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BOUNDARY VALUE PROBLEM FOR THE TWO-DIMENSIONAL LAPLACE EQUATION WITH TRANSMISSION CONDITION ON THIN INCLUSION

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Резюме. Розглянуто задачу для рівняння Лапласа в обмеженій двовимірній Ліпшицевій області з тонким включенням, на якому задана трансмісійна гранична умова, тобто умова, що містить як стрибок нормальної похідної, так і граничне значення шуканої функції. Доведено еквівалентність задачі у диференціальному формулюванні та відповідної варіаційної задачі. Досліджено питання існування та єдиності розв'язку поставленої задачі у відповідних функціональних просторах. На основі інтегрального подання розв'язку вихідна диференціальна задача зведена до системи граничних інтегральних рівнянь. Побудовано алгоритм чисельного розв'язування отриманої системи інтегральних рівнянь методом колокації. Представлено чисельні результати наближеного розв'язуваня деяких конкретних граничних задач.

Abstract. We consider boundary value problem for Laplace equation in bounded two-dimensional Lipschitz domain with thin inclusion. Transmission boundary condition upon it consists of the jump of normal derivative and the meaning of boundary value of seeking function. We prove the equivalence of initial boundary value problem and connected variational problem. As a result we obtain existence and uniqueness of solution of the posed problem in appropriate functional spaces. Based on the integral representation formula the considered boundary value problem is reduced to the system of boundary integral equations. We construct the algorithm of numerical solution of obtained system by collocation method. Our approach is illustrated by some numerical examples.

The numerical results show that the proposed methods give a good accuracy of reconstructions with an economical computational cost.

1. Introduction

Boundary value problems for the second order elliptic equations with transmission boundary conditions in nonsmooth domains are important class of boundary value problems and were considered by many authors [1]- [4], [7, 8].

We consider a special case of the transmission conditions when they are posed on an open Lipschitz curve. From the mathematical point of view such kind of problem describes stationary temperature field in domain with thin inclusion when the temperature passing through this inclusion is continuous and the heat flux is discontinuous and proportional to the boundary value of temperature.

Key words. Laplace equation; transmission condition; variational problem; open curve.

In oder to obtain convenient mathematical model for this physical problem it's useful to present thin objects as inclusion or crack like an open curve. As a result we get essentially unregular domain and need to introduce corresponding trace maps and appropriate functional spaces [1, 6].

In present paper we use a variational formulation of the posed boundary value problem with transmission condition which gives us opportunity to obtain the existence and uniqueness of solution.

2. Functional spaces and trace operators

Let $\Omega_+ \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain. This means that its boundary curve Σ is locally the graph of a Lipschitz function [5, 6]. Let us note that Σ can be piecewise smooth and have corner points. $\overline{\Omega}_+ = \Omega_+ \cup \Sigma$. We suppose that S is an open Lipschitz curve with the end points c_1 and c_2, $\overline{S} = S \cup \{c_1, c_2\}$ and $\overline{S} \subset \Omega_+$. We denote $\Omega = \Omega_+ \setminus \overline{S}$ and consider S as a part of a some closed bounded Lipschitz curve $\Sigma_0 = \overline{S} \cup S_0$, $\Sigma_0 \subset \Omega_+$.

Since Σ and S are Lipschitz almost everywhere we can define outward pointing vector of the normal $\vec{n}_x$, $x \in \Sigma$ or $x \in S$. Depend on the direction of $\vec{n}_x$, $x \in S$, we consider S as a double sided curve with sides S_+ and S_-.

In Ω_+ we consider the Laplace operator

$$Lu = -\Delta u = -\sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2$$

and connected bilinear form

$$a(u, v) = (\nabla u, \nabla v)_{L_2(\Omega_+)} = \int_{\Omega_+} \left\{ \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right\} dx.$$

We use the Hilbert spaces $H^1(\Omega_+)$ and $H^1(\Omega_+, L)$ of real functions with norms and inner products

$$\|u\|^2_{H^1(\Omega_+)} = \int_{\Omega_+} \left\{ |\nabla u|^2 + u^2 \right\} dx, \quad (u, v)_{H^1(\Omega_+)} = \int_{\Omega_+} \{(\nabla u, \nabla v) + uv\} \, dx,$$

$$\|u\|^2_{H^1(\Omega_+, L)} = \|u\|^2_{H^1(\Omega_+)} + \|Lu\|^2_{L_2(\Omega_+)},$$

$$(u, v)_{H^1(\Omega_+, L)} = (u, v)_{H^1(\Omega_+)} + (Lu, Lv)_{L_2(\Omega_+)}.$$

The trace operators $\gamma^+_{0,\Sigma} : H^1(\Omega_+) \to H^{1/2}(\Sigma)$ and $\gamma^+_{1,\Sigma} : H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ are continuous and surjective [5, 6]. Here $\gamma^+_{1,\Sigma} u \in H^{-1/2}(\Sigma) = (H^{1/2}(\Sigma))'$ and coincides with $\frac{\partial u}{\partial n_x}$ for $u \in C^1(\overline{\Omega}_+)$.

Let us denote by $C_0^\infty(\Omega)$ the class of infinitely differentiable functions with compact support in Ω. We introduce the Hilbert spaces $H^1(\Omega)$ and $H^1(\Omega, L)$ of real functions with norms

$$\|u\|^2_{H^1(\Omega)} = \int_{\Omega} \left\{ |\nabla u|^2 + u^2 \right\} dx, \tag{1}$$

$$\|u\|^2_{H^1(\Omega, L)} = \|u\|^2_{H^1(\Omega)} + \|Lu\|^2_{L_2(\Omega)},$$

where derivatives $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$ are defined as

$$\left(\frac{\partial u}{\partial x_i}, \varphi\right)_{L_2(\Omega)} = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\left(u, \frac{\partial \varphi}{\partial x_i}\right)_{L_2(\Omega)}$$

for all $\varphi \in C_0^\infty(\Omega)$.

We consider some trace maps in Ω. We denote $\gamma_{0,S}^\pm$ and $\gamma_{1,S}^\pm$ the restrictions of trace maps $\gamma_{0,\Sigma_0}^\pm$ and $\gamma_{1,\Sigma_0}^\pm$ on S respectively [9]. Then we have $\gamma_{0,S}^\pm : H^1(\Omega) \to H^{1/2}(S)$ and $\gamma_{1,S}^\pm : H^1(\Omega, L) \to H^{-1/2}(S)$.

We introduce the space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \gamma_{0,S}^\pm u = 0, \gamma_{0,\Sigma}^+ u = 0\}$$

and denote dual space $H^{-1}(\Omega) = (H_0^1(\Omega))'$. We also have that $H_0^1(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in the norm (1).

In what follows we use the next trace maps: $[\gamma_{0,S}] = \gamma_{0,S}^+ - \gamma_{0,S}^-$, $[\gamma_{1,S}] = \gamma_{1,S}^+ - \gamma_{1,S}^-$. Analogously as it was obtained in [9, 10] for $\mathbb{R}^3$ we can show that

$$[\gamma_{0,S}] : H^1(\Omega) \to H_{00}^{1/2}(S), \quad [\gamma_{1,S}] : H^1(\Omega, L) \to H_{00}^{-1/2}(S),$$

where $H_{00}^{1/2}(S) = \{g \in H^{1/2}(S) : p_0 g \in H^{1/2}(\Sigma_0)\}$. Here $p_0 g$ is extension by zero of the function g on S_0. The norm in $H_{00}^{1/2}(S)$ is given as

$$\|g\|_{H_{00}^{1/2}(S)} = \|p_0 g\|_{H^{1/2}(\Sigma_0)}.$$

$H_{00}^{-1/2}(S) = (H^{1/2}(S))'$, $H^{-1/2}(S) = (H_{00}^{1/2}(S))'$.

We have the first Green's formula for bounded domain with an open curve which in presented case for $u \in H^1(\Omega, L)$ and $v \in H^1(\Omega)$ has the following form:

$$a(u, v) = (Lu, v)_{L_2(\Omega)} + \langle \gamma_{1,S}^+ u, [\gamma_{0,S}] v\rangle + \langle [\gamma_{1,S}] u, \gamma_{0,S}^- v\rangle + \langle \gamma_{1,\Sigma}^+ u, \gamma_{0,\Sigma}^+ v\rangle. \quad (2)$$

Here $\langle \cdot, \cdot \rangle$ are relations of duality between $H_{00}^{1/2}(S)$ and $H^{-1/2}(S)$, $H^{1/2}(S)$ and $H_{00}^{-1/2}(S)$, $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ respectively.

Let $\overline{\Omega}_1 \subset \Omega_+$ be a Lipschitz domain bounded by the closed curve Σ_0. $\overline{\Omega}_1 = \Omega_1 \cup \Sigma_0$, $\Omega_2 = \Omega_+ \setminus \overline{\Omega}_1$. We denote by u_i the restriction of $u \in H^1(\Omega)$ to Ω_i, $i = 1, 2$. It's obviously that $u_i \in H^1(\Omega_i)$, $i = 1, 2$.

Lemma 1. *The trace map $\gamma_{0,S}^- : H^1(\Omega_+) \to H^{1/2}(S)$ is continuous and surjective.*

Proof. Let $g \in H^{1/2}(S)$ be an arbitrary function. We denote by $pg \in H^{1/2}(\Sigma_0)$ the extension of g on Σ_0. The trace map $\gamma_{0,\Sigma_0}^- : H^1(\Omega_1) \to H^{1/2}(\Sigma_0)$ is continuous and surjective. Thus there exists function $u_1 \in H^1(\Omega_1)$ with trace $\gamma_{0,\Sigma_0}^- u_1 = pg$ and

$$\|pg\|_{H^{1/2}(\Sigma_0)} \leq c\|u_1\|_{H^1(\Omega_1)}. \quad (3)$$

Analogously there exists the function $u_2 \in H^1(\Omega_2)$ that $\gamma_{0,\Sigma_0}^+ u_2 = pg$. Thus we have function $u \in H^1(\Omega_+)$ where u_i are the restrictions of u to Ω_i, $i = 1, 2$.

Then from (3) we obtain

$$\|g\|_{H^{1/2}(S)} = \inf_{pg\in H^{1/2}(\Sigma_0)} \|pg\|_{H^{1/2}(\Sigma_0)} \le c\|u_1\|_{H^1(\Omega_1)} \le c\|u\|_{H^1(\Omega_+)}.$$

Here c - some positive constant. □

3. Boundary value problem with transmision boundary condition and it's variational formulation

Let us state the following boundary value problem in domain Ω.

Problem T. Find a function $u \in H^1(\Omega, L)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega,$$

$$[\gamma_{0,S}]u = 0, \quad [\gamma_{1,S}]u + \lambda\gamma^-_{0,S}u = f,$$

$$\gamma^+_{0,\Sigma}u = g.$$

Here $f \in H^{-1/2}_{00}(S)$, $g \in H^{1/2}(\Sigma)$ and $\lambda \in C(\bar{S})$ are given.

A partial case of the problem T when $\gamma^+_{0,\Sigma}u = 0$ we denote as problem T_0.

We can connect with problem T_0 the next variational problem.

Problem VT_0. Find a function $u \in H^1_0(\Omega_+) = \{u \in H^1(\Omega_+) : \gamma^+_{0,\Sigma}u = 0\}$ that satisfies

$$b(u, v) = l(v)$$

for every $v \in H^1_0(\Omega_+)$.

Here

$$b(u, v) = (\nabla u, \nabla v)_{L_2(\Omega_+)} + (\lambda\gamma^-_{0,S}u, \gamma^-_{0,S}v)_{L_2(S)},$$

$$l(v) = \langle f, \gamma^-_{0,S}v\rangle. \tag{4}$$

Lemma 2. *If $\lambda \in C(\bar{S})$, $\lambda(x) \ge 0$, $x \in \bar{S}$, then bilinear form $b(u, v) : H^1_0(\Omega_+)\times H^1_0(\Omega_+) \to \mathbb{R}$ is continuous and $H^1_0(\Omega_+)$-elliptic.*

Proof. Since trace map $\gamma^-_{0,S} : H^1_0(\Omega_+) \to H^{1/2}(S)$ is continuous we have

$$|(\lambda\gamma^-_{0,S}u, \gamma^-_{0,S}v)_{L_2(S)}| \le M\|\gamma^-_{0,S}u\|_{L_2(S)}\|\gamma^-_{0,S}v\|_{L_2(S)} \le$$

$$\le M\|\gamma^-_{0,S}u\|_{H^{1/2}(S)}\|\gamma^-_{0,S}v\|_{H^{1/2}(S)} \le Mc\|u\|_{H^1_0(\Omega_+)}\|v\|_{H^1_0(\Omega_+)},$$

where $M = \max_{x\in\bar{S}} |\lambda(x)|$.

$$|(\nabla u, \nabla v)_{L_2(\Omega_+)}| \le \|\nabla u\|_{L_2(\Omega_+)}\|\nabla v\|_{L_2(\Omega_+)} \le \|u\|_{H^1_0(\Omega_+)}\|v\|_{H^1_0(\Omega_+)}.$$

Thus we obtain

$$|b(u, v)| \le (Mc + 1)\|u\|_{H^1_0(\Omega_+)}\|v\|_{H^1_0(\Omega_+)}.$$

If $\lambda(x) \ge 0$, $x \in \bar{S}$, then using Friedreich's inequality in $H^1_0(\Omega_+)$ we can get

$$b(u, u) = \|u\|^2_{L_2(\Omega_+)} + \|\lambda^{1/2}\gamma^-_{0,S}u\|^2_{L_2(S)} \ge c\|u\|^2_{H^1_0(\Omega_+)}.$$

Thus $b(u, v)$ is $H^1_0(\Omega_+)$ - elliptic. Here c - some positive constants which don't depend on u and v. □

Theorem 1. *Problems T_0 and VT_0 are equivalent.*

Proof. Let u be a solution of the problem T_0. It means that $u \in H^1(\Omega, L)$ and $[\gamma_{0,S}]u = 0$, $\gamma^+_{0,\Sigma}u = 0$. Thus $u \in H^1_0(\Omega_+)$. From the first Green's formula (2) we have $b(u,v) = l(v)$ for every $v \in H^1_0(\Omega_+)$. Thus u is a solution of the problem VT_0.

Let now $u \in H^1_0(\Omega_+)$ be a solution of the problem VT_0. Then for every $v \in H^1_0(\Omega_+)$ we have

$$(\nabla u, \nabla v)_{L_2(\Omega_+)} = \langle f - \lambda\gamma^-_{0,S}u, \gamma^-_{0,S}v\rangle. \tag{5}$$

By definition $\langle Lu, v\rangle = (\nabla u, \nabla v)_{L_2(\Omega_+)}$ for every $u \in H^1(\Omega_+)$ and $v \in H^1_0(\Omega_+)$. Here $Lu \in H^{-1}(\Omega_+) = (H^1_0(\Omega_+))'$. If $v \in C^\infty_0(\Omega)$ from (5) we can get the following relation:

$$(\nabla u, \nabla v)_{L_2(\Omega_+)} = \langle Lu, v\rangle = 0.$$

It means that $Lu \in H^{-1}(\Omega) = (H^1_0(\Omega))'$ and $Lu = 0$ in Ω.

Since $u \in H^1_0(\Omega_+)$ it follows that $[\gamma_{0,S}]u = 0$. Then from the first Green's formula (2) for arbitrary $v \in H^1_0(\Omega_+)$ we can get:

$$\langle [\gamma_{1,S}]u - f + \lambda\gamma^-_{0,S}u, \gamma^-_{0,S}v\rangle = 0.$$

The trace map $\gamma^-_{0,S} : H^1_0(\Omega_+) \to H^{1/2}(S)$ is surjective. Thus $\langle [\gamma_{1,S}]u - f + \lambda\gamma^-_{0,S}u, g\rangle = 0$ for arbitrary $g \in H^{1/2}(S)$. It gives us that $[\gamma_{1,S}]u + \lambda\gamma^-_{0,S}u = f$ and as a consequence we obtain that function u is a solution of the problem T_0. □

Theorem 2. *If $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, then problem VT_0 has a unique solution for arbitrary $f \in H^{-1/2}_{00}(S)$.*

Proof. Lemma 2 gives us that the bilinear form $b(u,v) : H^1_0(\Omega_+) \times H^1_0(\Omega_+) \to \mathbb{R}$ is continuous and $H^1_0(\Omega_+)$-elliptic

It's easy to show that the functional $l : H^1_0(\Omega_+) \to \mathbb{R}$ given by (4) is continuous. Since the trace map $\gamma^-_{0,S} : H^1_0(\Omega_+) \to H^{1/2}(S)$ is continuous we have:

$$|l(v)| = |\langle f, \gamma^-_{0,S}v\rangle| \leq \|f\|_{H^{-1/2}_{00}(S)} \|\gamma^-_{0,S}v\|_{H^{1/2}(S)} \leq c\|f\|_{H^{-1/2}_{00}(S)} \|v\|_{H^1_0(\Omega_+)},$$

where c - some positive constant which does not depend on v. Then by the Lax-Milgram Lemma we obtain what was to be proved. □

Theorem 3. *If $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$, then problem T has a unique solution for arbitrary $f \in H^{-1/2}_{00}(S)$ and $g \in H^{1/2}(\Sigma)$.*

Proof. Let function $w \in H^1(\Omega_+)$ satisfies $Lw = 0$ in Ω_+ and $\gamma^+_{0,\Sigma}w = g$. Then $[\gamma_{0,S}]w = 0$ and $[\gamma_{1,S}]w = 0$. As a corollary of theorem 1 and theorem 2 we obtain that the problem T_0 has a unique solution for arbitrary $f \in H^{-1/2}_{00}(S)$ if $\lambda \in C(\bar{S})$, $\lambda(x) \geq 0$, $x \in \bar{S}$. It means that there exists a solution u_0 of the problem T_0 with boundary condition $[\gamma_{1,S}]u_0 + \lambda\gamma^-_{0,S}u_0 = f - \lambda\gamma^-_{0,S}w$. Then it's easy to verify that the function $u = u_0 - w \in H^1(\Omega)$ is a solution of the problem T. □

Let us note that our approach remains true when $S = \bigcup_{i=1}^n S_i$, where S_i are open Lipshitz curves without common points.

4. System of boundary integral equations

Let $Q(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}$ - be fundamental solution of the operator $L = -\Delta$. Then the solution u of the problem T with condition $\gamma_{0,\Sigma}^{-} u = \gamma_{0,\Sigma}^{+} u$ has the following integral representation

$$u(x) = V\tau(x) + V_{\Sigma}\mu(x), \qquad x \in \Omega_{+},$$

where $\tau = [\gamma_{1,S}]u$, $\mu = [\gamma_{1,\Sigma}]u$,

$$V\tau(x) = \int_S Q(x,y)\tau(y)ds_y, \quad V_{\Sigma}\mu(x) = \int_{\Sigma} Q(x,y)\mu(y)dy.$$

Using boundary conditions we can reduce problem T to the following system of boundary integral equations:

$$\begin{cases} \tau + \lambda K\tau + \lambda \gamma_{0,S}^{+} V_{\Sigma}\mu = f, \\ \gamma_{0,\Sigma}^{+} V\tau + K_{\Sigma}\mu = g, \end{cases} \tag{6}$$

where

$$K\tau(x) = \int_S Q(x,y)\tau(y)dS_y, \quad \gamma_{0,S}^{+} V_{\Sigma}\mu(x) = \int_{\Sigma} Q(x,y)\mu(y)dS_y, \quad x \in S,$$

$$K_{\Sigma}\mu(x) = \int_{\Sigma} Q(x,y)\mu(y)dS_y, \quad \gamma_{0,\Sigma}^{+} V\tau(x) = \int_S Q(x,y)\tau(y)dS_y, \quad x \in \Sigma.$$

We use collocation method for solving of obtained system (6). Let us denote by N_S and N_{Σ} number of boundary elements of the second order given upon curves S and Σ respectively. Finally we derive the following system of linear algebraic equations:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \widetilde{\tau} \\ \widetilde{\mu} \end{pmatrix} = \begin{pmatrix} \widetilde{f} \\ \widetilde{g} \end{pmatrix}.$$

Here

$$A_{11} = \left\{ \delta_{ij} + \lambda(x_i) \int_{S_j} Q(x_i,y)ds_y \right\}, \; i,j = \overline{1,N_S},$$

$$A_{12} = \left\{ \lambda(x_i) \int_{\Sigma_j} Q(x_i,y)ds_y \right\}, \; i = \overline{1,N_S}, \; j = \overline{1,N_{\Sigma}},$$

$$A_{21} = \left\{ \int_{S_j} Q(x_i,y)ds_y \right\}, \; i = \overline{1,N_{\Sigma}}, \; j = \overline{1,N_S},$$

$$A_{22} = \left\{ \int_{\Sigma_j} Q(x_i,y)ds_y \right\}, \; i,j = \overline{1,N_{\Sigma}},$$

$$\widetilde{\tau} = (\tau_1, \cdots, \tau_{N_S}), \qquad \widetilde{\mu} = (\mu_1, \cdots, \mu_{N_\Sigma}),$$
$$\widetilde{f} = (f(x_1), \cdots, f(x_{N_S})), \qquad \widetilde{g} = (g(x_1), \cdots, g(x_{N_\Sigma})),$$

x_i – collocation points on S or Σ.

Approximate meaning of searching solution of the problem T we can get from the next expression:

$$u(x) = \sum_{i=1}^{N_S} \tau_i \int\limits_{S_i} Q(x,y) ds_y + \sum_{i=1}^{N_\Sigma} \mu_i \int\limits_{\Sigma_i} Q(x,y) ds_y.$$

5. Numerical examples

Example 1. We consider the domain Ω bounded by circle Σ of the radius $R = 2$ and with open curve $S = \{(x_1, x_2) : x_2 = x_1, -1 < x_1 < 1\}$ (see Fig. 1):

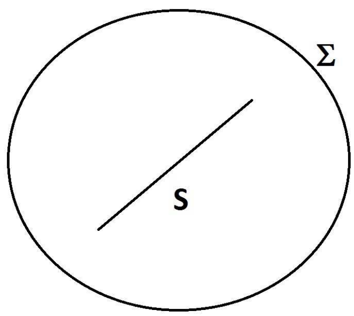
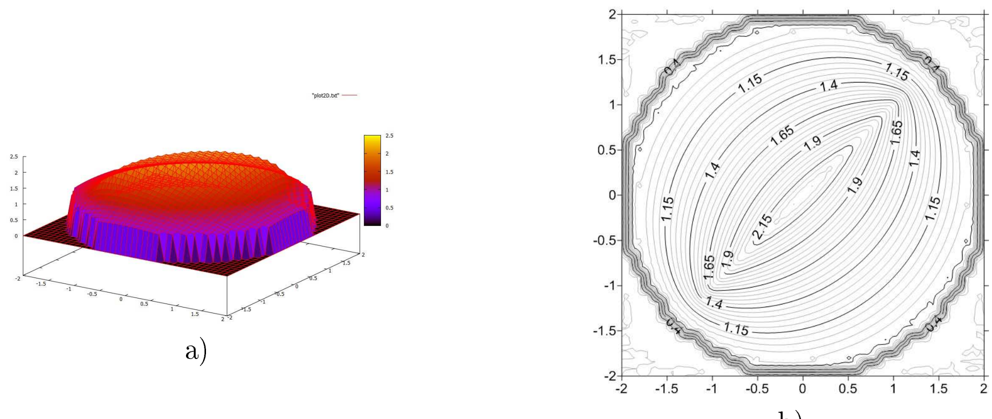


Fig. 1

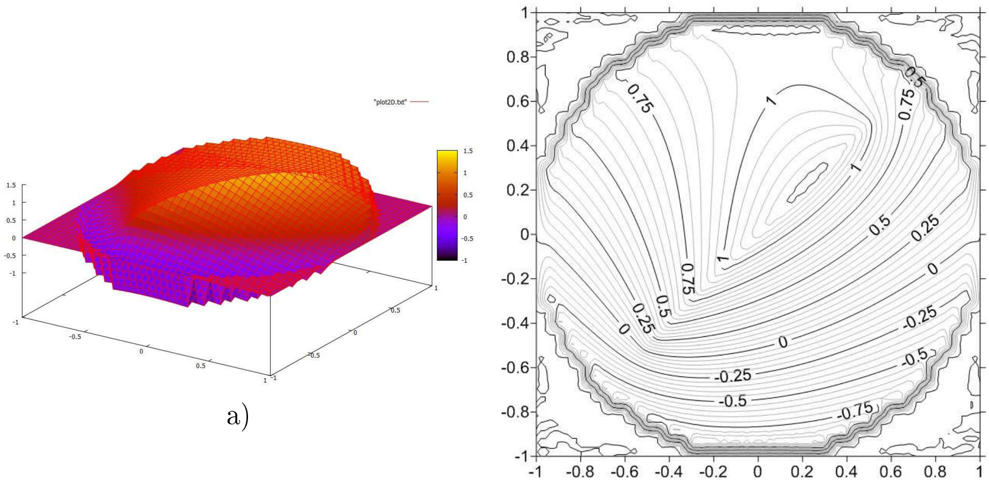
The obtained numerical result for given meaning of λ, f and g is presented in Fig. 2a and Fig. 2b.



a) b)

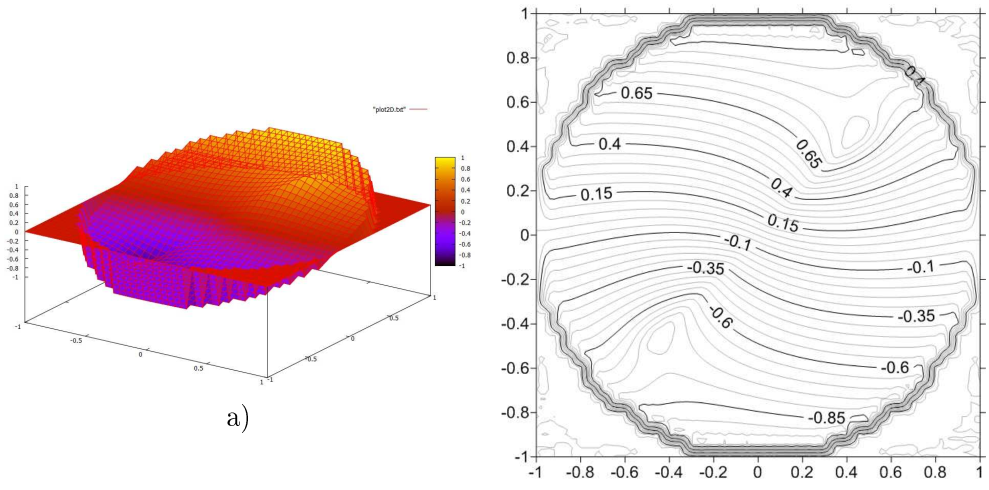
Fig. 2. $\lambda = 1$, $g = 1$, $f = 5$, $N_\Sigma = 800$, $N_S = 160$

If we take another meanings of functions g and f we can get the following results (see Fig. 3a, Fig. 3b, Fig. 4a and Fig. 4b):



a) b)

FIG. 3. $\lambda = 1$, $g = x_2$, $f = 5$, $N_\Sigma = 800$, $N_S = 160$



a) b)

FIG. 4. $\lambda = 1$, $g = x_2$, $f = 10x_2$, $N_\Sigma = 800$, $N_S = 160$

Example 2. We consider the next domain where S consists of two parts as it presented on Fig. 5

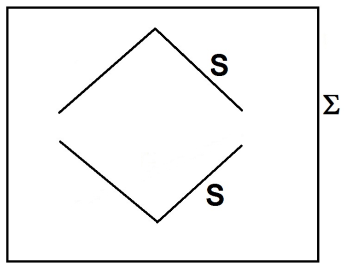
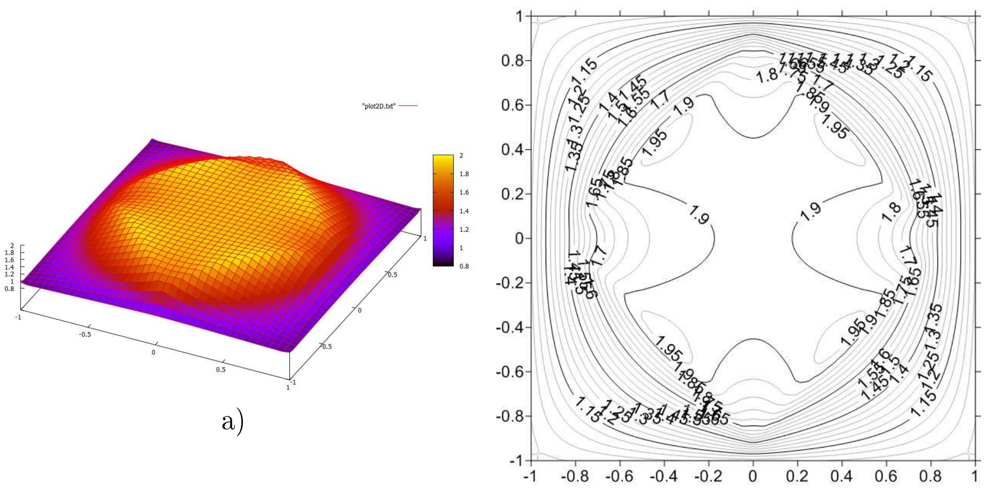


Fig. 5

Numerical result for given meanings of λ, f and g for this example is presented in Fig. 6a and Fig. 6b.



a) b)

Fig. 6. $\lambda = 1$, $g = 1$, $f = 1$, $N_\Sigma = 640$, $N_S = 320$

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