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AN ITERATIVE METHOD FOR THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN THREE-DIMENSIONAL DOMAINS

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Резюме. Ми розглядаємо ітераційний узагальнений метод Ландвебера для задачі Коші для рівняння Лапласа у двозв'язних тривимірних областях. Цей метод є регуляризуючою процедурою для отримання стабільного розв'язку. На кожному кроці ітераційного методу потрібно розв'язати дві коректні прямі задачі для рівняння Лапласа. Кожна пряма задача вирішується методом граничних інтегральних рівнянь із застосуванням проекційного методу Гальоркіна для дискретизації. Наприкінці наведені деякі чисельні результати.

Abstract. We consider an iterative generalized Landweber method for the Cauchy problem for the Laplace equation in doubly connected 3-dimensional domains. This method is a regularizing procedure for obtaining a stable solution to the Cauchy problem, and consists of solving two well-posed direct problems for the Laplace equation at each iteration step. Each direct problem is solved by a boundary integral equations method with a projection Galerkin method for the discretisation. Some numerical results are given and discussed as well at the end.

1. Introduction

The Cauchy problem for the Laplace equation has important applications. For example, it occurs in electrostatics, non-destructive testing, cardiology, leak identification, etc. This problem belongs to the class of ill-posed linear inverse problems, since it is unstable with respect to input data [7] (a small remark here, the input Cauchy data should be compatible [6]). We focus on the numerical solution of this Cauchy problem in three-dimensional doubly connected domains.

The Cauchy problem can be solved numerically in a stable way by combining direct methods, such as for example the boundary integral equations method [4,10–12,15] or the method of fundamental solutions [14] etc, with some regularization strategy, for example, Tikhonov regularization with an appropriate way of selecting the regularization parameter like the Morozov discrepancy principle or the L-curve method [4, 12, 15]. Another approach for numerically solving the Cauchy problem is to use iterative methods, where the choice of the termination of the iterations is part of the regularization. Numerical examples show that iterative methods give good results in the case of noisy data, namely,

Key words. Laplace equation, Cauchy problem, Landweber method, Robin boundary problem, boundary integral equations, projection Galerkin method, Wienert's method, R^3 domains.

we can calculate an approximation with an error being equal to the noise level or even smaller, by selecting a good strategy for the numerical implementation of the iterative approach. Commonly used methods are the alternating method [5, 8, 12] and the Landweber procedure [12] in combination with the boundary integral equations method for solving the direct problems needed in both these iterative algorithms.

In this paper, we apply one recent approach being a generalized Landweber method proposed in [2], for 3-dimensional doubly connected domains. The main difference from the standard Landweber method is that we do not need to use any adjoint operator, that is we do not need to involve any adjoint differential equation.

We then describe more on the problem formulation. Let $D_1 \subset \mathbb{R}^3$, $D_2 \subset \mathbb{R}^3$ be simply connected smooth bounded domains with boundary surfaces Γ_1 and Γ_2, respectively, that satisfy: $\overline{D}_1 \subset D_2$. Let $D = D_2 \setminus \overline{D}_1$ be the solution domain and $\nu = (\nu_1, \nu_2, \nu_3)^t$ the outward unit normal to the boundary of D; this boundary is denoted by $\partial D = \Gamma_1 \cup \Gamma_2$.

The Cauchy problem is then as follows. We need to find a classical solution $u \in C^2(D) \cap C^1(\overline{D})$ of the Laplace equation:

$$\Delta u = 0 \quad \text{in } D \tag{1}$$

that satisfies the boundary conditions:

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_2. \tag{2}$$

It is not the full solution in D that is of prime interest, it is instead to find (reconstruct) the corresponding Cauchy data $\left\{ u, \frac{\partial u}{\partial \nu} \right\}$ on the interior boundary surface Γ_1.

As mentioned, for the numerical solution of the above problem, we apply one adjoint-free Landweber method [2] being a regularizing procedure for obtaining a stable numerical solution [2]. At each step of the iterative procedure, we need to solve the Dirichlet respectively the Robin direct problems for the Laplace equation. We use the boundary integral equations method for solving the required direct problems in the iterative method, and this choice is based on good numerical results for domains in $\mathbb{R}^2$, see [11, 12] as well as for domains in $\mathbb{R}^3$ [4, 5], together with advantages such as reduction of the dimension of the problem and the flexibility in terms of the form of the boundary surfaces. As a stopping rule for the iterations, the Morozov discrepancy principle is used.

The solution of each direct problem is represented as a combination of potentials [4,9,12]. Based on this representation, we obtain a system of linear integral equations for finding the unknown densities by requiring that the given Cauchy data should be satisfied. For discretization Wienert's method is applied; it is a Galerkin discrete projection method, where the unknown densities are represented as a linear combination of spherical harmonics [1] and the boundary integrals are rewritten over the unit sphere, and to those obtained integrals certain cubature rules are then applied [13, 16].

An outline of this work is: in Section 2, we consider the iterative algorithm, the boundary integral equations method for one of the direct problems in the procedure (having boundary conditions of Robin type) is given in Section 3 and in Section 4 some numerical results are shown and discussed.

2. The iterative algorithm

We consider one of the iterative methods proposed in [2], in three-dimensional doubly connected domains. At each iteration step, we need to solve one Dirichlet and one Robin boundary value problem for the Laplace equation. The algorithm is as follows:

- The first approximation u_0 of the solution u is calculated by solving the Dirichlet boundary value problem:

$$\Delta u_0 = 0 \quad \text{in } D, \tag{3}$$

$$u_0 = \eta_0 \text{ on } \Gamma_1 \quad \text{and} \quad u_0 = f \quad \text{on } \Gamma_2, \tag{4}$$

where η_0 is an arbitrary initial starting approximation on the boundary Γ_1.

- Then the element v_0 is obtained by solving the Robin boundary value problem:

$$\Delta v_0 = 0 \quad \text{in } D, \tag{5}$$

$$\frac{\partial v_0}{\partial \nu} + \kappa v_0 = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial v_0}{\partial \nu} + \kappa v_0 = g - \frac{\partial u_0}{\partial \nu} \quad \text{on } \Gamma_2. \tag{6}$$

- Having obtained u_{k-1} and v_{k-1}, the approximation u_k is obtained from the Dirichlet boundary value problem:

$$\Delta u_k = 0 \quad \text{in } D, \tag{7}$$

$$u_k = \eta_k \text{ on } \Gamma_1 \quad \text{and} \quad u_k = f \quad \text{on } \Gamma_2, \tag{8}$$

where

$$\eta_k = \eta_{k-1} + \gamma v_{k-1}|_{\Gamma_1}, \quad \gamma > 0. \tag{9}$$

- Then the solution v_k is obtained by solving the following Robin boundary value problem:

$$\Delta v_k = 0 \quad \text{in } D, \tag{10}$$

$$\frac{\partial v_k}{\partial \nu} + \kappa v_k = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial v_k}{\partial \nu} + \kappa v_k = g - \frac{\partial u_k}{\partial \nu} \quad \text{on } \Gamma_2. \tag{11}$$

The iterative procedure then continues by iterating in the last two steps. The stopping rule is the Morozov discrepancy principle. The initial approximation is arbitrary for linear problems, and we select it as the zero-function.

The parameter κ in the Robin boundary condition is positive: $\kappa > 0$. The parameter $\gamma > 0$ in the iterative procedure is a relaxation parameter, which is needed for convergence of the algorithm [2].

The Dirichlet and Robin boundary value problems are well-posed in $L^2(D)$ for boundary data from $L^2(\Gamma_1)$ and $L^2(\Gamma_2)$. Moreover, given $f, g \in L^2(\Gamma_2)$ one can show that $\lim_{k\to\infty} \|u - u_k\|_{L^2(D)} = 0$, where u_k is the k-th approximation generated from the above algorithm and u is the solution of the Cauchy problem (1)–(2). Furthermore, for noisy data $\{f^\delta, g^\delta\}$, with $\delta > 0$, we have

$\left\| f^\delta - u_k^\delta \right\|_{L^2(\Gamma_2)} \le \tau\delta$, for $\tau > 1$, where u_k^δ is the k-th approximation obtained from the iterative algorithm using the noisy data. For further information and details on these estimates, see [2].

3. Numerical solution of the boundary value problems

To solve each of the boundary value problems used in the iterative procedure, we use the boundary integral equations method. In the introduction, we mentioned some advantages of this approach such as reducing the dimension of the problem compared with the dimension of the solution domain, the flexibility of applying it for domains of different shapes or even to unbounded domains, its super-algebraic convergence for analytical data etc.

In [3], it is demonstrated how to solve the Dirichlet boundary value problem using a single-layer representation of the solution. The similar ideas can be applied to the Dirichlet boundary value problem by instead using a combination of single- and double-layer potentials to represent the solution, thereby obtaining an integral equation of the second kind to solve [9].

We then turn to the Robin boundary value problem:

$$\Delta u = 0 \quad \text{in } D, \tag{12}$$

$$\frac{\partial u}{\partial \nu} + \kappa u = h \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu} + \kappa u = w \quad \text{on } \Gamma_2, \tag{13}$$

where $\kappa > 0$, $h \in L_2(\Gamma_1)$, $w \in L_2(\Gamma_2)$ are given.

To obtain an integral equation of the second kind, we represent the solution of (12)–(13) as a sum of two single-layer potentials:

$$u(x) = \sum_{l=1}^{2} \int_{\Gamma_l} \varphi_l(y)\Phi(x,y)\, ds(y), \quad x \in D, \tag{14}$$

where $\Phi(x,y) = \dfrac{1}{4\pi|x-y|}$ is a fundamental solution of the Laplace equation in $\mathbb{R}^3$ and $\varphi_l \in C(\Gamma_l)$, $l = 1,2$, are unknown densities.

From the representation of the solution (14) requiring the boundary conditions (13) to be satisfied, invoking properties of single-layer potentials [9], we obtain a system of linear integral equations for finding the unknown densities:

$$\begin{cases} -\dfrac{1}{2}\varphi_1 + K_{11}\varphi_1 + K_{12}\varphi_2 + \kappa\left(S_{11}\varphi_1 + S_{12}\varphi_2\right) = h, & \text{on } \Gamma_1, \\ \dfrac{1}{2}\varphi_2 + K_{21}\varphi_1 + K_{22}\varphi_2 + \kappa\left(S_{21}\varphi_1 + S_{22}\varphi_2\right) = w, & \text{on } \Gamma_2, \end{cases} \tag{15}$$

where we used the following boundary integral operators for $l, r = 1,2$:

$$(S_{lr}\psi)(x) = \int_{\Gamma_r} \psi(y)\Phi(x,y)\, ds(y),\ x \in \Gamma_l,\ \psi \in C(\Gamma_r), \tag{16}$$

$$(K_{lr}\psi)(x) = \int_{\Gamma_r} \psi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}\, ds(y),\ x \in \Gamma_l,\ \psi \in C(\Gamma_r). \tag{17}$$

Notice here that for the Robin boundary problem the approximation of the solution on the internal boundary surface Γ_1 needed in the above generalized Landweber algorithm, can be obtained as

$$u(x) = (S_{11}\varphi_1)(x) + (S_{12}\varphi_2)(x), \quad x \in \Gamma_1. \tag{18}$$

We assume that the two boundary surfaces can be smoothly mapped one-to-one to the unit sphere $\mathbb{S}^2 = \left\{\widehat{x} \in \mathbb{R}^3 : |\widehat{x}| = 1\right\}$. In that case there exist one-to-one mappings $q_l : \mathbb{S}^2 \to \Gamma_l$, $l = 1, 2$, having smoothly varying Jacobian J_{q_l}, $l = 1, 2$. Therefore, based on (16) and (17), we can rewrite the system of integral equations (15) over the unit sphere:

$$\begin{cases} -\dfrac{1}{2}\phi_1 + \tilde{K}_{11}\phi_1 + \tilde{K}_{12}\phi_2 + \kappa\left(\tilde{S}_{11}\phi_1 + \tilde{S}_{12}\phi_2\right) = \tilde{h}, & \text{on } \mathbb{S}^2, \\ \dfrac{1}{2}\phi_2 + \tilde{K}_{21}\phi_1 + \tilde{K}_{22}\phi_2 + \kappa\left(\tilde{S}_{21}\phi_1 + \tilde{S}_{22}\phi_2\right) = \tilde{w}, & \text{on } \mathbb{S}^2, \end{cases} \tag{19}$$

where $\phi_l(\widehat{x}) = \varphi_l(q_l(\widehat{x}))$, $l = 1, 2$, $\tilde{h}(\widehat{x}) = h(q_1(\widehat{x}))$, $\tilde{w}(\widehat{x}) = w(q_2(\widehat{x}))$ for $\widehat{x} \in \mathbb{S}^2$ and the parametrised integral operators are for $l, r = 1, 2$:

$$(\tilde{S}_{lr}\psi)(\widehat{x}) = \int\limits_{\mathbb{S}^2} \psi(\widehat{y}) L_{lr}(\widehat{x}, \widehat{y})\, ds(y), \quad \psi(\widehat{x}) \in C(\mathbb{S}^2),\, \widehat{x} \in \mathbb{S}^2, \tag{20}$$

and

$$(\tilde{K}_{lr}\psi)(\widehat{x}) = \int\limits_{\mathbb{S}^2} \psi(\widehat{y}) M_{lr}(\widehat{x}, \widehat{y})\, ds(y), \quad \psi(\widehat{x}) \in C(\mathbb{S}^2),\, \widehat{x} \in \mathbb{S}^2, \tag{21}$$

with

$$L_{lr}(\widehat{x}, \widehat{y}) = \begin{cases} J_{q_r}(\widehat{y})\Phi(q_l(\widehat{x}), q_r(\widehat{y})), & l \neq r, \\ \dfrac{R_l(\widehat{x}, \widehat{y})}{|\widehat{x} - \widehat{y}|}, & l = r, \end{cases}$$

$$M_{lr}(\widehat{x}, \widehat{y}) = \begin{cases} -J_{q_r}(\widehat{y}) \dfrac{(q_l(\widehat{x}) - q_r(\widehat{y}))^T \nu(q_l(\widehat{x}))}{4\pi|q_l(\widehat{x}) - q_r(\widehat{y})|^3}, & l \neq r, \\ \dfrac{\tilde{R}_l(\widehat{x}, \widehat{y})}{|\widehat{x} - \widehat{y}|}, & l = r, \end{cases}$$

where

$$R_l(\widehat{x}, \widehat{y}) = \frac{J_{q_l}(\widehat{y})}{4\pi} \begin{cases} \dfrac{|\widehat{x} - \widehat{y}|}{|q_l(\widehat{x}) - q_l(\widehat{y})|}, & \widehat{x} \neq \widehat{y} \\ \dfrac{1}{J_{q_l}(\widehat{x})}, & \widehat{x} = \widehat{y} \end{cases}$$

and

$$\tilde{R}_l(\widehat{x}, \widehat{y}) = -R_l(\widehat{x}, \widehat{y}) \begin{cases} \dfrac{(q_l(\widehat{x}) - q_r(\widehat{y}))^T \nu(q_l(\widehat{x}))}{4\pi|q_l(\widehat{x}) - q_r(\widehat{y})|^2}, & \widehat{x} \neq \widehat{y}\,, \\ -\dfrac{2\sum\limits_{j=1}^{3} q'_{jl}(\widehat{x})\nu_j(\widehat{x}) - \sum\limits_{j=1}^{3} q''_{jl}(\widehat{x})\nu_j(\widehat{x})}{2J^2_{q_l}(\widehat{x})}, & \widehat{x} = \widehat{y}\,. \end{cases}$$

From this representation, it can be seen that the integral operators S_{ll} and K_{ll}, $l = 1, 2$, each have a weak singularity.

For the numerical approximation of the integrals in (20) and (21), we next use the following cubature rules for $n^{'} > 0$, see [13,16]:

– cubature for integrals with a continuous integrand:

$$\int\limits_{\mathbb{S}^2} f(\widehat{y})\, ds(\widehat{y}) \approx \sum_{p^{'}=0}^{2n^{'}+1} \sum_{s^{'}=1}^{n^{'}+1} \widetilde{\mu}_{p^{'}} \widetilde{a}_{s^{'}} f(\widehat{y}_{s^{'}p^{'}}); \tag{22}$$

– cubature for integrals with a weak singularity in the integrand:

$$\int\limits_{\mathbb{S}^2} \frac{f(\widehat{y})}{|\widehat{x}-\widehat{y}|}\, ds(\widehat{y}) \approx \sum_{p^{'}=0}^{2n^{'}+1} \sum_{s^{'}=1}^{n^{'}+1} \widetilde{\mu}_{p^{'}} \widetilde{b}_{s^{'}} f(T_{\widehat{x}}^{-1} \widehat{y}_{s^{'}p^{'}}). \tag{23}$$

In the cubature rules (22)–(23), we use the following cubature points:

$$\widehat{y}_{s^{'}p^{'}} = \Big(\sin\theta_{s^{'}} \cos\varphi_{p^{'}}, \sin\theta_{s^{'}} \sin\varphi_{p^{'}}, \cos\theta_{s^{'}}\Big),$$

with $\varphi_{p^{'}} = \dfrac{p^{'}\pi}{n^{'}+1}$, $\theta_{s^{'}} = \arccos z_{s^{'}}$, where $z_{s^{'}}$ are the zeros of the Legendre polynomials $P_{n^{'}+1}$ [1]. The weights of the cubature rules are: $\widetilde{\mu}_{p^{'}} = \dfrac{\pi}{n^{'}+1}$, $\widetilde{a}_{s^{'}} = \dfrac{2(1-z_{s^{'}}^2)}{((n^{'}+1)P_{n^{'}}(z_{s^{'}}))^2}$, $\widetilde{b}_{s^{'}} = \widetilde{a}_{s^{'}} \sum\limits_{l=0}^{n^{'}} P_l(z_{s^{'}})$. Following [13], we use an orthogonal transformation $T_{\widehat{x}}$ to move the weak singularity in the integrands to appear at the north pole of the sphere; it is present in (23). The transformation $T_{\widehat{x}}$ is defined as follows:

$$T_{\widehat{x}} = D_F(\varphi) D_T(\theta) D_F(-\varphi), \quad x \in \mathbb{S}^2$$

with

$$D_F(\psi) \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_T(\psi) \begin{pmatrix} \cos(\psi) & 0 & -\sin(\psi) \\ 0 & 1 & 0 \\ \sin(\psi) & 0 & \cos(\psi) \end{pmatrix}.$$

The cubatures (22)–(23) have exponential convergence for a sufficiently smooth integrand f, see [16].

For discretisation of the system (19), we use a Galerkin projection method. The unknown densities ϕ_l, $l = 1, 2$, are first approximated by a linear combination of real-valued spherical harmonics:

$$\phi_l \approx \widetilde{\phi}_l = \sum_{k=0}^{n} \sum_{m=-k}^{k} \phi_{k,m}^l Y_{k,m}^R, \quad l = 1, 2, \tag{24}$$

where $\phi_{k,m}^l$ are the unknown coefficients, and the real-valued spherical harmonics are:

$$Y_{k,m}^R = \begin{cases} \operatorname{Im} Y_{k,|m|}, & 0 < m \le k, \\ \operatorname{Re} Y_{k,|m|}, & -k \le m \le 0 \end{cases}$$

with $Y_{k,m}$ the spherical harmonics [1].

We consider the following discrete inner product, defined from the cubature rule (22):

$$(v,d) = \sum_{p=0}^{2n+1}\sum_{s=1}^{n+1} \mu_p a_s v(\widehat{y}_{sp}) d(\widehat{y}_{sp}), \quad v,d \in C(\mathbb{S}^2), \tag{25}$$

where the weights and points are generated from (22) for the parameter $n > 0$.

After approximating the unknown densities in (19) by (24), and by applying $(n+1)^2$ times the inner product (25) to (19) with $Y^R_{k,m}$, $k = 0,\dots,n$, $m = -k,\dots,k$, taking into account the representation of the integral operators (20) and (21), we obtain a linear system of equations for finding the unknown coefficients in the representation (24):

$$\begin{cases} \sum\limits_{k=0}^{n}\sum\limits_{m=-k}^{k}\left(\phi^1_{k,m}A^{11}_{kk'mm'} + \phi^2_{k,m}A^{12}_{kk'mm'}\right) = \\ \qquad = \sum\limits_{p=0}^{2n+1}\sum\limits_{s=1}^{n+1}\mu_p a_s \tilde{h}(\widehat{x}_{sp})Y^R_{k,m}(\widehat{x}_{sp}), \\ \sum\limits_{k=0}^{n}\sum\limits_{m=-k}^{k}\left(\phi^1_{k,m}A^{21}_{kk'mm'} + \phi^2_{k,m}A^{22}_{kk'mm'}\right) = \\ \qquad = \sum\limits_{p=0}^{2n+1}\sum\limits_{s=1}^{n+1}\mu_p a_s \tilde{w}(\widehat{x}_{sp})Y^R_{k,m}(\widehat{x}_{sp}), \end{cases} \tag{26}$$

for $k' = 0,\dots,n$, $m = -k,\dots,k$, $n = 0,1,\dots$, with coefficients for $l,r = 1,2$ given by:

$$\begin{aligned} A^{lr}_{kk'mm'} &= \sum_{p=0}^{2n+1}\sum_{s=1}^{n+1}\sum_{p'=0}^{2n'+1}\sum_{s'=1}^{n'+1}\mu_{p'}\mu_p a_s Y^R_{k',m'}(\widehat{x}_{sp})\times \\ &\times\left(\begin{cases} \widetilde{a}'_s Y^R_{k,m}(\widehat{y}_{s'p'})\left(M_{lr}(\widehat{x}_{sp},\widehat{y}_{s'p'}) + \kappa L_{lr}(\widehat{x}_{sp},\widehat{y}_{s'p'})\right), & l \neq r \\ \widetilde{b}'_s Y^R_{k,m}(\widehat{y}^{s'p'}_{sp})\left(\tilde{R}_l(\widehat{x}_{sp},\widehat{y}^{s'p'}_{sp}) + \kappa R_l(\widehat{x}_{sp},\widehat{y}^{s'p'}_{sp})\right), & l = r \end{cases}\right. \\ &+\left.\begin{cases} 0, & l \neq r \\ (-1)^l \dfrac{1}{2} Y^R_{k,m}(\widehat{x}_{sp}), & l = r \end{cases}\right), \end{aligned}$$

where $\widehat{y}^{s'p'}_{sp} = T^{-1}_{\widehat{x}_{sp}}\widehat{y}_{s'p'}$.

Calculation of the coefficients $A^{lr}_{kk'mm'}$ requires many operations. We can reduce the number of operations by using sequential calculation of smaller additional matrices [4, 5]. Employing this strategy, we can reduce the number of operations from $O(n^8)$ to $O(n^5)$. The coefficients $A^{lr}_{kk'mm'}$ of the system (26) need only to be calculated once, and can then be used at each step of the generalized Landweber iterative algorithm. In fact, we only need to calculate the right-hand side of the system (26) at each step for different functions $\tilde{h}$ and $\tilde{w}$.

After finding the unknown coefficients $\phi^l_{k,m}$, $l = 1,2$, from (26), we can find an approximation of the unknown densities $\tilde{\phi}_l$, $l = 1,2$, from (24).

The solution of the Robin boundary value problem (12)–(13) on the interior surface Γ_1 is given by (18); using the approximation of the densities (24), the cubature rule (22) and the representation of the integral operator (16), an approximation of the solution on Γ_1 is then given by:

$$u(\widehat{x}) \approx \sum_{s'=1}^{n'+1} \sum_{\rho'=0}^{2n'+1} \Big(\widetilde{b}_{s'} \widetilde{\mu}_{\rho'} \tilde{\phi}_1(T_{\widehat{x}}^{-1} \widehat{y}_{s'\rho'}) R_1(\widehat{x}, T_{\widehat{x}}^{-1} \widehat{y}_{s'\rho'}) + \\ + \widetilde{a}_{s'} \widetilde{\mu}_{\rho'} \tilde{\phi}_2(\widehat{y}_{s'\rho'}) L_{12}(\widehat{x}, \widehat{y}_{s'\rho'}) \Big), \quad \widehat{x} \in \Gamma_1.$$

4. Numerical experiments

In this section, we give some numerical examples. The main example is the numerical solution of the Cauchy problem (1)–(2) by using the iterative generalized Landweber algorithm with exact and noisy data. However, we first start by giving results for the Robin boundary value problem (12)–(13) needed in the iterative algorithm, to see how our proposed boundary integral equations method and discretisation perform for this direct problem.

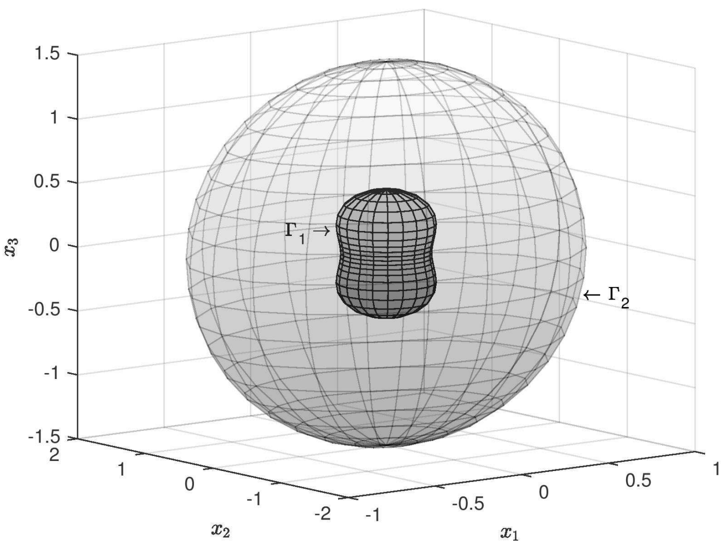


Fig. 1. The solution domain D in Ex. 1

Example 1 (Robin problem (12)–(13)). Let the doubly connected domain D (see Fig. 1) be bounded by the two surfaces:

$\Gamma_l = \{x(\theta,\varphi) = r_1(\theta,\varphi)\,(\sin\theta\cos\varphi, 2\sin\theta\sin\varphi, \cos\theta)\,,\ \theta \in [0,\pi],\ \varphi \in [0,2\pi]\}\,,$

where radial function r_1 is:

$$r_1(\theta,\varphi) = \frac{1}{2\sqrt{1+\sqrt{2}}} \sqrt{\cos(2\theta) + \sqrt{2 - \sin^2(2\theta)}},$$

and

$$\Gamma_2 = \big\{x(\theta,\varphi) = (\sin\theta\cos\varphi, 1.5\sin\theta\sin\varphi, 1.5\cos\theta)\,,\ \theta \in [0,\pi],\ \varphi \in [0,2\pi]\big\}.$$

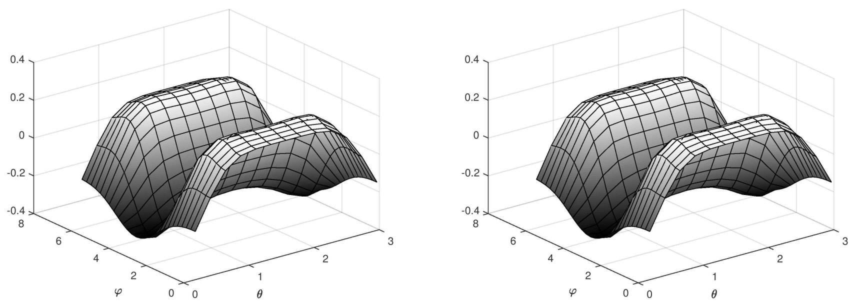
TABL. 1. L_2-errors for the Robin boundary value problem in Ex. 1

$n = n'$	$\dfrac{\lVert u_{ex} - u_n \rVert_{L_2(\Gamma_1)}}{\lVert u_{ex} \rVert_{L_2(\Gamma_1)}}$
2	5.13-E01
4	1.27-E02
6	6.56-E04
8	2.81-E05
10	1.91-E06
12	1.50-E07

The boundary data needed in the Robin boundary problem are generated from the exact solution: $u_{ex}(x) = x_2^2 - x_3^2 + x_1$, $x = (x_1, x_2, x_3)$, thus we get:

$$\frac{\partial u}{\partial \nu}(x) + \kappa u(x) = \nu_1(x) + 2x_2\nu_2(x) - 2x_3\nu_3(x) + \\ + \kappa(x_2^2 - x_3^2 + x_1), \quad x \in \Gamma_l, \quad l = 1, 2.$$

Values of the relative L_2-errors for the Robin boundary value problem (12)–(13) are presented in Table 1. As we can see from this table, super-algebraic convergence is present. In Fig. 2 are the exact and the numerical approximation for the function values on the internal boundary surface Γ_1, obtained with the discretisation parameters being $n = n' = 12$.



a). exact solution b). approximate solution

FIG. 2. Exact and numerical approximation for the function values on the internal boundary Γ_1 for the solution of the Robin boundary problem in Ex. 1

Example 2 (Cauchy problem (1)–(2)). Let the domain D (see Fig. 3) be bounded by the two surfaces:

$$\Gamma_l = \big\{ x(\theta, \varphi) = r_l(\theta, \varphi)\,(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,, \\ \theta \in [0, \pi],\ \varphi \in [0, 2\pi] \big\}, \quad l = 1, 2,$$

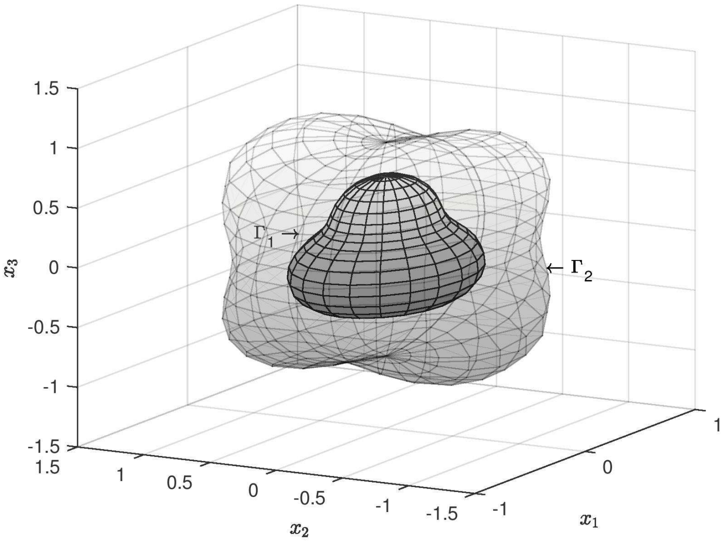
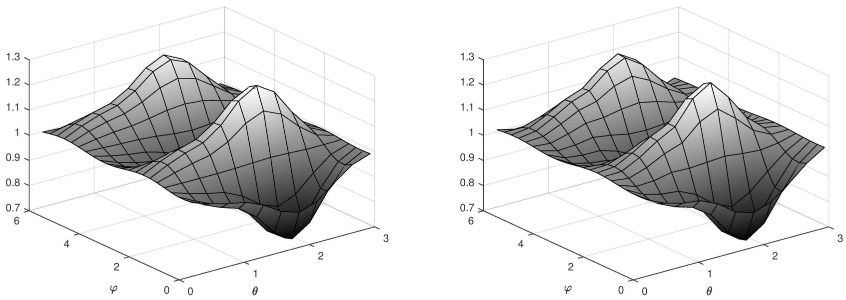


FIG. 3. The solution domain D in Ex. 2



a). exact solution b). approximate solution

FIG. 4. Reconstruction of the solution on the boundary Γ_1 in Ex. 2 (exact data)

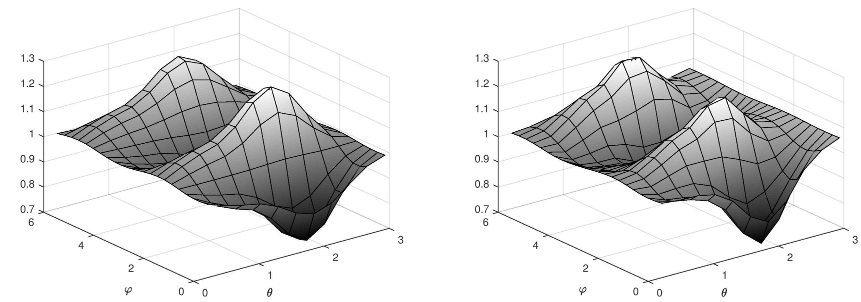
where the radial functions are as follows:

$$r_1(\theta, \varphi) = 0.2 \left(0.6 + \sqrt{4.25 + 2\cos(3\theta)}\right)$$

and

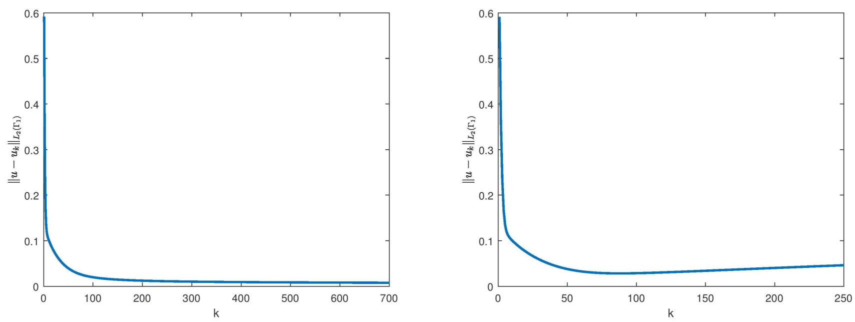
$$r_2(\theta, \varphi) = \sqrt{0.8 + 0.2\,(\cos(2\varphi) - 1)\,(\cos(4\theta) - 1)}\,.$$

We take a harmonic function $u_{ex}(x) = e^{x_2} \cos x_1 - e^{x_1} \sin x_2$ as an exact solution of the Cauchy problem (1)–(2). The necessary data for the Cauchy problem are generated from the exact solution u_{ex} on the external boundary Γ_2, as in Example 1.



a). exact solution b). approximate solution

FIG. 5. Reconstruction of the solution on the boundary Γ_1 in Ex. 2 (3% noise)



a). exact data b). 3% noisy data

FIG. 6. L_2-errors in Ex. 2

The results of the numerical reconstruction of the function u_{ex} by the generalized Landweber algorithm on the boundary Γ_1, for the cases of exact and noisy data, are shown in Figs. 4–5. Values of the relative L_2-errors at each iteration are presented in Fig. 6. In the case of exact data, after 700 iterations, we get

$$\frac{\|u_{ex} - u_{700}\|_{L_2(\Gamma_1)}}{\|u_{ex}\|_{L_2(\Gamma_1)}} = 0.0078$$

and for noisy data after 88 iterations (noise is 3%) we obtain

$$\frac{\|u_{ex} - u_{88}\|_{L_2(\Gamma_1)}}{\|u_{ex}\|_{L_2(\Gamma_1)}} = 0.0283,$$

in both cases the discretisation parameters for the direct boundary value problems are $n' = n = 10$. The relaxation parameter γ for the generalized Landweber method is selected as 0.5 (both for exact and noisy data).

5. Conclusion

We employed a generalized iterative Landweber algorithm, which can be applied to obtain a stable solution to the Cauchy problem, in particular it was used to find a stable approximation of the function values of the solution on the interior boundary surface of doubly connected three-dimensional domains. At each iteration step of the algorithm, we need to solve one Dirichlet and one Robin boundary value problem. Each of these direct boundary problems is solved by an indirect integral equations method in conjunction with a Galerkin method for the discretisation. Applicability of proposed algorithm and discretisation are highlighted by some numerical examples both for direct problems as well as for the Cauchy problem.

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