeta_1} = \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}$ and $\boldsymbol{\zeta_2} = \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}$ is performed respectively:

$$\begin{aligned} \omega\tau b(\boldsymbol{q_2}, \boldsymbol{\zeta_1}) + b(\boldsymbol{q_1}, \boldsymbol{\zeta_1}) + g(\boldsymbol{\zeta_1}, \theta_1) = 0,\ \boldsymbol{\zeta_1} = \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \\ -\omega\tau b(\boldsymbol{q_1}, \boldsymbol{\zeta_2}) + b(\boldsymbol{q_2}, \boldsymbol{\zeta_2}) + g(\boldsymbol{\zeta_2}, \theta_2) = 0,\ \boldsymbol{\zeta_2} = \boldsymbol{q_2} - \omega\tau\boldsymbol{q_1}. \end{aligned} \tag{43}$$

After simplifying the first equation of (43) with taking into account the relations (42) we obtain:

$$\begin{aligned} &\omega\tau b(\boldsymbol{q_2}, \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}) + b(\boldsymbol{q_1}, \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}) + g(\boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \theta_1) = 0, \\ &b(\boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}, \boldsymbol{q_1} + \omega\tau\boldsymbol{q_2}) = k(\theta_1, \theta_1). \end{aligned} \tag{44}$$

Similarly, simplifying the second equation of (43) with taking into account the relations (42) we get:

$$\begin{array}{l} -\omega\tau b(\boldsymbol{q_1}, \boldsymbol{q_2} - \omega\tau \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2} - \omega\tau \boldsymbol{q_1}) + g(\boldsymbol{q_2} - \omega\tau \boldsymbol{q_1}, \theta_2) = 0, \\ b(\boldsymbol{q_2} - \omega\tau \boldsymbol{q_1}, \boldsymbol{q_2} - \omega\tau \boldsymbol{q_1}) = k(\theta_2, \theta_2). \end{array} \tag{45}$$

The last equations of the relations (44) and (45) can be rewritten in the following way:

$$\begin{array}{l} b(\boldsymbol{q_1}, \boldsymbol{q_1}) + 2\omega\tau b(\boldsymbol{q_1}, \boldsymbol{q_2}) + \omega^2\tau^2 b(\boldsymbol{q_2}, \boldsymbol{q_2}) = k(\theta_1, \theta_1), \\ \omega^2\tau^2 b(\boldsymbol{q_1}, \boldsymbol{q_1}) - 2\omega\tau b(\boldsymbol{q_1}, \boldsymbol{q_2}) + b(\boldsymbol{q_2}, \boldsymbol{q_2}) = k(\theta_2, \theta_2). \end{array} \tag{46}$$

After summarizing these 2 equations of (46) we obtain:

$$(1 + \omega^2\tau^2)[b(\boldsymbol{q_1}, \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2})] = k(\theta_1, \theta_1) + k(\theta_2, \theta_2). \tag{47}$$

□

Let us introduce a scalar product on the space W in the following way:

$$\begin{array}{l} ((y, w)) = \sum\limits_{i=1}^{2} [a(\boldsymbol{u_i}, \boldsymbol{v_i}) + z(p_i, \xi_i) + \frac{1}{2} b(\boldsymbol{q_i}, \boldsymbol{\zeta_i}) + \frac{1}{2(1+\omega^2\tau^2)} k(\theta_i, \eta_i)] \\ \quad \forall\ y = (\boldsymbol{u_1}, p_1, \theta_1, \boldsymbol{q_1}, \boldsymbol{u_2}, p_2, \theta_2, \boldsymbol{q_2}) \in W, \\ \quad \forall\ w = (\boldsymbol{v_1}, \xi_1, \eta_1, \boldsymbol{\zeta_1}, \boldsymbol{v_2}, \xi_2, \eta_2, \boldsymbol{\zeta_2},) \in W. \end{array} \tag{48}$$

We also introduce a norm generated by the scalar product (48):

$$|||y|||^2 = (y, y)\ \forall y \in W. \tag{49}$$

Then the following estimations are easy noticed:

$$\begin{array}{l} |\Pi_\omega(y, w)| \leq M_1(\omega)|||y||| \cdot |||w|||, \\ M_1(\omega) = C\ max\{\omega^{-1}, 1, \omega, \omega^2\},\ \forall y, w \in W, \end{array} \tag{50}$$

and

$$\begin{array}{l} | < \chi_\omega, w > | \leq M_2(\omega)||\chi_\omega||_* \cdot |||w|||, \\ M_2(\omega) = C\ max\{\omega^{-1}, 1\},\ \forall w \in W. \end{array} \tag{51}$$

Here and everywhere the symbol C means a positive constant value, which is not dependent on solutions of variational problem (36).

Consider now the expression for $\Pi_\omega(w, w)$:

$$\begin{array}{l} \Pi_\omega(w, w) = -\omega^2[m(\boldsymbol{u_1}, \boldsymbol{u_2}) - m(\boldsymbol{u_2}, \boldsymbol{u_1})] + \\ +\omega[a(\boldsymbol{u_1}, \boldsymbol{u_1}) + a(\boldsymbol{u_2}, \boldsymbol{u_2})] + [c(\boldsymbol{u_1}, \boldsymbol{u_2}) - c(\boldsymbol{u_2}, \boldsymbol{u_1})] + \\ +[e(p_2, \boldsymbol{u_1}) - e(p_1, \boldsymbol{u_2}) + e(p_1, \boldsymbol{u_2}) - e(p_2, \boldsymbol{u_1})] + \\ +[\gamma(\theta_2, \boldsymbol{u_1}) - \gamma(\theta_1, \boldsymbol{u_2}) + \gamma(\theta_1, \boldsymbol{u_2}) - \gamma(\theta_2, \boldsymbol{u_1})] + \\ +[\pi(\theta_2, p_1) - \pi(\theta_1, p_2) + \pi(\theta_1, p_2) - \pi(\theta_2, p_1)] + \\ +[\chi(p_2, p_1) - \chi(p_1, p_2)] + \\ +\omega^{-1}[z(p_1, p_1) + z(p_2, p_2)] + [s(\theta_2, \theta_1) - s(\theta_1, \theta_2)] + \\ +\omega^{-1}[g(\boldsymbol{q_1}, \theta_1) + g(\boldsymbol{q_2}, \theta_2) - g(\boldsymbol{q_1}, \theta_1) - g(\boldsymbol{q_2}, \theta_2)] + \\ +\tau[b(\boldsymbol{q_2}, \boldsymbol{q_1}) - b(\boldsymbol{q_1}, \boldsymbol{q_2})] + \omega^{-1}[b(\boldsymbol{q_1}, \boldsymbol{q_1}) + b(\boldsymbol{q_2}, \boldsymbol{q_2})] = \\ = \sum\limits_{i=1}^{2} [\omega a(\boldsymbol{u_i}, \boldsymbol{u_i}) + \omega^{-1} z(p_i, p_i) + \omega^{-1} b(\boldsymbol{q_i}, \boldsymbol{q_i})] = \\ = \sum\limits_{i=1}^{2} [\omega a(\boldsymbol{u_i}, \boldsymbol{u_i}) + \omega^{-1} z(p_i, p_i) + \\ +\omega^{-1}(\frac{1}{2} b(\boldsymbol{q_i}, \boldsymbol{q_i}) + \frac{1}{2(1+\omega^2\tau^2)} k(\theta_i, \theta_i))] \geq \alpha(\omega) \cdot |||w|||^2, \end{array} \tag{52}$$

where $\alpha(\omega) = min\{\omega^{-1}, \omega\} \quad \forall w \in W$.

Since the statements (50 - 52) are held and they are actually the conditions of Lions-Lax-Milgram theorem, the following theorem is then proved:

Theorem 2. *For each $w > 0$ and $\tau > 0$ variational problem (36) has a unique solution $\psi \in W$, which satisfy the relation:*

$$|||\psi||| \leq \alpha^{-1}(\omega)M_2(\omega)||\chi_\omega||_*. \tag{53}$$

5. Galerkin discretization

Galerkin scheme makes a transition of the solution of variational problem (33) from space $W := \Phi \times \Phi$ to its finite-dimensional subspace $W_h := \Phi_h \times \Phi_h$, $\Phi_h \subset \Phi$, $\dim W_h = N(h) < +\infty$. Thus, discretized variational problem (36) looks in the following way:

$$\left\{\begin{array}{l} \textit{given angular frequency} \;\; \omega > 0, \;\; \chi_\omega \in W', \\ \textit{approximations space} \;\; W_h \subset W, \; \dim W_h < +\infty; \\ \textit{find vector} \;\; \psi_h = (\boldsymbol{u_{1h}}, \boldsymbol{u_{2h}}, p_{1h}, p_{2h}, \theta_{1h}, \theta_{2h}, \boldsymbol{q_{1h}}, \boldsymbol{q_{2h}}) \in W_h \\ \textit{such that} \;\; \Pi_\omega(\psi_h, \varphi) =< \chi_\omega, \varphi > \quad \forall \, \varphi \in W_h. \end{array}\right. \tag{54}$$

Since problem (36) is well-posed, the same applies to its discretized counterpart (54).

In the space W we select some basis functions $\{w_i\}_{i=1}^{\infty}$. For each natural number $m \geq 1$, $h = 1/m$ a sequence of approximation spaces W_h and operators of orthogonal projection $\mathrm{Pr}_h : W \to W_h$ are defined so that a set $\{w_i\}_{i=1}^{m}$ is a base of W_h, $((\psi - \mathrm{Pr}_h \psi, w)) = 0 \; \forall \, \psi \in W, \; \forall \, w_h \in W_h$.

Now variational problem (36) is replaced by a sequence of the following problems:

$$\left\{\begin{array}{l} \textit{given } \omega > 0, \; \chi_\omega \in W' \textit{and } h > 0, \; W_h \subset W, \; \dim W_h = m < +\infty; \\ \textit{find vector } \psi_h \in W_h \textit{ such that} \\ \Pi_\omega(\psi_h, w) =< \chi_\omega, w > \;\; \forall w \;\in W_h. \end{array}\right. \tag{55}$$

Theorem 3. *Let $\boldsymbol{\psi} \in W$ be a solution of problem (36) with parameter $\omega > 0$. Then a sequence of Galerkin approximations $\{\boldsymbol{\psi_h}\} \subset W$ is unambiguously defined by the solutions of problems (55) and has the following properties:*

$$||\psi - \psi_h||_W \leq \alpha^{-1} M_1(\omega) \inf_{w \in W_h} ||\psi - w||_W \;\; \forall \; h > 0; \tag{56}$$

$$\lim_{h \to 0} ||\psi - \psi_h||_W = 0. \tag{57}$$

Proof. The correctness of the inequality (56) is based on the fact that

$$\Pi_\omega(\psi - \psi_h, w) = 0 \; \forall \; w \in W_h,$$

and the estimation

$$\begin{array}{l} \alpha||\psi - \psi_h||_W^2 \leq \Pi_\omega(\psi - \psi_h, \psi - \psi_h) = \Pi_\omega(\psi - \psi_h, \psi - w) \leq \\ \leq M_1(\omega)||\psi - \psi_h||_W||\psi - w||_W \;\; \forall \; w \in W_h. \end{array}$$

Taking into account the density of sequence of spaces $\{W_h\}$ in the separable space W

$$\lim_{h \to 0} ||w - Pr_h w||_W = 0 \; \forall \; w \in W.$$

Therefore, basing on the equality

$$\inf_{w\in W_h} ||\psi - w||_W = ||\psi - \Pr_h \psi||_W$$

and (56) we can conclude the correctness of (57), when $\omega > 0$. □

Theorem 4. *on the convergence of FEM approximations.*

Let $\psi \in W$ be a solution of problem (36) and exists a natural number $k \geq 1$ such that $\psi \in W \cap [H^{k+1}(\Omega)]^{2(d+1)}$. Let approximations ψ_h be defined by solving problem (55) in the spaces $W_h \subset W$, which are constructed with making use of piecewise-polynomial functions of FEM and have the following property:

for each $\varphi \in W \cap [H^{k+1}(\Omega)]^{2(d+1)}$, $k \geq 1$ there exist $\varphi_h \in W_h$ and $C = const > 0$ such that $\|\varphi - \varphi_h\|_{m,\Omega} \leq C \cdot h^{k+1-m}\|\varphi\|_{k+1,\Omega}$, $0 \leq m \leq k$, where h is the diameter of finite element mesh and k is the greatest degree of full polynomial of d variables, which is precisely defined by basis functions of W_h on each finite element.

Then the convergence of sequence $\psi_h \subset W$ is characterized by the estimation

$$\|\psi - \psi_h\| \leq C \cdot h^k ||\psi||_{k+1,\Omega}, \tag{58}$$

where $C = const > 0$ is not dependent on values we are looking for.

Proof. The estimation (58) is implied from the inequality (56), the equivalence of norms $|| \cdot ||_W$ and $|| \cdot ||_{1,\Omega}$ on W and the density properties defined in the theorem body.

$$\|\psi - \psi_h\|_W \leq \alpha^{-1} M_1(\omega) \inf_{w\in W_h} \|\psi - w\| = \|\psi - w\|_{1,\Omega} \leq C \cdot h^k ||\psi||_{k+1,\Omega}$$

□

Let us now pay a deeper attention to the aforementioned selection of finite-dimensional subspace $W_h \in W$. Taking into account the definition of W_h that is $W_h = V_h \times X_h \times Y_h \times Z_h \times V_h \times X_h \times Y_h \times Z_h$, where

$$\begin{aligned} &V_h \subset V,\ X_h \subset X,\ Y_h \subset Y,\ Z_h \subset Z, \\ &dimV_h < +\infty,\ dimX_h < +\infty,\ dimY_h < +\infty,\ dimZ_h < +\infty. \end{aligned} \tag{59}$$

we can write the expansions of solution amplitudes as following:

$$\begin{aligned} \boldsymbol{u_{\alpha h}} &\simeq \sum_{i=0}^{N} \boldsymbol{U_\alpha} \phi_i^V(x), \\ p_{\alpha h} &\simeq \sum_{i=0}^{N} \boldsymbol{P_\alpha} \phi_i^X(x), \\ \theta_{\alpha h} &\simeq \sum_{i=0}^{N} \boldsymbol{\Theta_\alpha} \phi_i^Y(x), \\ \boldsymbol{q_{\alpha h}} &\simeq \sum_{i=0}^{N} \boldsymbol{Q_\alpha} \phi_i^Z(x), \alpha = 1, 2, \end{aligned} \tag{60}$$

where $\phi_i^V(x), \phi_i^X(x), \phi_i^Y(x)$ and $\phi_i^Z(x)$ are the basis functions of spaces V, X, Y and Z respectively. Then we obtain the system of linear equations for finding

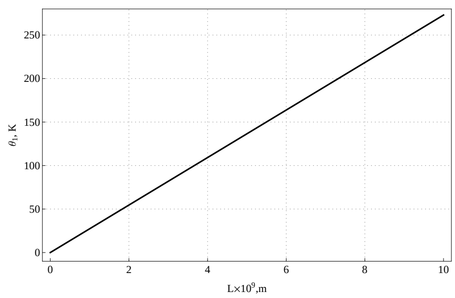

FIG. 1. Amplitude of temperature θ_1

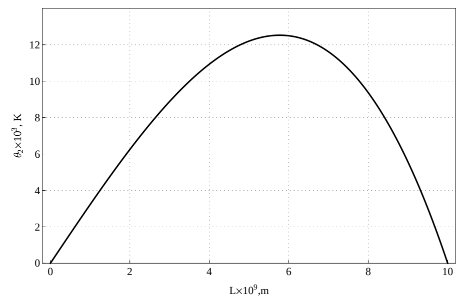

FIG. 2. Amplitude of temperature θ_2

nodal values of the unknown amplitudes:

$$\begin{bmatrix} \omega\boldsymbol{A} & -[-\omega^2\boldsymbol{M}+\boldsymbol{C}] & 0 & \boldsymbol{E^T} & 0 & \boldsymbol{Y^T} & 0 & 0 \\ [-\omega^2\boldsymbol{M}+\boldsymbol{C}] & \omega\boldsymbol{A} & -\boldsymbol{E^T} & 0 & -\boldsymbol{Y^T} & 0 & 0 & 0 \\ 0 & \boldsymbol{E} & \omega^{-1}\boldsymbol{Z} & \omega\boldsymbol{X} & 0 & \boldsymbol{\Pi^T} & 0 & 0 \\ -\boldsymbol{E} & 0 & -\boldsymbol{X} & \omega^{-1}\boldsymbol{Z} & -\boldsymbol{\Pi^T} & 0 & 0 & 0 \\ 0 & \boldsymbol{Y} & 0 & \boldsymbol{\Pi} & 0 & \boldsymbol{S} & -\omega^{-1}\boldsymbol{G^T} & 0 \\ -\boldsymbol{Y} & 0 & -\boldsymbol{\Pi} & 0 & -\boldsymbol{S} & 0 & 0 & -\omega^{-1}\boldsymbol{G^T} \\ 0 & 0 & 0 & 0 & \omega^{-1}\boldsymbol{G} & 0 & \omega^{-1}\boldsymbol{B} & \tau\boldsymbol{B} \\ 0 & 0 & 0 & 0 & 0 & \omega^{-1}\boldsymbol{G} & -\tau\boldsymbol{B} & \omega^{-1}\boldsymbol{B} \end{bmatrix}.$$

$$\begin{aligned} &\cdot [\boldsymbol{U_1}, \boldsymbol{U_2}, \boldsymbol{P_1}, \boldsymbol{P_2}, \boldsymbol{\Theta_1}, \boldsymbol{\Theta_2}, \boldsymbol{Q_1}, \boldsymbol{Q_2}]^T = \\ &= \left[-\boldsymbol{L_2}, \boldsymbol{L_1}, \omega^{-1}\boldsymbol{R_1}, \omega^{-1}\boldsymbol{R_2}, \omega^{-1}\boldsymbol{F_1}, \omega^{-1}\boldsymbol{F_2}, 0, 0\right]^T. \end{aligned} \tag{61}$$

Here the elements of the matrices and vectors are computed using the bilinear and linear forms defined in (30), for example $\boldsymbol{A} = \{a_{ij}\} = \{a(\phi_i^V, \phi_j^V)\}$. The matrix of the system of equations (61) is positively defined, but not the symmetric one. More precisely, it can be represented as the sum of positively defined symmetric matrix and a skew-symmetric one.

6. NUMERICAL EXPERIMENTS

We consider a piezoelectric bar with length $L = 10^{-8}m$ made of PZT-4 ceramics.A harmonic thermal loading with angular frequency $\omega = 3 \cdot 10^6 rad/s$ is applied to the right edge of the bar. So, the boundary conditions for thermal field are:

$$\theta_1(0) = 0K,\ \theta_1(L) = 273K,\ \theta_2(0) = 0K,\ \theta_2(L) = 0K. \tag{62}$$

On the left edge of the bar the boundary conditions for mechanical and electric fields are homogeneous and of Dirichlet type :

$$u_1(0) = 0m,\ u_2(0) = 0m,\ p_1(0) = 0V,\ p_2(0) = 0V. \tag{63}$$

On the right edge of the bar the boundary conditions for mechanical and electric fields are homogeneous and of Neumann type :

$$\sigma_1(L) = 0N \cdot m^{-2},\ \sigma_2(L) = 0N \cdot m^{-2},\ J_1(L) = 0A,\ J_2(L) = 0A. \tag{64}$$

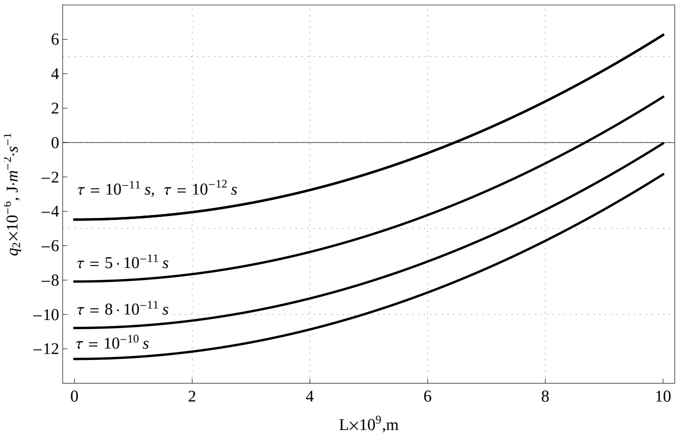

FIG. 3. Amplitude of heat flux component q_2 for the PZT-4 bar for relaxation times $\tau = 10^{-10}, 8\cdot10^{-11}, 5\cdot10^{-11}, 10^{-11}, 10^{-12}s$

We take the coefficients of PZT-4 as in [20]:

$\rho = 7500[kg/m^3]$	$e = 15.1[C/m^2]$
$c_v = 350[J/kg\cdot K]$	$\pi = 2.7\times10^{-4}[C/K\cdot m^2]$
$\lambda = 1.1[W/m\cdot K]$	$\chi = 6.46\times10^{-9}[C^2/N\cdot m^2]$
$c = 115\times10^9[N/m^2]$	$\alpha = 3.13\times10^{-5}[K^{-1}]$

Also we take $z = 5\times10^{-12}[\Omega^{-1}\cdot m^{-1}]$, $a = 40[m^2\cdot s^{-1}]$ and $T_0 = 298[K]$. As mentioned in [20], the value of the relaxation time τ for PZT-4 cannot be found in the literature. However, the relation time τ is determined for different type of materials, ranging from 10^{-10} for gases to 10^{-14} for metals. Therefore, in our numerical experiments we will use the values of relaxation time $\tau = 10^{-10}, 8\cdot10^{-11}, 5\cdot10^{-11}, 10^{-11}, 10^{-12}s$. For discretization by spatial variable we divide the interval $[0, L]$ into $N = 256$ finite elements with piecewise linear solution approximations on them.

Fig. 1 shows that under these boundary conditions and angular frequency $\omega = 3\cdot10^6 rad\cdot s^{-1}$ the calculated temperature increment θ_1 is changing linearly along the bar, regardless of the value of relaxation time τ. Fig. 2 depicts the calculated amplitude θ_2 of the temperature increment. It is also not dependent on the value of relaxation time τ.

On the other hand, as Fig.3 shows, the amplitude of heat flux q_2 is dependent on the parameter τ. It worth mentioning, that the amplitude calculated with $\tau = 10^{-12}s$ is almost identical to the one obtained as a solution of the classical thermopiezoelectricity problem for forced harmonic vibrations (when no modified Fourier law is taken into account).

7. Conclusions

The harmonic vibrations of the pyroelectric materials have been studied under generalized Lord-Shulman thermopiezoelectricity theory. The variational problem for this special case has been formulated and its well-posedness has been proved. Then the discretization of the problem using Galerkin-method has been performed. The finite element method has been utilized to construct the bases of approximation spaces of the discretized problem. The rate of convergence of FEM-approximations has been determined. After the discretization we obtain the system of linear algebraic equations with positively defined matrix in its left part. Therefore, we can be sure that the solution of that system always exists. The numerical experiment of applying a harmonic thermal loading to the pyroelectric bar has been set up and studied. The results of the experiment showed the significant influence of the "relaxation time" parameter on the nodal values of solution amplitudes.

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