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ON A BOUNDARY INTEGRAL EQUATION METHOD FOR ELASTOSTATIC CAUCHY PROBLEMS IN ANNULAR PLANAR DOMAINS

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Резюме. Розглянуто задачу Коші реконструкції поля зсуву (переміщення) планарного кільцеподібного лінійного пружного тіла, коли відомо вектор переміщення та напружень на зовнішній границі. Шукане значення представлене у вигляді еластостатичного потенціалу простого шару по двох границях тіла, що містить дві невідомі густини. Використовуючи задані граничні умови, отримано систему інтегральних рівнянь для знаходження цих густин. Досліджено властивості системи, здійснено дискретизацію за схемою Нистьома та регуляризацію Тіхонова. Наведені чисельні результати показують, що переміщення та відповідне поле напружень на границі, де не задано початкових значень, можна достатньо точно реконструювати як для точних вхідних даних, так і для даних з похибкою.
Abstract. The Cauchy problem of reconstructing the displacement field of a planar annular linear elastic body from knowledge of the displacement vector and normal stress (traction) on the outer boundary is considered. The sought field is represented in terms of a single-layer elastic potential over the two boundary curves of the body involving two unknown densities. These densities are found by imposing the given boundary conditions, rendering a system of two boundary integrals to be solved for the densities. Properties of this system is investigated, and discretisation is done via a Nyström scheme together with Tikhonov regularization. Numerical results are included showing that the displacement can be accurately reconstructed in a stably way both for exact and noisy data together with the corresponding stress field on the boundary part where no information is initially given.

1. Introduction

Let $D \subset \mathbb{R}^2$ be an annular planar domain with sufficiently smooth boundaries Γ_1 and Γ_2. Each boundary part is a simple closed curve, and Γ_1 is contained in the bounded interior of Γ_2. The domain D is then the bounded region in-between Γ_1 and Γ_2 as illustrated in Fig. 1. We consider D to be a representative for a planar linear isotropic elastic body.

In some applications it is not possible to take measurements throughout the boundary of D. There can be a hostile environment or the body can be partly buried making only a part of the boundary accessible for measurements.

We assume that the external boundary Γ_2 is accessible for measurements but not Γ_1. Our aim is to reconstruct the missing data on Γ_1. We work in the setting of elastostatics (static elastic deformation), and, as mentioned, D is

Key words. Elastostatics, Cauchy problem, boundary integral equation method, trigonometrical quadrature method.

considered as a planar linear isotropic material. The displacement vector $u = (u_1, u_2) \in C^2(D) \cap C^1(\overline{D})$ describes the deformation of D. Under the standard assumptions of elastostatics (in particular small deformations of an isotropic and homogeneous linear elastic material) the displacement field satisfies the Navier equation

$$\mu \Delta u + (\lambda + \mu)\text{grad div} u = 0 \quad \text{in} \quad D, \tag{1}$$

with the constants μ and λ ($\mu > 0, \lambda > -\mu$) being the Lamé coefficients characterizing physical properties of the body.

We assume that the displacement and normal stress (the traction field) can be measured on Γ_2, giving respectively the Dirichlet boundary condition

$$u = f \quad \text{on} \quad \Gamma_2 \tag{2}$$

and Neumann boundary condition

$$Tu = g \quad \text{on} \quad \Gamma_2. \tag{3}$$

The vector functions f and g are given, and are commonly termed as Cauchy data. The element Tu is the stress tensor (due to molecular interactions from the deformation) in the outward unit normal direction to the boundary and is denoted as the traction. The traction can be expressed as

$$Tu = \lambda \text{div}\, u\, \nu + 2\mu(\nu \cdot \text{grad})u + \mu \text{div}(Qu)Q\nu,$$

where ν is the outward unit normal vector to the boundary, and the matrix Q is given by $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The introduction of the matrix Q makes for an easy way to express the last term in the right-hand side in the definition of Tu in the planar case, which otherwise has to be written in terms of a projection of a rotational field.

The Cauchy problem in elastostatics is then to solve (1)–(3), and in particular to find the displacement and traction on the boundary part Γ_1. Uniqueness is clear from standard results of elliptic equations such as the Holmgren theorem. However, the solution will not in general depend continuously on the data, that is the Cauchy problem is ill-posed. We tactically assume that the data are compatibly such that there exists a displacement field u.

In [3], an overview is given of a regularizing method based on a single-layer approach for the stable numerical solution to the corresponding classical Cauchy problem for the Laplace equation (for both two and three dimensional regions). The method surveyed builds on ideas given in [6] and [1]. We continue the work of [3], by extending the single-layer approach to the above Cauchy problem in elastostatics.

The Cauchy problem for elliptic equations is classical, and it is not possible in this work to give adequate overview and references. To at least guide the reader to some works, see the introduction in [2]. It is stationary heat transfer problems that make up the majority of the works on numerical methods for Cauchy problems, the corresponding results for elasticity is more limited. However, the first and third author of the work [8] have been active on inverse problems in elasticity, see for example [8, 9] and references therein (there are plenty more from these authors). However, the numerics is via the boundary element method or the method of fundamental solutions for simply connected domains. In [4] an iterative regularizing method is developed for the Cauchy problem of elastostatics in a half-plane containing a bounded inclusion.

For the outline of the work, in Section 2, we recall the fundamental solution to the Navier equation and discuss some classical integral formulations. In Section 3, the Cauchy problem is reduced to a system of boundary integral equations by representing the solution in terms of a single-layer solution over the boundary curves giving two unknown densities to determine. Furthermore, by parameterising the boundary curves, a parameterised system of integral equations is obtained. Properties of system is stated, see Theorem 1. Then, in Section 4, the parameterised system is discretised using a Nyström scheme. The discrete linear system obtained is ill-conditioned due to the ill-posedness of the Cauchy problem, hence Tikhonov regularization is invoked for its solution. In Section 5, numerical examples are presented for two different planar regions, showing that accurate and stable numerical results can be obtained both for the displacement and traction on the boundary part Γ_1. Some conclusions are given in the final section, Section 6.

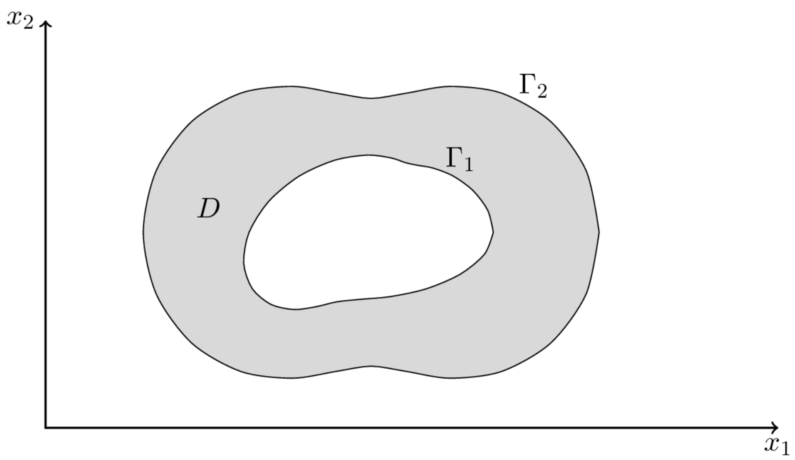


Fig. 1. Example of an annular planar domain D with boundary parts Γ_1 and Γ_2

2. Reduction to integral equations by Betti's formula

Reduction of the Cauchy problem (1)–(3) to a system of integral equations involves the use of the fundamental solution to the equation (1). In this section, we recall that fundamental solution, and for the sake of completeness, we state some direct representation formulas for the solution of (1)–(3). However, these representation formulas will not be further used, instead, in the next section, we introduce an alternative single-layer approach.

It is known [7] that the fundamental solution of the Navier equation (1) is given by

$$\Phi(x,y) = \frac{C_1}{2\pi}\Psi(x,y)I + \frac{C_2}{2\pi}J(x-y), \tag{4}$$

where

$$C_1 = \frac{\lambda+3\mu}{\mu(\lambda+2\mu)}, \quad C_2 = \frac{\lambda+\mu}{\mu(\lambda+2\mu)},$$

and

$$\Psi(x,y) = \ln\frac{1}{|x-y|}, \quad x,y\in\mathbb{R}^2, \quad x\neq y.$$

Here, I is identity matrix (of size 2×2), J is defined by the formula

$$J(\omega) = \frac{\omega\omega^\top}{|\omega|^2}, \quad \omega\in\mathbb{R}^2\setminus\{\bar{0}\}.$$

An analogue of the Green's formula for the Laplace equation is the so-called Betti's formula for the Navier equation; details and derivation of this formula can be found in for example [7]. Using Betti's formula, we seek the solution of (1)–(3) in the form

$$u(x) = \int\limits_{\Gamma_1} \left[T_y\Phi(x,y)\right]^\top \psi_1(y) - \Phi(x,y)\psi_2(y)\,ds(y) + B(x), \quad x\in D, \tag{5}$$

where

$$B(x) = \int\limits_{\Gamma_2} \Phi(x,y)g(y) - \left[T_y\Phi(x,y)\right]^\top f(y)\,ds(y).$$

The unknown vector-densities ψ_1 and ψ_2 represent the sought values (Cauchy data) on the inner inaccessible boundary Γ_1, that is

$$\psi_1(x) = u(x) \quad \text{and} \quad \psi_2(x) = Tu(x), \quad x\in\Gamma_1.$$

The representation (5) is then matched against the Cauchy data, that is against the displacement $u(x)$ respectively traction $Tu(x)$ on Γ_2. Using classical jump relations for the potentials in (5), we obtain the following system of integral equations of the second kind,

$$\begin{aligned} &\frac{1}{2}\psi_1(x) - \int\limits_{\Gamma_1}\left[T_y\Phi(x,y)\right]^\top\psi_1(y)\,ds(y) + \int\limits_{\Gamma_2}\Phi(x,y)\psi_2(y)\,ds(y) = B(x),\\ &\frac{1}{2}\psi_2(x) - T_x\int\limits_{\Gamma_1}\left[T_y\Phi(x,y)\right]^\top\psi_1(y)\,ds(y) + \\ &\qquad + \int\limits_{\Gamma_2} T_x\Phi(x,y)\psi_2(y)\,ds(y) = TB(x), \end{aligned} \tag{6}$$

where $x\in\Gamma_1$.

The described method of reducing the problem (1)–(3) to the above system of integral equations (IE) is naturally denoted the *direct* integral equation approach. We do not employ this but consider a related alternative strategy based on single-layer potentials.

3. Reduction to integral equations by potential theory

To reduce the Cauchy problem (1)–(3) to a the system of integral equations, we apply what is termed as an *indirect* integral equations approach based on potential theory.

We seek the solution of (1)–(3) as a single–layer elastic potential

$$u(x) = \int\limits_{\Gamma_1} \Phi(x,y)\varphi_1(y)\,ds(y) + \int\limits_{\Gamma_2} \Phi(x,y)\varphi_2(y)\,ds(y), \quad x \in D \tag{7}$$

with unknown vector-densities φ_1 and φ_2. We have the following result.

Proposition 1. *The single-layer potential* (7) *is the solution of the Cauchy problem* (1)–(3) *provided that the densities* φ_1 *and* φ_2 *are solutions of the following system of integral equations*

$$\begin{aligned} &\int\limits_{\Gamma_1} \Phi(x,y)\varphi_1(y)\,ds(y) + \int\limits_{\Gamma_2} \Phi(x,y)\varphi_2(y)\,ds(y) = f(x), \quad x \in \Gamma_2, \\ &\int\limits_{\Gamma_1} T_x\Phi(x,y)\varphi_1(y)\,ds(y) + \frac{1}{2}\varphi_2(x) + \\ &+ \int\limits_{\Gamma_2} T_x\Phi(x,y)\varphi_2(y)\,ds(y) = g(x), \quad x \in \Gamma_2. \end{aligned} \tag{8}$$

A proof of the proposition is obtained by matching the representation against the given Cauchy data involving classical jump relations for elastic single-layer potentials (for formulas, see [5, 7]).

There are singularities present in kernels in the above system. It is advantageous, both for theoretical and numerical investigations, to parameterise the system and make the singularities explicit. For the parameterisation, assume that the boundary curves Γ_1 and Γ_2 each have a parametric representation

$$\Gamma_i := \{x_i(t) = (x_{i1}(t), x_{i2}(t)) : t \in [0, 2\pi]\}, \quad i = 1, 2,$$

where x_{i1} and x_{i2} are both 2π–periodic and twice continuously differentiable.

Using the representation of the boundary curves, we obtain from (8) the parameterised system of integral equations,

$$\begin{cases} \frac{1}{2\pi}\int\limits_0^{2\pi} K_{21}(t,\tau)\mu_1(\tau)\,d\tau + \frac{1}{2\pi}\int\limits_0^{2\pi} K_{22}(t,\tau)\mu_2(\tau)\,d\tau = f(t), \\ \frac{1}{2\pi}\int\limits_0^{2\pi} N_{21}(t,\tau)\mu_1(\tau)\,d\tau + \frac{1}{2}\frac{\mu_2(t)}{|x_2'(t)|} + \frac{1}{2\pi}\int\limits_0^{2\pi} N_{22}(t,\tau)\mu_2(\tau)\,d\tau = g(t), \end{cases} \tag{9}$$

where

$$K_{ij}(t,\tau) = 2\pi\Phi(x_i(t), x_j(\tau)), \quad i,j = 1,2,$$

$$N_{ij}(t,\tau) = \frac{1}{|x_i'(t)|}\left\{M_{ij}^1(t,\tau) + M_{ij}^2(t,\tau)\right\}, \quad i,j = 1,2,$$

$$M_{ij}^1(t,\tau) = C_3\frac{(x_i(t) - x_j(\tau)) \cdot x_i'(t)}{|x_i(t) - x_j(\tau)|^2}Q, \quad i \neq j,$$

$$M_{ij}^2(t,\tau) = -\frac{(x_i(t)-x_j(\tau))\cdot Qx_i'(t)}{|x_i(t)-x_j(\tau)|^2}\left[C_3 I + C_4 \widetilde{J}(x_i(t),x_j(\tau))\right],$$
$$t \neq \tau \text{ when } \quad i=j,$$

$$\widetilde{J}(x_i(t),x_j(\tau)) = J(x_i(t)-x_j(\tau)), \quad t\neq\tau \text{ when } \quad i=j,$$

and

$$\widetilde{J}(x_i(t),x_i(t)) = \frac{x_i'(t)\,[x_i'(t)]^\top}{|x_i'(t)|^2}.$$

We have used the notation

$$f(t)=f(x_2(t)), \quad g(t)=g(x_2(t)), \quad \mu_i(\tau)=\varphi_i(x_i(\tau))|x_i'(\tau)|, \quad i=1,2,$$

and defined

$$C_3 = -\frac{2\mu}{\lambda+2\mu} \quad \text{and} \quad C_4 = \frac{4(\lambda+\mu)}{\lambda+2\mu}.$$

The kernels K_{22} and N_{22} (to be precise the component M_{22}^1) have singularities that can be written in an additive way using special weight functions. Put

$$K_{ii}(t,\tau) = \widetilde{K}_i(t,\tau) - \frac{C_1}{2}\ln\left\{\frac{4}{e}\sin^2\frac{t-\tau}{2}\right\}I, \quad i=1,2, \tag{10}$$

where

$$\widetilde{K}_i(t,\tau) = \begin{cases} K_{ii}(t,\tau) + \frac{C_1}{2}\ln\left\{\frac{4}{e}\sin^2\frac{t-\tau}{2}\right\}I, & t\neq\tau, \\ \frac{C_1}{2}\ln\frac{1}{e|x_i'(t)|^2}I + C_2\widetilde{J}(x_i(t),x_i(t)), & t=\tau. \end{cases}$$

Similar manipulations can be done for the kernels N_{11} and N_{22}. Denote by

$$M_{ii}^1(t,\tau) = M_i^3(t,\tau) + \frac{C_3}{2}\cot\frac{t-\tau}{2}Q, \quad i=1,2.$$

Then,

$$M_i^3(t,\tau) = \begin{cases} M_{ii}^1(t,\tau) - \frac{C_3}{2}\cot\frac{t-\tau}{2}Q, & t\neq\tau, \\ -\frac{C_3}{2}\frac{x_i'(t)\cdot x_i''(t)}{|x_i'(t)|^2}Q, & t=\tau. \end{cases}$$

As a result of these expressions, we obtain

$$N_{ii}(t,\tau) = \widetilde{N}_i(t,\tau) + \frac{C_3}{2|x_i'(t)|}\cot\frac{t-\tau}{2}Q, \quad i=1,2, \tag{11}$$

where

$$\widetilde{N}_i(t,\tau) = \begin{cases} N_{ii}(t,\tau) - \frac{C_3}{2|x_i'(t)|}\cot\frac{t-\tau}{2}Q, & t\neq\tau, \\ \frac{1}{|x_i'(t)|}\left\{M_i^3(t,t) + M_{ii}^2(t,t)\right\}, & t=\tau. \end{cases}$$

Using for example L'Hopital's rule, it is straightforward to verify that the components M_{ii}^2 are at least continuous across $t=\tau$:

$$M_{ii}^2(t,t) = -\frac{x_i'(t)\cdot Qx_i''(t)}{2|x_i'(t)|^2}\left[C_3 I + C_4\widetilde{J}(x_i(t),x_i(t))\right], \quad i=1,2.$$

Introduce the integral operators:

$$(S_{ii}\mu_i)(t) = \frac{1}{2\pi}\int_0^{2\pi}\left[\widetilde{K}_i(t,\tau) - \frac{C_1}{2}\ln\left\{\frac{4}{e}\sin^2\frac{t-\tau}{2}\right\}I\right]\mu_i(\tau)\,d\tau, \quad i = 1,2,$$

$$(S_{ij}\mu_j)(t) = \frac{1}{2\pi}\int_0^{2\pi} K_{ij}(t,\tau)\mu_j(\tau)\,d\tau, \quad i,j = 1,2, \quad i \neq j,$$

$$(L_{ii}\mu_i)(t) = \frac{1}{2\pi}\int_0^{2\pi}\left[\widetilde{N}_i(t,\tau) + \frac{C_3}{2|x_i'(t)|}\cot\frac{t-\tau}{2}Q\right]\mu_i(\tau)\,d\tau, \quad i = 1,2,$$

$$(L_{ij}\mu_j)(t) = \frac{1}{2\pi}\int_0^{2\pi} N_{ij}(t,\tau)\mu_j(\tau)\,d\tau, \quad i,j = 1,2, \quad i \neq j.$$

Taking into account the above expressions for the singularities in the kernels, the system of integral equations (9) can be written in operator form:

$$\begin{cases} (S_{21}\mu_1)(t) + (S_{22}\mu_2)(t) = f(t), \\ (L_{21}\mu_1)(t) + \left(\left(\frac{1}{2}I + L_{22}\right)\mu_2\right)(t) = g(t). \end{cases} \tag{12}$$

It can then be shown that for the operator corresponding to this system, the following holds.

Theorem 1. *The operator* $\mathbf{M} : L_2[0,2\pi] \times L_2[0,2\pi] \to L_2[0,2\pi] \times L_2[0,2\pi]$ *defined as*

$$\mathbf{M} = \begin{pmatrix} S_{21} & S_{22} \\ L_{21} & \frac{1}{2}I + L_{22} \end{pmatrix}$$

is injective and has a dense range.

This follows in the same way as for the corresponding theorem for the Laplace operator; for details in the case of the Laplace operator, see [3].

4. Full discretization and Tikhonov regularization

For the discretization of the system (12) of integral equations, we use quadratures rules that are based on trigonometric interpolation. The quadrature rules presume introducing an equidistant mesh of nodal points,

$$t_j = \frac{\pi}{n}j, \quad j = \overline{0, 2n-1}, \quad n \in \mathbb{N}. \tag{13}$$

The operator S_{22} in (12) contains a logarithmic singularity. We therefore use the quadrature

$$\frac{1}{2\pi}\int_0^{2\pi}\ln\left\{\frac{4}{e}\sin^2\frac{t-\tau}{2}\right\}f(\tau)\,d\tau \approx \sum_{j=0}^{2n-1} R_j(t)f(t_j), \tag{14}$$

where $R_j(t)$ is a weight function given by

$$R_j(t) := -\frac{1}{2n}\left\{1 + 2\sum_{k=1}^{n-1}\frac{\cos k(t-t_j)}{k} + \frac{\cos n(t-t_j)}{n}\right\}.$$

For a singularity of the kind contained in the operator L_{22} in (12), we apply instead the quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2} f(\tau), d\tau \approx \sum_{j=0}^{2n-1} \widetilde{T}_j(t) f(t_j), \tag{15}$$

with a weight function

$$\widetilde{T}_j(t) := -\frac{1}{n}\sum_{k=1}^{n-1} \sin k(t-t_j) - \frac{1}{2n}\sin n(t-t_j).$$

Since we work with 2π-periodic functions, it natural to use the trapezoidal rule

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\, d\tau \approx \frac{1}{2n}\sum_{j=0}^{2n-1} f(t_j). \tag{16}$$

Derivation of the quadrature formulas (14)–(16), and proof of their order of convergence can be found in [5].

For a partial discretization of the system of integral equations (12), we apply the quadrature formulas (14)–(16) on the equidistant nodal points (13). After then also collocating at these points, we obtain a system of linear equations

$$\begin{cases} \frac{1}{2n}\sum\limits_{j=0}^{2n-1} K_{21}(t_i,t_j)\mu_{1j} + \sum\limits_{j=0}^{2n-1}\left[\frac{1}{2n}\widetilde{K}_2(t_i,t_j) - \frac{C_1}{2}R_j(t_i)I\right]\mu_{2j} = f(t_i), \\ \frac{1}{2n}\sum\limits_{j=0}^{2n-1} N_{21}(t_i,t_j)\mu_{1j} + \\ \left\{\frac{1}{2|x_2'(t_i)|}I + \sum\limits_{j=0}^{2n-1}\left[\frac{1}{2n}\widetilde{N}_2(t_i,t_j) - \frac{C_3}{2|x_2'(t_i)|}\widetilde{T}_j(t_i)Q\right]\right\}\mu_{2j} = g(t_i), \end{cases} \tag{17}$$

where $i=\overline{0,2n-1}$, and

$$\mu_{kj} \approx \mu_k(t_j), \quad k=1,2, \quad j=\overline{0,2n-1}.$$

In a matrix-vector form, the system (17) can be written as

$$\mathbf{A}\bar{\mu} = \mathbf{F}. \tag{18}$$

As noted earlier, the problem (1)–(3) is ill-posed (there is no continuous dependence with respect to the input data). Hence, the system (12) is also ill-posed. A consequence of this is that the discrete linear system (17) is ill-conditioned, since it is obtained from (12). In order to obtain a stable numerical solution to (12), a regularizing method is needed. One such method is, for example, the classical Tikhonov regularization.

Tikhonov regularization for a linear system $Ax=b$ is based on minimizing the functional

$$\min_x \|Ax-b\|_2^2 + \alpha\|x\|_2^2,$$

where the number $\alpha > 0$ is the regularization parameter to be appropriately chosen.

The minimization problem is reduced to the approximation of x_α from the equality

$$(\alpha I + A^* A)x_\alpha = A^* b,$$

where A^* is adjoint operator of A.

In the case of a discrete system as (17), the usual transposed matrix $A^\top$ acts as an adjoint operator to the matrix A. Therefore, the regularization for (17) consists in finding $\bar{\mu}_\alpha$ from the system

$$(\alpha I + \mathbf{A}^\top \mathbf{A})\bar{\mu}_\alpha = \mathbf{A}^\top \mathbf{F}, \tag{19}$$

where the matrix $\mathbf{A}$ and vector $\mathbf{F}$ are determined in accordance with (17).

Taking into account the representation (7) of the solution to the Cauchy problem (1)–(3) and classical properties of the single-layer potential, the displacement vector u and traction Tu can be constructed on the inner boundary Γ_1 by the formulas

$$u(x) = (S_{11}\varphi_1)(x) + (S_{12}\varphi_2)(x), \quad x \in \Gamma_1$$

and

$$Tu(x) = \left(\left(-\frac{1}{2}I + L_{11}\right)\varphi_1\right)(x) + (L_{12}\varphi_2)(x), \quad x \in \Gamma_1.$$

We generate an approximation to the quantities in discrete form by the formulas

$$u(x_1(t_i)) \approx \sum_{j=0}^{2n-1}\left[\frac{1}{2n}\widetilde{K}_1(t_i,t_j) - \frac{C_1}{2}R_j(t_i)I\right]\mu_{1j} + \frac{1}{2n}\sum_{j=0}^{2n-1}K_{12}(t_i,t_j)\mu_{2j}, \quad i = \overline{0, 2n-1} \tag{20}$$

and

$$Tu(x_1(t_i)) \approx -\frac{1}{2}\frac{\mu_{1i}}{|x_1'(t_i)|} + \sum_{j=0}^{2n-1}\left[\frac{1}{2n}\widetilde{N}_1(t_i,t_j) - \frac{C_3}{2|x_1'(t_i)|}\widetilde{T}_j(t_i)Q\right]\mu_{1j} + \frac{1}{2n}\sum_{j=0}^{2n-1}N_{12}(t_i,t_j)\mu_{2j}, \quad i = \overline{0, 2n-1}, \tag{21}$$

where μ_{kj} is the solution of the regularized system (19).

5. Numerical experiments

We shall present numerical results for two different configurations.

Example 1. Consider the annular domain of Fig. 2 having boundary curves

$$\Gamma_1 = \left\{x_1(t) = (1.2\cos t, 1.6\sqrt{0.4\sin^2 t + \cos^2 t}\sin t) : t \in [0, 2\pi]\right\},$$
$$\Gamma_2 = \left\{x_2(t) = (3\cos t, 4\sqrt{0.4\sin^2 t + \cos^2 t}\sin t) : t \in [0, 2\pi]\right\}.$$

As the exact solution to compare our numerical reconstructions with, we take

$$u_{ex}(x) = \Phi_1(x, y^*), \quad x \in D,$$

where Φ_1 is the first column of the matrix constituting the fundamental solution Φ in (4), and y^* is an arbitrary point which does not belong to the domain D.

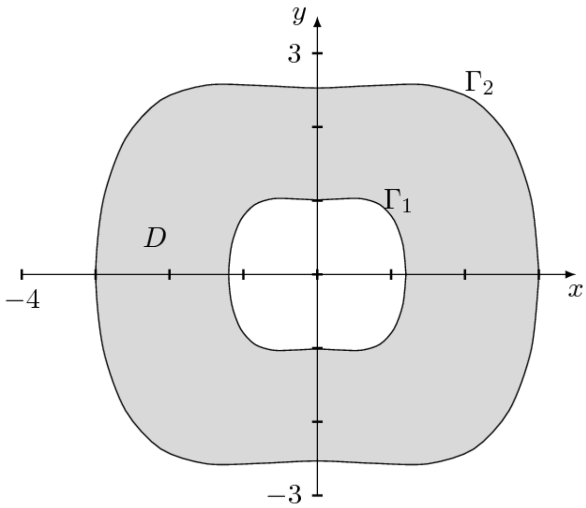


FIG. 2. Domain in Example 1

Then boundary values of the solution u_{ex} can be calculated exactly by the formulas

$$f_{ex\_i}(x) = \Phi_1(x, y^*) \quad \text{and} \quad g_{ex\_i}(x) = T\Phi_1(x, y^*), \quad x \in \Gamma_i,\ i = 1, 2.$$

(A)

α	$\delta = 0$
E-10	3.94E-4
E-11	9.37E-5
E-12	2.92E-5
E-13	**2.59E − 5**
E-14	1.49E-4
E-15	1.33E-3

(B)

α	$\delta = 0.03$	$\delta = 0.05$
E-2	3.97E-2	4.18E-2
E-3	2.81E-2	4.92E-2
E-4	**3.65E − 3**	**5.36E − 3**
E-5	7.39E-3	8.56E-3
E-6	1.02E-2	1.76E-2
E-7	3.32E-2	5.33E-2

TABL. 1. Error in the reconstructed element f_{11} compared with the exact solution, for different parameters α in the case of (A) exact and (B) noisy data with noise level δ

Let the Cauchy data (2) and (3) be generated as $f = f_{ex\_2}$ and $g = g_{ex\_2}$, respectively. Concerning parameters, we use $y^* = (0, 0)$, the Lamé coefficients are $\lambda = 2$, $\mu = 1$, and the discretization parameter $n = 32$ in (13).

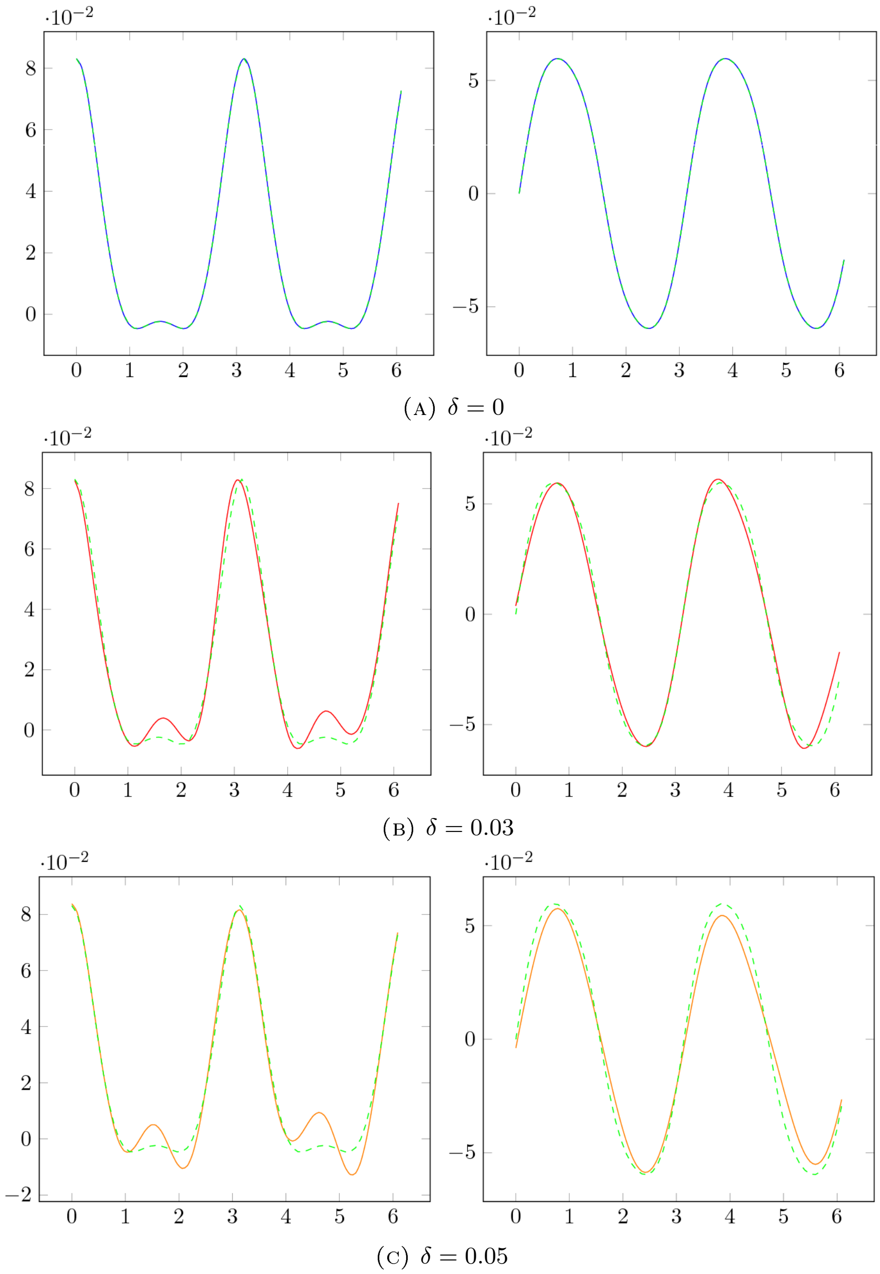
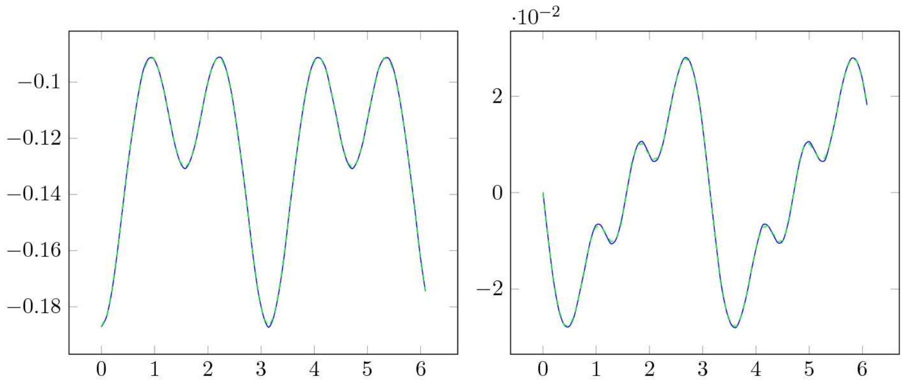
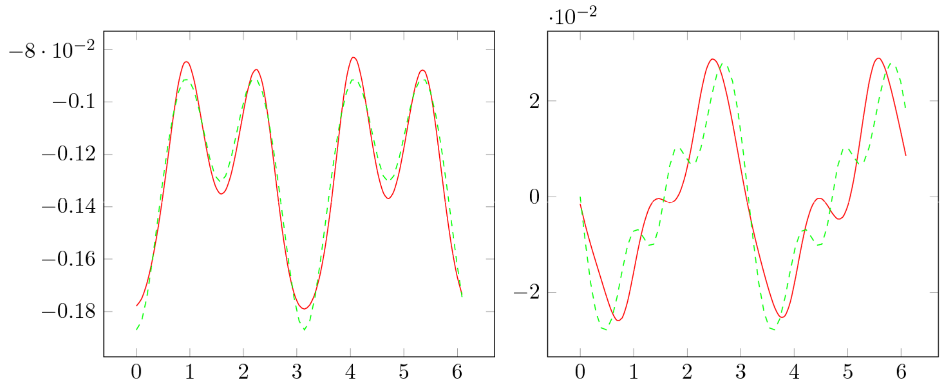


FIG. 3. Approximated (———) and exact (- - -) solutions of f_{11} (left) and f_{12} (right) for noise level δ

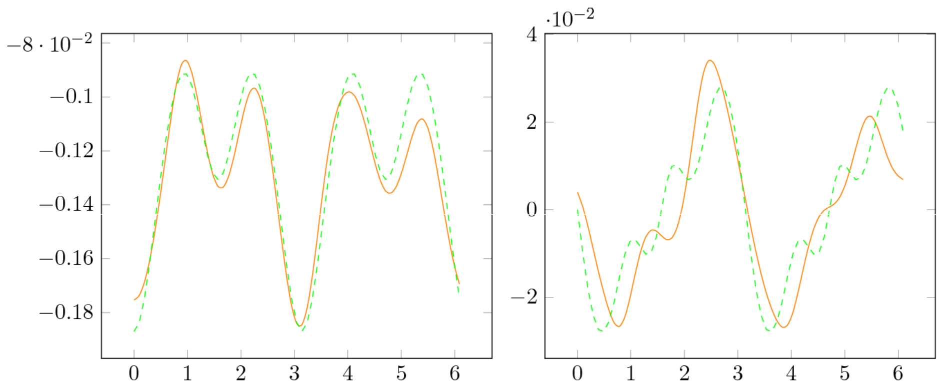
Due to the ill-posedness of the Cauchy problem, we apply Tikhonov regularization as mentioned in the previous section. The regularizing parameter α is chosen by trial and error. The optimal regularization parameter used is as given in Table 1 for exact data and for noisy data having 3% and 5% random pointwise error added into the data, respectively.



(A) $\delta = 0$



(B) $\delta = 0.03$



(C) $\delta = 0.05$

FIG. 4. Approximated (———) and exact (- - -) solutions of g_{11} (left) and g_{12} (right) for noise level δ

The number in bold is the value chosen for α.

To be more precise about noisy data, we point out that noisy data g_δ is generated from the exact value g as follows

$$g_\delta = g + \delta(2\eta - 1)\|g\|_{L_2},$$

with noise level δ and a random value $\eta \in (0,1)$.

The approximation of the displacement $f_1 = (f_{11}, f_{12})$ and traction $g_1 = (g_{11}, g_{12})$ on the inner boundary Γ_1, are calculated according to the formulas (20) and (21). The obtained results are shown in the Fig. 3 and Fig. 4.

As expected, the displacement vector is more accurately reconstructed than the traction. However, it is pleasing to see that also with noisy data, the reconstructions of the traction components follow the exact values. When more noise is added, the accuracy decreases but in a stable manner meaning that the results still resembles the exact values.

To convince the reader that the results presented are not optimised but are of the form to be expected for other configurations and data, we present results for a different domain and set of Cauchy data.

Example 2. In this example, we consider the doubly connected planar domain shown in Fig. 5. The boundary curves have parametric representation given by:

$$\Gamma_1 = \left\{x_1(t) = (0.7\cos t, 0.72\sin t + 0.6\cos^2 t) : \, t \in [0, 2\pi]\right\},$$
$$\Gamma_2 = \left\{x_2(t) = (1.8\cos t, 1.68\sin t + 1.4\cos^2 t) : \, t \in [0, 2\pi]\right\}.$$

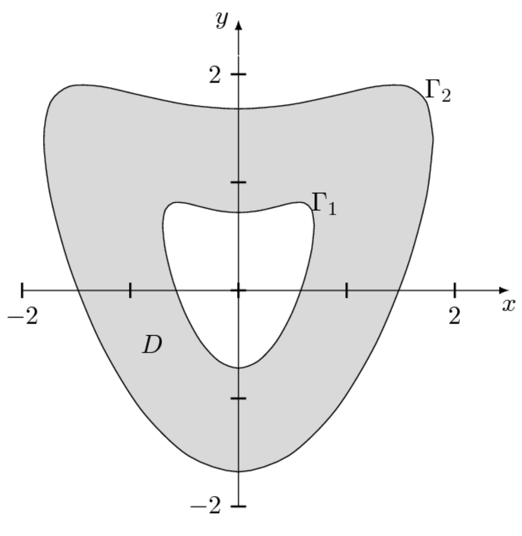


FIG. 5. Domain in Example 2

To have some data to compare against, we generate the Cauchy data artificially. This means that we first solve a Dirichlet boundary value problem, with values on the boundary curves as

$$f_i(x) = \begin{pmatrix} x_1 + x_2 \\ 5x_1 - x_2 \end{pmatrix}, \quad x = (x_1, x_2) \in \Gamma_i, \quad i = 1, 2.$$

Let the Lamé parameters be $\lambda = 2$, $\mu = 2$, and the discretization parameter is set to $n = 32$ in (13).

(A)

α	$\delta = 0$
E-7	6.47E-5
E-8	1.42E-5
E-9	4.27E-6
E-10	**1.22E − 6**
E-11	2.61E-6
E-12	2.59E-5

(B)

α	$\delta = 0.03$	$\delta = 0.05$
E-1	1.52E-1	2.11E-1
E-2	2.45E-1	2.97E-1
E-3	3.78E-2	5.66E-2
E-4	**2.55E − 2**	**3.13E − 2**
E-5	8.57E-2	8.01E-2
E-6	1.71E-1	2.08E-1

TABL. 2. Error in the reconstructed element f_{12} compared with the exact solution, for different parameters α in the case of (a) exact and (b) noisy data with noise level δ

Let the solution of the above Dirichlet problem be given as a single-layer elastic potential (7). After performing the similar manipulations that have been described for the Cauchy problem (that is parameterisation of the obtained system, making singularities explicit and then discretize), we obtain a system of linear equations

$$\sum_{j=0}^{2n-1}\left[\frac{1}{2n}\widetilde{K}_m(t_i,t_j)-\frac{C_1}{2}R_j(t_i)I\right]\mu_{mj}+\frac{1}{2n}\sum_{j=0}^{2n-1}K_{ml}(t_i,t_j)\mu_{lj}=f_m(x_m(t_i)),$$
$$i=\overline{0,2n-1},\quad m=1,2,\quad l=3-m.$$

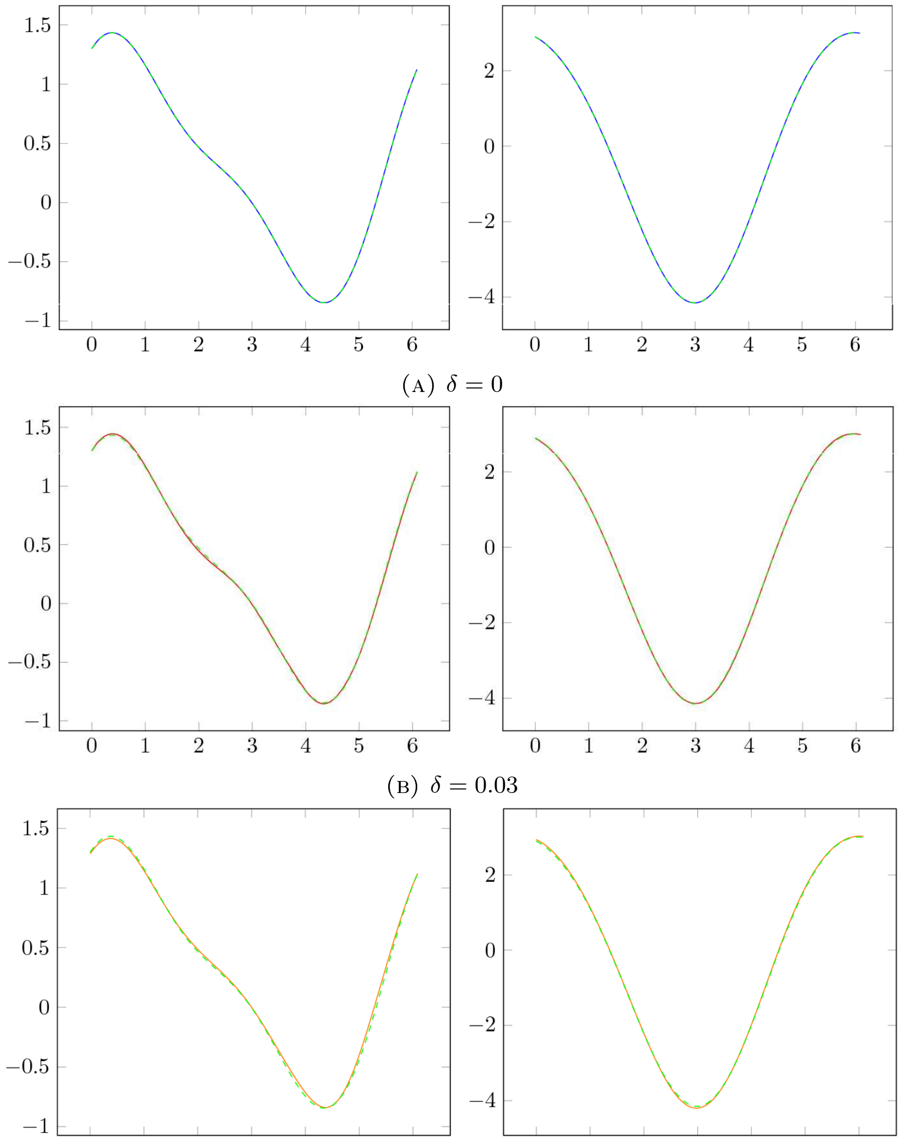
Solving for μ_{mj}, we can then calculate the Neumann boundary values by the formula

$$\begin{aligned}&g_m(x_m(t_i))\approx\\&\approx(-1)^m\frac{1}{2}\frac{\mu_{mi}}{|x'_m(t_i)|}+\sum_{j=0}^{2n-1}\left[\frac{1}{2n}\widetilde{N}_m(t_i,t_j)-\frac{C_3}{2|x'_m(t_i)|}\widetilde{T}_j(t_i)Q\right]\mu_{mj}+\\&+\frac{1}{2n}\sum_{j=0}^{2n-1}N_{ml}(t_i,t_j)\mu_{lj},\quad i=\overline{0,2n-1},\quad m=1,2,\quad l=3-m.\end{aligned}\tag{22}$$

The Cauchy data in (2) and (3) is then generated as $f=f_2$ and $g=g_2$.

As in the previous example, we have to choose a regularization parameter α. The values used are given in bold in Table 2.

The numerical approximation of the Cauchy data on the inner boundary Γ_1 is found via the formulas (20) and (21). The results obtained are shown in



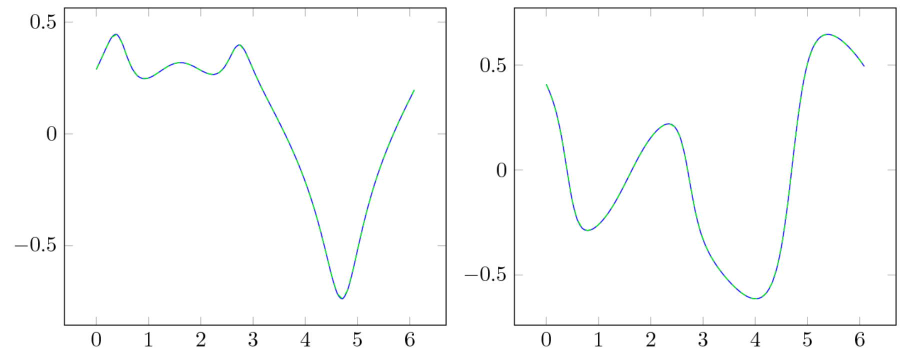
(A) $\delta = 0$

(B) $\delta = 0.03$

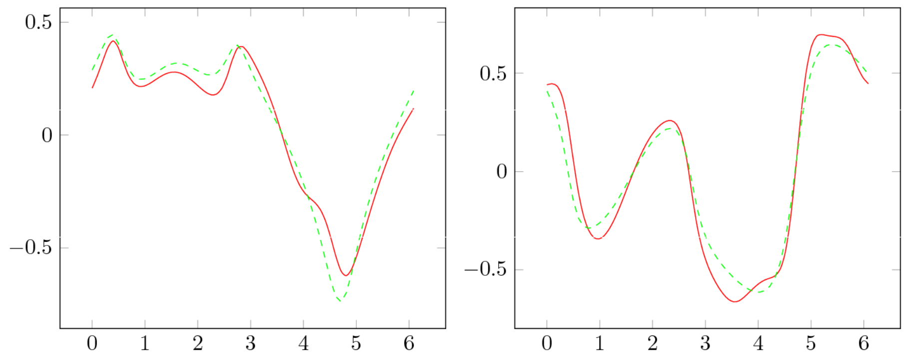
(C) $\delta = 0.05$

FIG. 6. Approximated (——) and exact (- - -) solutions of f_{11} (left) and f_{12} (right) for noise level δ

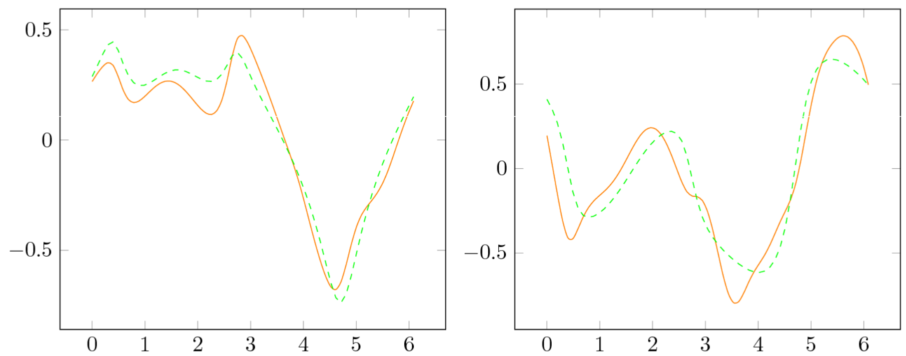
Fig. 6 and Fig. 7. It should be noted that in this example what is denoted as the exact Neumann data in the Cauchy problem is in fact an approximation since it is generated via solving the Dirichlet problem as explained above. But since the direct Dirichlet problem is well-posed and the discretization parameter is sufficiently large ($n = 32$), a high-order accuracy of the data generated by (22) is expected.



(A) $\delta = 0$



(B) $\delta = 0.03$



(C) $\delta = 0.05$

FIG. 7. Generated (- - -) and approximated (———) solutions of g_{11} (left) and g_{12} (right) for noise level δ

The obtained results are similar to those found in the previous example. The traction vector is also here reconstructed with less accuracy than the placement as expected but follows the exact solution.

6. Conclusion

A regularizing method based on the elastic single-layer potential was derived for the Cauchy problem in elastostatics. The Cauchy data in the form of the displacement and traction is given on the outer boundary curve of a planar annular and linear isotropic body. From the single-layer representation, a system of boundary integrals to be solve for two unknown densities were obtained by matching against the data. It was shown that the system has at most one solution, and that there exists a solution for a dense set of square integrable data. Discretisation was done via a Nyström scheme in conjunction with Tikhonov regularization. Special care was taken to handle the various singularities in the kernels. The suggested approach performs well as verified by two numerical examples. The reconstructions corroborated well both for the displacement vector and traction with the sought solutions, also in the case of noisy data. The traction vector is naturally found with less accuracy. Overall, the outlined approach is a lightweight and flexible method for elastostatic Cauchy problems, and generalizes naturally earlier work [3] on a single-layer approach for the Cauchy problem for the Laplace equation.

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