Журнал обчислювальної та прикладної математики | **2016** **№ 2 (122)** | Journal of Numerical & Applied Mathematics

UDC 519.6

ON THE APPLICATION OF MULTIPARAMETER INVERSE EIGENVALUE PROBLEM AND NUMERICAL METHODS FOR FINDING ITS SOLUTION

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Резюме. У роботі здійснено огляд відомих прикладів практичних застосувань оберненої задачі на власні значення у різних наукових та інженерних сферах досліджень. Крім того, представлено існуючі чисельні методи та різноманітні техніки відшукання розв'язку оберненої спектральної задачі.

Abstract. This survey collects the known examples of practical application of inverse eigenvalue problems in different scientific and engineering areas. It also provides an overview of the existing numerical methods and different techniques for finding the solution of the inverse eigenvalue problem.

1. Introduction

An inverse eigenvalue problem is a subject of interest of different authors. There are numerous examples of practical application of this problem and of the analysis of its partial cases. In this article we try to make an overview of the most known and interesting examples of practical application of this type of problems.

Let $A(c)$ be an afinne family

$$A(c) = A_0 + \sum_{k=1}^{n} c_k A_k, \tag{1}$$

where $c \in R^n$, and $\{A_k\}$ are real symmetric matrices of dimension $n \times n$.

Let's also denote the eigenvalues of the matrix $A(c)$ as $\{\lambda_i(c)\}_1^n$, where $\lambda_1(c) \leq ... \leq \lambda_n(c)$.

The following problem is known as *the general inverse eigenvalue problem*:

Problem 1. *Provided real numbers* $\lambda_1^* \leq ... \leq \lambda_n^*$ *find* $c \in R^n$ *such that the eigenvalues of* (1) *satisfy the condition* $\lambda_i(c) = \lambda_i^*$, $i = 1, ..., n$.

One of the partial cases of the Problem 1 is *the additive inverse eigenvalue problem*:

Problem 2. *Let the linear family* (1) *be defined as* $A_k = e_k e_k^T$, $k = 1, ..., n$ *where* e_k *is a k-th unit vector such, that*

$$A(c) = A_0 + D, where D = diag(c_k) \tag{2}$$

Key words. Inverse eigenvalue problem; inverse spectral problem; Sturm-Liouville problem; eigenvalue; eigenvector; numerical method; iteration procedure; Newton-like methods.

Provided the real values $\lambda_1^* \leq ... \leq \lambda_n^*$ *find* $c \in R^n$ *such, that the eigenvalues of the matrix* (2) *satisfy the condition* $\lambda_i(c) = \lambda_i^*, i = 1, ..., n$.

Another partial case of general problem, that is considered in this survey, is *the multiplicative inverse eigenvalue problem*:

Problem 3. *Given a real symmetric matrix* A *and its eigenvalues* $\lambda_1^* \leq ... \leq \lambda_n^*$, *find an additive diagonal matrix* $D = diag(c_k)$, $c \in R^n$, *such that the result matrix* AD *has the given eigenvalues.*

Both additive and multiplicative inverse eigenvalue problems have been formulated by Downing and Householder (1956).

It is known that the inverse eigenvalue problems arise in different scientific areas, including systems of identification, seismic topography, geophysics, molecular spectroscopy, structural analysis, mechanic systems simulation and so on. Some of the partial cases of inverse eigenvalue problem appear in factor analysis, educational testing problem, etc (see [1] and the cited references).

2. Examples and practical application of the inverse eigenvalue problems

The classical example of inverse eigenvalue problem is the problem of finding a solution of inverse Sturm-Liouville problem. The continuous problem has been investigated by, for example, Borh, Gelfand, Levitan and Hald. The discrete analog can be found in the survey [3], a more detailed overview is presented below.

Let's consider a boundary problem [3]:

$$-u''(x) + p(x)u(x) = \lambda u(x),$$

$$u(0) = u(\pi) = 0.$$

The task is to find the potential $p(x)$ by using the given spectrum $\{\lambda_i^*\}_1^\infty$. In order to build the discrete analog, the authors [3] use a uniform mesh, defining $h = \frac{\pi}{n+1}$, $u_k = u(kh)$, $p_k = p(kh)$, $k = 1, ..., n$, and make a suggestion that the values $\{\lambda_i^*\}_1^\infty$ are known. By using the finite differences for the approximation u'', the following equation is received:

$$\frac{-u_{k+1} + 2u_k - u_{k-1}}{h^2} + p_k u_k = \lambda_j^* u_k, k = 1, ..., n, u_0 = u_{n+1} = 0,$$

where λ_j^* is an eigenvalue from the set $\{\lambda_i^*\}_1^n$.

Thus,it is obtained the additive inverse eigenvalue problem (2) with the matrix

$$A_0 = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & & 2 \end{pmatrix} \tag{3}$$

and $D = diag(p_k)$.

Another well known example is the inverse spectral problem which arises in the analysis of string vibrations. A reference to this example can be found, for example, in [1], [3]. Let's briefly explain the content of this problem.

Consider the corresponding boundary problem [3]:

$$-u''(x) = \lambda p(x) u(x),$$
$$u(0) = u(\pi) = 0.$$

It is needed to find the density function $p(x) > 0$, under the condition that the fixed eigenvalues $\{\lambda_i^*\}_1^\infty$ are known. In order to proceed to the discrete analog of this problem, the transformations, similar to the case of Sturm-Liouville problem, are performed. As a result, the following equation is obtained:

$$Au = \lambda_i^* Du, i = 1, ..., n,$$

or, if reformulating a bit:

$$D^{-1}Au = \lambda_i^* u, i = 1, ..., n,$$

where $D = diag(p(kh)) > 0$, and the matrix A is defined by the correlation (3).

It can be easily seen that the obtained problem is the multiplicative inverse eigenvalue problem.

It is also possible to rewrite this problem in the form (1), where $A_0 = 0$, $A_k = e_k a_k^T$, $k = 1, ..., n$, and the a_k^T is a k-th row if the matrix A.

There are several inverse spectral problems with a matrix of a specific structure. For example, the problem of reconstructing the Jacobi matrix from the given spectral data. Briefly speaking, the inverse eigenvalue problem with the Jacobi matrix consists in defining the elements of the matrix from the given spectral data. This problem plays an important role in different applications, including vibration theory and structural design [10]. In some cases only a limited number of eigenvalues of the Jacobi matrix is provided. For example, four or five, as in the problem, presented in [10].

An interesting partial case of the general inverse spectral problem is the inverse Toeplitz problem (see [6]). According to the author, it is important, that although the Toeplitz matrices have such special structure, the question of solvability is opened for the case $n \geq 5$.

An inverse eigenvalue problem with a symmetrix matrix arises, for example, in the applied physics and the theory of control. This problem is investigated in the survey [9] and the cited references.

The other areas where the Problem 1 arises are nuclear spectroscopy and molecular spectroscopy. In practice the formulation of such problem often includes less parameters that there are eigenvalues. In such cases it makes sense to consider the problem formulation in least squares:

$$\min_{c \in R^n} \sum_{i=1}^{m} (\lambda_i(c) - \lambda_i^*)^2.$$

An important type of problems arising in the engineer researches can be described with the following formula

$$\min_{c \in R^m} f(c) \text{ by } l \leq \lambda_i(c) \leq u, i = 1, ..., n,$$

where $f(c)$ is a real-valued function of purpose, l and u are fixed lower and upper boundaries of eigenvalues of matrix $A(c)$, which is defined by the correlation (1). It's interesting to mention that the solution of the given problem often includes multiple eigenvalues, because the minimization of the function of purpose can simultaneously conduct several eigenvalues to the same boundary. This is why it's very important to choose the numerical method of solving the inverse spectral problem so that it correctly handles the case of multiple eigenvalues.

3. Numerical methods for solving the inverse eigenvalue problems

There is the rich literature dedicated to the question of numerical methods for finding an approximate solution of the inverse spectral problem. One of the creators of this theory is Friendland, who developed four quadratically convergent numerical methods together with his colleagues [3]. One of the methods, presented in [3], is, basically, the Newton method for solving the following system of nonlinear equations:

$$f(c) = \begin{bmatrix} \lambda_1(c) - \lambda_1^* \\ \dots \\ \lambda_n(c) - \lambda_n^* \end{bmatrix} = 0,$$

where $\lambda^* = [\lambda_1^*, ..., \lambda_n^*]^T \in R$, and $\lambda(c) = [\lambda_1(c), ..., \lambda_n(c)]^T$ is the vector of unique eigenvalues of the matrix $A(c)$. Each $\lambda_i(c)$ is a real-valued function, differentiable in some neighborhood of the point c^*, if c^* is the solution of Problem 1.

Note, that each iteration of this method involves solving a full spectral problem for the matrix $A(c)$.

Two other methods from [3] are considered to be the modifications of the Newton method, where the calculation of eigenvectors is simplified. This means that instead of calculating the exact eigenvectors, or in other words, solving the corresponding spectral problem, the approximation of these eigenvectors is calculated. The fourth method from [3] originally is based on the work of Biegler-Konig, (see [4] and the cited references), and uses the idea of calculating the determinant.

Based on the methods developed by Friedland and others [3], there have been constructed new methods for solving some inverse eigenvalue problems by other scientists. For example, in the paper [6] there are presented two methods for finding the solution of an inverse singular problem: one of the methods is continuous, the other – discrete. The discrete method generalizes the iteration process, originally proposed by Friedland for solving an inverse spectral problem. The new method converges locally under the condition of existence of the problem's solution.

Different authors have investigated this methods. Ones of the firsts who used it, where Downing and Householder – for solving the additive and the multiplicative inverse spectral problems. For a long time this method was also used by the physics in the nuclear spectroscopy calculations.

Instead of calculating the exact eigenvectors of the matrix $A(c)$ on each iteration of the method, it is possible to approximate them by using, for example, the inverse iteration. On this idea the Method II [3] is based.

The Method III is built on the idea of using a matrix of exponentials and the Cayley transform.

As explained by the authors in the survey [6], from the geometric point of view, the Method III [3] can be interpreted as the classical Newton method. This means that the geometry which is involved in the Method III, is closely bound to the geometry of the Newton method for the nonlinear equations with one variable. Consequently, the Method III can be generalized to the iteration process for calculation the approximate solution of the inverse singular problem.

Investigation of the methods, described in [3], can be found in other various articles, for example – in [1]. As it is stated by the author [1], in case of a matrix of big dimensions, the Method III has an obvious disadvantage: constructing an inverse matrix on each step is an expensive operation. These expenses can be decreased by using the iteration procedures (inner iterations). Because of it, usually the Method III, as the other methods of this type, is too expensive in such sense that the number of performed iterations (inner iterations) is much bigger then the number of iterations needed for convergence of the Newton method (outer iterations).

In order to calculate the solution of the classic additive and multiplicative inverse eigenvalue problems the Newton-like methods are also fine to use.

Among the known methods of this type it is worth mentioning the algorithm suggested by Kublanovskaya [2]. This algorithm calculates the solution as a zero of the function

$$H(c) = \begin{bmatrix} \lambda_1(c) - \lambda_1 \\ \vdots \\ \lambda_n(c) - \lambda_n \end{bmatrix},$$

where $\lambda_1(c) < ... < \lambda_n(c)$ are the eigenvalues of the matrix $A(c)$, and $\lambda_1 \leq ... \leq \lambda_n$ are the given eigenvalues.

As an alternative to the Kublanovskaya method, there is another algorithm presented in [2]. This one is also a Newton-like method and it calculates the solution of the initial problem, as the zeros of the function

$$F(c) = \begin{bmatrix} det(A(c) - \lambda_1 I) \\ \vdots \\ det(A(c) - \lambda_n I) \end{bmatrix}.$$

In order to reduce extra expenses of the exact iteration methods and to increase the effectiveness, the scientists Chan, Chung and Xu (see [1] and the cited references) suggested in inexact Newton-like method, which is used for the matrices of big dimensions. The inexact Newton method stops the inner iteration process before it converges. Thus, it is possible to decrease the total number of both, inner and outer, iterations, by choosing a proper stop condition.

In the paper [1] another approach is put forward – an inexact method of Cayley transform for the inverse eigenvalue probelm. This methods also minimizes the extra expenses and increases the productivity.

Based on the differentiation theory and on the QR-decomposition of a matrix, Li suggested a numerical method for solving the inverse spectral problems, which works for the case of unique eigenvalues (see [4] and the cited references).

In the same paper [4] there is examined the formulation and local convergence of a quadratically convergent method for solving the general inverse eigenvalue problem provided that its solution exists. The proposed method is based on the mentioned QR-decomposition of a matrix and the ideas of Li and Dai (see [4] and the cites references). As it is stated by the authors, this method is applicable for the case of unique eigenvalues as well as for multiple eigenvalues of the matrix.

One more approach to building a numerical method for solving an inverse spectral problem is suggested in the survey [9]. This approach is based on the analysis of analyticity of eigenvalues and eigenvectors of matrix of the problem. The examination of analyticity of spectral problems has a long history (see [9] and the cited references). However, according to the author, relatively small attention has been paid to the examination of analyticity of matrix spectra in the case when the matrix analytically depends on several parameters. Thereby, in [9] a new method is proposed. This is another modification of the known Newton method and allows to find the approximate solution of an inverse eigenvalue problem with a real symmetric matrix, which depends on several parameters.

Recently another approach type of methods – the gradient methods – gained the attention of scientists. For example, a variation-gradient method for solving multiparameter eigenvalue problems has been developed by Klobystov and Podlevkyi (see [5], [7]). The proposed method was later modified and extended to the inverse spectral problem by Podlevskyi and Yaroshko (see [8]). The idea of these methods, for both direct and inverse multiparameter eigenvalue problems, is to replace the spectral problem with an equivalent variation problem and applying the iterative method to find the solution of this variation problem. The mentioned method is based on the gradient procedure and the Newton method.

Let's consider the following multiparameter spectral problem in the Euclidian space E^n:

$$T(\lambda)\, x \equiv Ax - \lambda_1 B_1 x - \cdots - \lambda_m B_m x = 0, \tag{4}$$

where $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$ – are spectral parameters, $x = (x_1, \ldots, x_n) \in E^n$, and $A, B_1, ..., B_m$ – are some linear operators that act in the Euclidian space E^n.

Let's place in correspondence to the spectral problem (4) the variation problem of minimization of a functional:

$$F(u) = \frac{1}{2} \left\| T(\lambda)\, x \right\|_H^2, \forall u = \{x, \lambda\} \in H. \tag{5}$$

The problem (5) consists in finding such set of parameters $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$ and the corresponding vector $x \in E^n \setminus \{0\}$ on which the functional $F(u)$ reaches its minimal value:

$$F(u) \to \min_u, u \in U \subset H, \tag{6}$$

where U is a set of points $u = \{x, \lambda\}$, that satisfies the equation (4), H is an Euclidian space.

It can be shown that the spectral problem (4) and the variation problem (6) are equivalent. This means that each eigenpair $\{x, \lambda\}$ of the problem (4) is a point of minimum $u = \{x, \lambda\}$ of the functional (5), and vice versa.

This result allows us to build the gradient procedure for the numerical solving of the problem (6) and, therefore, the problem (4):

$$u_{k+1} = u_k - \gamma(u_k) \nabla F(u_k), \quad k = 0, 1, 2, ... \tag{7}$$

The formula (7) describes the whole class of methods, which differ one from another only by the choice of the step $\gamma(u_k)$.

In our method we suggest calculating the value $\gamma_k = \gamma(u_k)$ on each step of the iteration process by the formula:

$$\gamma_k = \frac{F(u_k)}{\|\nabla F(u_k)\|_H^2}.$$

To conclude, the iteration process can be written in the form:

$$u_{k+1} = u_k - \frac{F(u_k)}{\|\nabla F(u_k)\|_H^2} \nabla F(u_k). \tag{8}$$

So far we have described the method for solving the direct eigenvalue problem. Let's explain the algorithm of solving the inverse spectral problem, which is based on the described gradient procedure.

Consider the inverse eigenvalue problem (1) with the real matrices A_0, $A_1, ..., A_m \in E^{n \times n}$, and where the pairs $\{\lambda_k, x^k\}_{k=1}^m$ are the eigenpairs of the matrix $A(p)$. Here $\lambda = \{\lambda_1, ..., \lambda_m\} \in E^m$, $x^k \in H = E^n \setminus \{0\}$, $k = 1, ..., m$, and E is a real Euclidian space.

Using the definition of the eigenvalue and the corresponding eigenvector, we can build the system of m equations for finding the parameters $p_1, ..., p_m$:

$$\begin{cases} ((A_0 - \lambda_1 I) + p_1 A_1 + ... + p_m A_m) x^1 = 0, \\ \dots \\ ((A_0 - \lambda_m I) + p_1 A_1 + ... + p_m A_m) x^m = 0. \end{cases} \tag{9}$$

Now lets transform this system by introducing the matrix operators $\mathbf{A}, \mathbf{B}_i : \mathbf{H} \to \mathbf{H}$, $\mathbf{H} = \oplus_{k=1}^m E^{n \times n}$ $(i = 1, ..., m.)$,

$$A = \begin{pmatrix} (A_0 - \lambda_1 I) & & 0 \\ & \ddots & \\ 0 & & (A_0 - \lambda_m I) \end{pmatrix}, \quad B_i = \begin{pmatrix} -A_i & & 0 \\ & \ddots & \\ 0 & & -A_i \end{pmatrix},$$

In case $\mathbf{x} = \left(x^1, x^2, ..., x^m\right)^T \in \mathbf{H}$, we get

$$\mathbf{A}\mathbf{x} = \left((A_0 - \lambda_1 I) x^1, (A_0 - \lambda_2 I) x^2, ..., (A_0 - \lambda_m I) x^m\right),$$

$$\mathbf{B}_i\mathbf{x} = \left(-A_i x^1, -A_i x^2, ..., -A_i x^m\right).$$

Now it is possible to pass from the problem (9) to the problem in the form (4) in the space $\mathbf{H}$

$$T(p) \equiv \mathbf{A}\mathbf{x} - p_1\mathbf{B}_1\mathbf{x} - ... - p_m\mathbf{B}_m\mathbf{x} = 0. \tag{10}$$

Therefore, we retrieved the problem of finding the set of parameters $p_1, ..., p_m$, such that the equation (10) has a non-trivial solution $\mathbf{x} \in \mathbf{H} \setminus \{\mathbf{0}\}$.

In correspondence to the problem (10) we put the variation problem:

$$F(\mathbf{u}) \to \min_{\mathbf{u}}, \mathbf{u} \in \mathbf{U} \subset \mathbf{H},$$

where $F(\mathbf{u}) = \frac{1}{2}\left\|T(p)\,\mathbf{x}\right\|_H^2, \quad \forall \mathbf{u} = \{\mathbf{x}, p\} \in H = \mathbf{H} \oplus E^m$.

As expected, the task is to find the set of parameters $p = \{p_1, ..., p_m\} \in E^m$ and the corresponding vector $\mathbf{x} \in \mathbf{H}\setminus\{\mathbf{0}\}$, on which the functional $F(\mathbf{u})$ reaches its minimal value.

In order to solve the variation problem we use the iteration process (8). Consequently, we obtain the solution of the initial inverse eigenvalue problem.

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Received 08.08.2016