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HETEROGENEOUS MODEL OF THE PROCESS OF THERMAL CONDUCTIVITY IN A MULTILAYERED MEDIUM WITH THIN LAYERS

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Резюме. Ми розглядаємо початково крайову задачу теплопровідності в багатошаровому середовищі з малими товщинами шарів. Побудовано комп'ютерну модель, що дозволяє враховувати малі товщини шарів та уникати труднощів, які пов'язані з чисельною реалізацією задачі. Доведено теорему про неперервність та еліптичність білінійних форм варіаційних рівнянь. Для чисельного дослідження розв'язку використано напіваналітичний метод скінченних елементів.

Abstract. We consider initially the boundary value problem of thermal conductivity in a multilayered medium with small layer thicknesses. A computer model has been constructed, which allows to take into account the small thicknesses of the layers and avoid the difficulties associated with the numerical implementation of the problem. The theorem on the continuity and ellipticity of bilinear forms of variational equations is proved. The semi-analytic finite elements method used for numerical investigation of the solution.

1. Introduction

Modern materials and constructions, that are used in an instrument making, often have a difficult, heterogeneous structure. Natural environments physical processes areinvestigated in that, too in swingeing majority is heterogenous. It is known that at the mathematical design of problems in such environments there are two going neartaking into account of them difficult structure. The first approach envisages the use of process of homogenization, and second – in development of multiscale strategy. At development of the second approach, that allows more exactly to take into account the features of structure of environment, often there are the difficulties, constrained with the use of numeral methods(in particular, at application of Finite Elements Method) in areas that contain the thin including. In such cases build various-scale mathematical models to development of that the devoted works of many authors, in particular [1], [3], [4], [5] [7].

In this work we numerically construct a heterogeneous mathematical model of the process of heat and mass transfer in multilayer environments, where the thicknesses of layers are much smaller than other characteristic sizes.

Key words. Heat equation, heterogeneous model, finite elements method.

2. Formulation of problem

Let's consider the problem of heat conductivity for a multilayered medium of complex shape, which occupies an area

$$V = \bigcup V_k, k = 1, n; V_i \bigcap V_j = \oslash, i \neq j$$

with different thermal characteristics of the material of each layer. Boundary V_k of each regions consists of the lateral surface S_k and front surfaces S_k^- and S_k^+ and is considered Lipschitz (Fig.1).

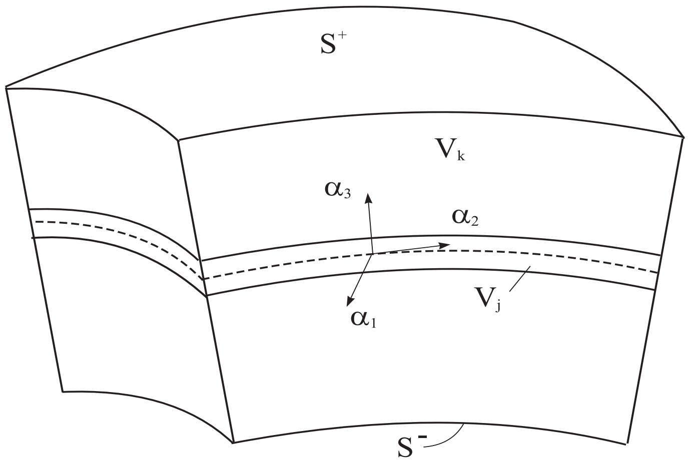

Fig. 1. The domain with thin layer

We denote J_2 the set of indexes of the regions V_k corresponding to "thin" layers whose thickness is small in comparison with other characteristic sizes. We will denote J_3 the set of indices of other areas. We associate each of the regions with some curvilinear coordinate systems related to the median surface of the area. The Lame coefficients of these regions are given by the relations

$$H_{1j} = A_{1j}(1 + k_{1j}\alpha_3^j), H_{2j} = A_{2j}(1 + k_{2j}\alpha_3^j), H_{3j} = 1.$$

Here A_1^j, A_2^j are the Lame coefficients of median surface, k_1^j, k_2^j are coefficient of curvature of the median surface. Let's consider the process of heat conduction in the described region, assuming that on the outer boundary there is a heat exchange according to Newton's law, and on the interfaces there is an ideal contact [2].

3. Transformation the three-dimensional heat transfer problem to two-dimensional in a thin layer

Consider a thin layer, where thickness is small compared with other characteristic of its size, occupying the area V_j. Let us itroduce the curvilinear coordinate system $(\alpha_1^j, \alpha_2^j, \alpha_3^j)$ associated with the median surface Ω_j of the region with the boundary Γ_j. The coordinate lines of this surface are the lines of major curvature. This

$$V_j = \{\alpha_1^j, \alpha_2^j, \alpha_3^j : \ (\alpha_1^j, \alpha_2^j) \in \Omega_j, -\frac{h_j}{2} \le \alpha_3^j \le \frac{h_j}{2}\},$$

where Ω_j is a two-dimensional region with a Lipschitz boundary on the median surface of the layer, h_j is the thickness of the layer. We will assume that on the facial surfaces $\alpha_3^j = \frac{h_j}{2}$ and $\alpha_3^j = -\frac{h_j}{2}$ the heat fluxes q_n^+ and q_n^- are given respectively, and on the lateral surface there is a heat exchange according to Newton's law

$$-\lambda_j \frac{\partial T_j}{\partial n}|_S = (T_j - T_c), \tag{1}$$

where λ_j is the coefficient of thermal conductivity, n is the external normal to the surface, T_j is the temperature function of the layer, T_c is the ambient temperature. At the initial moment of time, the temperature distribution is given by the ratio

$$T_j(\alpha_1^j, \alpha_2^j, \alpha_3^j, 0) = T_0^j(\alpha_1^j, \alpha_2^j, \alpha_3^j). \tag{2}$$

The process of thermal conductivity in the orthogonal coordinate system associated with the median surface of the layer can be described with the following equation:

$$c_j \rho_j \frac{\partial T_j}{\partial \tau} = \sum_{l=1}^{2} \frac{1}{H_{1j} H_{2j}} (\frac{\partial}{\partial \alpha_l^j} \lambda_j \frac{H_{1j} H_{2j}}{H_{lj}} \frac{\partial T_j}{\partial \alpha_l}) + q_{vj}, \tag{3}$$

where c_j is a coefficient of specific heat capacity, ρ_j is a coefficient of density, q_{vj} is the density of internal heat sources, τ is the time parameter. Considering that the thickness of the layer is small, we assume that the distribution of the desired function of temperature over the thickness of the layer is according to the linear law. In accordance with this assumption, we will supply the temperature in the region in the form

$$T_j(\alpha_1^j, \alpha_2^j, \alpha_3^j, \tau) = t_1(\alpha_1^j, \alpha_2^j, \tau) + \frac{2\alpha_3^j}{h_j} t_2(\alpha_1^j, \alpha_2^j, \tau). \tag{4}$$

We substitute (4) into (3) and orthogonalize the non-relation of the Bubnov-Galerkin equation to functions $v_1(\alpha_1^j, \alpha_2^j)$ and $\alpha_3^j v_2(\alpha_1^j, \alpha_2^j)$, wheare $v_1(\alpha_1^j, \alpha_2^j)$, $v_2(\alpha_1^j, \alpha_2^j) \in W_2^1(\Omega_j)$.

We select and calculate the integral over the variable α_3^j in the interval $[-\frac{h_j}{2}, -\frac{h_j}{2}]$. At the same time, let's take into account that the element of the volume and we use development in the Macrolena series of quantities $1/A_1^j(1 + k_1^j)\alpha_3^j$, $1/A_2^j(1 + k_2^j)\alpha_3^j$. Having neglected the magnitude $O((h_j k_i^j)^2)$, $i = 1, 2$ and taking into account the fact that $v_1(\alpha_1^j, \alpha_2^j), v_2(\alpha_1^j, \alpha_2^j)$ are arbitrary functions, we obtain the following key equations with the respect to the unknown functions t_1^j, t_2^j :

$$c_j\rho_j h_j \frac{\partial t_1^j}{\partial \tau} + c_j\rho_j \frac{h_j^2}{6}(k_1^j + k_2^j)\frac{\partial t_2^j}{\partial \tau} = \sum_{i=1}^{2}(\frac{h_j}{A_1^j A_2^j}\frac{\partial}{\partial \alpha_i^j}(\lambda_j \frac{A_{3-i}^j}{A_i^j}\frac{\partial t_1^j}{\partial \alpha_i^j})+$$
$$+\frac{h_j^2}{6A_1^j A_2^j}\frac{\partial}{\partial \alpha_i^j}(\lambda_j \frac{A_{3-i}^j}{A_i^j}(k_{3-i}^j - k_i^j)\frac{\partial t_2^j}{\partial \alpha_i^j})) + (1 + k_1^j \frac{h_j}{2})(1 + k_2^j \frac{h_j}{2})q_n^+ + \tag{5}$$
$$+(1 - k_1^j \frac{h_j}{2})(1 - k_2^j \frac{h_j}{2})q_n^- - q_1 = 0,$$

$$c_j\rho_j \frac{h_j^2}{6}(k_1^j + k_2^j)\frac{\partial t_1^j}{\partial \tau} + c_j\rho_j \frac{h_j}{3}\frac{\partial t_2^j}{\partial \tau} =$$
$$= \sum_{i=}^{2}(\frac{h_j^2}{6A_1^j A_2^j}\frac{\partial}{\partial \alpha_i^j}(\lambda_j \frac{A_{3-i}^j}{A_i^j}(k_{3-i}^j - k_i^j)\frac{\partial t_1^j}{\partial \alpha_i^j})+$$
$$+\frac{h_j}{3A_1^j A_2^j}\frac{\partial}{\partial \alpha_i^j}(\lambda_j \frac{A_{3-i}^j}{A_i^j}\frac{\partial t_2^j}{\partial \alpha_i^j})) + (1 + k_1^j \frac{h_j}{2})(1 + k_2^j \frac{h_j}{2})q_n^+ + \tag{6}$$
$$+(1 - k_1^j \frac{h_j}{2})(1 - k_2^j \frac{h_j}{2})q_n^- + \frac{4\lambda_j}{h_j} - q_2 = 0.$$

We use the following notation

$$q_1 = \int_{-\frac{h_j}{2}}^{\frac{h_j}{2}} q_v(1 + k_1^j \alpha_3^j)(1 + k_2^j \alpha_3^j)d\alpha_3^j,$$

$$q_2 = \frac{2}{h_j}\int_{-\frac{h_j}{2}}^{\frac{h_j}{2}} q_v(1 + k_1^j \alpha_3^j)(1 + k_2^j \alpha_3^j)\alpha_3^j d\alpha_3^j,$$

$$-\lambda_j \frac{\partial T_j}{\partial \alpha_3^j} = q_n^+, \text{ for } \alpha_3^j = \frac{h_j}{2}.$$

By performing similar transformations to the boundary condition on the lateral cylindrical surface, we obtain boundary conditions for functions t_1^j, t_2^j in the form

$$-(\frac{\lambda_j h_j}{A_i^j}\frac{\partial t_1^j}{\partial \alpha_i^j} + \frac{1}{6}\frac{\lambda_j h_j^2}{A_i^j}(k_{3-i}^j - k_i^j)\frac{\partial t_2^j}{\partial \alpha_i^j})n_i = \alpha(h_j t_1^j + \frac{h_j^2}{6}k_\Gamma^j t_2^j - t_1^c),$$
$$-(\frac{1}{6}\frac{\lambda h_j^2}{A_i^j}(k_{3-i}^j - k_i^j)\frac{\partial t_1^j}{\partial \alpha_i^j} + \frac{h_j}{3}\frac{\lambda_j}{A_i^j}\frac{\partial t_2^j}{\partial \alpha_i^j})n_i = \alpha(\frac{h_j^2}{6}k_\Gamma^j t_1^j + \frac{h_j}{3}t_2^j - t_2^c), \tag{7}$$

$$t_1^c = \int_{-\frac{h_j}{2}}^{\frac{h_j}{2}} T_c(1 + k_\Gamma^j \alpha_3^j)d\alpha_3^j,$$

$$t_2^c = \frac{2}{h_j}\int_{-\frac{h_j}{2}}^{\frac{h_j}{2}} T_c(1 + k_\Gamma^j \alpha_3^j)\alpha_3^j d\alpha_3^j,$$

$k_\Gamma^j = k_1^j n_1^2 + k_2^j n_2^2$, (n_1, n_2) are the coordinates of the unit normal vector to Γ.

Here the element of the surface area

$$dS = A_\Gamma^j(1 + k_\Gamma^j \alpha_3^j)d\Gamma,$$

where $A_\Gamma^j = A_1^j n_1^2 + A_2^j n_2^2$.

In the same way from equation (6)–(7) we also obtain the initial conditions

$$\begin{aligned} &h_j t_1^j(\alpha_1^j, \alpha_2^j, 0) + \frac{h_j^2}{6}(k_1^j + k_2^j) t_2^j(\alpha_1^j, \alpha_2^j, 0) = t_1^0, \\ &\frac{h_j^2}{6}(k_1^j + k_2^j) t_1^j(\alpha_1^j, \alpha_2^j, 0) + \frac{h_j}{3} t_2(\alpha_1^j, \alpha_2^j, 0) = t_2^0, \end{aligned} \tag{8}$$

where

$$t_1^0 = \int_{-\frac{h_j}{2}}^{\frac{h_j}{2}} T_0^j(1 + k_1^j \alpha_3^j)(1 + k_2^j \alpha_3^j) d\alpha_3^j,$$

$$t_2^0 = \frac{2}{h_j}\int_{-\frac{h_j}{2}}^{\frac{h_j}{2}} T_0^j(1 + k_1^j \alpha_3^j)(1 + k_2^j \alpha_3^j)\alpha_3^j d\alpha_3^j.$$

Thus, for a thin layer it is possible to reduce the dimensionality of the problem to two-dimensional relative curvilinear coordinates on the median surface. As a result, we obtain a mathematical model of the process of thermal conductivity in a thin layer, consisting of equations (5), (6), boundary conditions (7) and initial conditions (8).

4. Description of the process of thermal conductivity in a multilayer area

Considering the results obtained in the previous section, the heterogeneous mathematical model of the heat conduction process in a multilayered medium, can be presented as the following system of differential equations of different measurements in spatial coordinate

$$c_j \rho_j \frac{\partial T_j}{\partial \tau} = \sum_{l=1}^{3} \frac{1}{H_{1j} H_{2j}} \left(\frac{\partial}{\partial \alpha_l^j} \lambda_j \frac{H_{1j} H_{2j}}{H_{lj}} \frac{\partial T_j}{\partial \alpha_l}\right) + q_{vj}, \; j \in V_j, \tag{9}$$

$$\begin{aligned} &c_j \rho_j h_j \frac{\partial t_1^{(j)}}{\partial \tau} + c_j \rho_j \frac{h_j^2}{6}(k_1^{(j)} + k_2^{(j)}) \frac{\partial t_2^{(j)}}{\partial \tau} = \\ &+ \sum_{i=1}^{2} \Big(\frac{h_j}{A_1^{(j)} A_2^{(j)}} \frac{\partial}{\partial \alpha_i^{(j)}} (\lambda_j \frac{A_{3-i}^{(j)}}{A_i^{(j)}} \frac{\partial t_1^{(j)}}{\partial \alpha_i^{(j)}}) + \\ &+ \frac{h_j^2}{6 A_1^{(j)} A_2^{(j)}} \frac{\partial}{\partial \alpha_i^{(j)}} (\lambda_j \frac{A_{3-i}^{(j)}}{A_i^{(j)}} (k_{3-i}^{(j)} - k_i^{(j)}) \frac{\partial t_2^{(j)}}{\partial \alpha_i^{(j)}}) + \\ &+ (1 + k_1^{(j)} \frac{h_j}{2})(1 + k_2^{(j)} \frac{h_j}{2}) q_n^{(j)+} + \\ &+ (1 - k_1^{(j)} \frac{h_j}{2})(1 - k_2^{(j)} \frac{h_j}{2}) q_n^{(j)-} - q_1 = 0, \end{aligned} \tag{10}$$

$$
\begin{gathered}
c_j\rho_j\frac{h_j^2}{6}(k_1^{(j)}+k_2^{(j)})\frac{\partial t_1^{(j)}}{\partial\tau}+c_j\rho_j\frac{h_j}{3}\frac{\partial t_2^{(j)}}{\partial\tau}= \\
=\sum_{i=1}^{2}(\frac{h_j^2}{6A_1^{(j)}A_2^{(j)}}\frac{\partial}{\partial\alpha_i^{(j)}}(\lambda_j\frac{A_{3-i}^{(j)}}{A_i^{(j)}}(k_{3-i}^{(j)}-k_i^{(j)})\frac{\partial t_1^{(j)}}{\partial\alpha_i^{(j)}})+ \\
+\frac{h_j}{3A_1^{(j)}A_2^{(j)}}\frac{\partial}{\partial\alpha_i^{(j)}}(\lambda_j\frac{A_{3-i}^{(j)}}{A_i^{(j)}}\frac{\partial t_2^{(j)}}{\partial\alpha_i^{(j)}})+ \\
+(1+k_1^{(j)}\frac{h_j}{2})(1+k_2^{(j)}\frac{h_j}{2})q_n^{(j)+}+ \\
+(1-k_1^{(j)}\frac{h_j}{2})(1-k_2^{(j)}\frac{h_j}{2})q_n^{(j)-}+\frac{4\lambda}{h_j}-q_2=0.
\end{gathered}
\tag{11}
$$

On the boundary with the external environment, the desired functions must satisfy the relation

$$
(-\lambda_j\frac{\partial T_j}{\partial n}-a(T_j-T_{c_k}))|_{S_j}=0, \tag{12}
$$

$$
\begin{gathered}
-\sum_{i=1}^{2}(\frac{\lambda_j h_j}{A_i^{(j)}}\frac{\partial t_1^(j)}{\partial\alpha_i^{(j)}}+\frac{1}{6}\frac{\lambda_j h_j^2}{A_i^{(j)}}(k_{3-i}^{(j)}-k_i^{(j)})\frac{\partial t_2^{(j)}}{\partial\alpha_i^(j)})n_i= \\
=a(h_j t_1^{(j)}+\frac{h_j^{(2)}}{6}k_\Gamma^{(j)}t_2^{(j)}-t_1^c),
\end{gathered}
\tag{13}
$$

$$
\begin{gathered}
-\sum_{i=1}^{2}(\frac{1}{6}\frac{\lambda_j h_j^2}{A_i^{(j)}}(k_{3-i}^{(j)}-k_i^{(j)})\frac{\partial t_1^(j)}{\partial\alpha_i^{(j)}}+\frac{1}{3}\frac{\lambda_j h_j}{A_i^{(j)}}\frac{\partial t_2^{(j)}}{\partial\alpha_i^(j)})n_i= \\
=a(\frac{1}{6}h_j^2k_\Gamma^{(j)}t_1^{(j)}+\frac{h_j^{(2)}}{3}t_2^{(j)}-t_2^c)
\end{gathered}
\tag{14}
$$

and initial conditions

$$
T_j(\alpha_1^j,\alpha_2^j,\alpha_3^j,0)=T_0^j(\alpha_1^j,\alpha_2^j,\alpha_3^j),\ \text{for } j\in J_3, \tag{15}
$$

$$
h_jt_1^{(j)}(\alpha_1,\alpha 2,0)+\frac{h_j^2}{6}(k_1^{(j)}+k_2^{(j)})t_2^{(j)}(\alpha_1,\alpha 2,0)=t_1^0,\ \text{for } j\in J_2, \tag{16}
$$

$$
\frac{h_j^2}{6}(k_1^{(j)}+k_2^{(j)})t_1^{(j)}(\alpha_1,\alpha_2,0)+\frac{h_j}{3}(t_2^{(j)}(\alpha_1,\alpha_2,0)=t_2^0,\ \text{for } j\in J_2. \tag{17}
$$

For a complete description of the mathematical model, it is necessary to introduce the conjugation conditions, which describe the equality of the functions of temperature distribution and heat fluxes on the boundary of the collisions of the regions:

$$
T_{1_{j_1}}|_{S_j}=T_{1_{j_2}}, \tag{18}
$$

$$
\lambda_{j_1}\frac{\partial T_{j_1}}{\partial n}|_{S_j}=\lambda_{j_2}\frac{\partial T_{j_2}}{\partial n}. \tag{19}
$$

If the contact layer is thin, it should be taken into account that the temperature on the upper face surface is given by the ratio

$$
T_{j_1}=t_1^{(j_2)}+t_2^{(j_2)}
$$

and on the bottom, respectively

$$T_{j_1} = t_1^{(j_2)} - t_2^{(j_2)}.$$

Taking into account all the described relations, we obtained a closed system of differential equations of the second order for the determination of unknown unknown temperature functions.

5. Variational formulation

According to the Bubnov-Galerkin method, we construct the variational equations of the problem (9)–(19). For the transformation of the integrals we use the following formula, derived from the Green's formula:

$$-\int_V div(\lambda \nabla u) v dV = \int_V \lambda \nabla u \nabla v dV - \int_S \lambda \frac{\partial u}{\partial n} v dS. \tag{20}$$

Consider the variational problem of thermal conductivity in a multilayered medium with subtle inclusions, that is to find functions u, which satisfy the equation

$$\begin{gathered}\sum_{k\in J_3} A_k(T_k, u^k) + \sum_{j\in J_2} a_j(t_j, u^j) + \sum_{j\in J_2} m^j(t'_j, u^j) = \\ = \sum_{k\in J_3} (f_k, u^k) + \sum_{j\in J_2} (f^j, u^j),\end{gathered} \tag{21}$$

$$M^k(T_k, u^k) = \int_{V_k} c^k \rho^k T_k u^k dv,$$

$$\begin{gathered}A^k(T_k, u^k) = \int_{V_k} \lambda^k \operatorname{grad} u \operatorname{grad} T_k dv - \int_{S_k} \lambda^k \frac{\partial T_k}{\partial v} u^k dS + \\ + g_{11} \int_{S^+} a T_{ck} u^{ck} ds + g_{12} \int_{S^-} a T_{ck} u^{ck} ds,\end{gathered}$$

$$\begin{gathered}a^j(t^j, u^j) = \int_{\Omega_j} t_p^{jT} A u^j d\Omega + \int_{\Gamma_j} t^{jT} G u^j d\Gamma + \\ + g_{21} \int_{\Omega_{j+}} (1 + k_1 \frac{h}{2})(1 + k_2 \frac{h}{2})(t_1^{j+} + t_2^{j+})(u_1^{j+} + u_2^{j+}) d\Omega + \\ + g_{22} \int_{\Omega_{j-}} (1 - k_1 \frac{h}{2})(1 - k_2 \frac{h}{2})(t_1^{j-} - t_2^{j-})(u_1^{j-} - u_2^{j-}) d\Omega,\end{gathered}$$

$$m^j(t^j, u^j) = \int_{\Omega_j} t^{jT'} \overline{M} u_j A_1^j A_2^j d\alpha_1 d\alpha_2, \tag{22}$$

$$t^T = (t_1, t_2),\ u^T = (u_1, u_2),\ t'^T = (\frac{\partial t_1}{\partial \tau}, \frac{\partial t_2}{\partial \tau}),$$

$$t_p^T = (\frac{\partial t_1}{\partial \alpha_1}, \frac{\partial t_1}{\partial \alpha_2}, \frac{\partial t_2}{\partial \alpha_1}, \frac{\partial t_2}{\partial \alpha_2})^T,\ u_p^T = (\frac{\partial u_1}{\partial \alpha_1}, \frac{\partial u_1}{\partial \alpha_2}, \frac{\partial u_2}{\partial \alpha_1}, \frac{\partial u_2}{\partial \alpha_2})^T,$$

$$\overline{A} = \begin{pmatrix} \lambda^j h^j & 0 & \frac{\lambda^j (h^j)^2 (k_2^j - k_1^j)}{6} & 0 \\ 0 & \lambda^j h^j & 0 & \frac{\lambda^j (h^j)^2 (k_2^j - k_1^j)}{6} \\ \frac{\lambda^j (h^j)^2 (k_2^j - k_1^j)}{6} & 0 & \lambda^j \frac{h^j}{3} & 0 \\ 0 & \frac{\lambda^j (h^j)^2 (k_2^j - k_1^j)}{6} & 0 & \lambda^j \frac{h^j}{3} \end{pmatrix},$$

$$\overline{M} = \begin{pmatrix} c^j \rho^j h^j & \frac{1}{6} c^j \rho^j (h^j)^2 (k_1^j + k_2^j) \\ \frac{1}{6} c^j \rho^j (h^j)^2 (k_1^j + k_2^j) & \frac{1}{3} c^j \rho^j h^j \end{pmatrix},$$

$$G = \begin{pmatrix} a h^j & a \frac{(h^j)^2}{6} k_\Gamma \\ a \frac{(h^j)^2}{6} k_\Gamma & a \frac{h^j}{3} \end{pmatrix},$$

$$g_{1_i} = 0, \quad g_{2_i} = 1,$$

$$g_{1_i} = 0, \quad g_{2_i} = 1,$$

if the layer containing the outer surface,

$$g_{1_i} = 1, g_{2_i} = 0$$

otherwise, conjugation condition

$$T_{k_1} = T_{k_2}, \tag{23}$$

the initial condition,

$$\sum_{k \in J_3} M^{(k)}(T_k - T_k^0, u^{(k)}) + \sum_{j \in J_2} m^{(j)}(t_j - t_0^j, u^{(j)}) = 0, \ \tau = 0 \tag{24}$$

for arbitrary functions $u^k \in U_k(\Omega_k), u_1^j, u_2^j \in U_j(\Omega_j)$, where, those that implement the main junction conditions. Let us prove the following lemma.

Lemma 1. *The bilinear forms associated with the operator of the problem* (20)–(23) *are symmetric under the homogeneous boundary condition of the third kind.*

Proof. Let us prove that for bilinear forms $a^j(u,v), m^j(u,v)$ the following statements hold true:

1) the domain of the operator of the problem is a dense set.

2) $a^{(j)}(u,v) = a^{(j)}(v,u); m^{(j)}(u,v) = m^{(j)}(v,u)$.

3)The first statement is executed because $C_0^\infty(V) \subset D$.

Obviously, the implementation of the second equality for bilinear forms $a(u,v), m(u,v)$ provided by the symmetry of the matrices $\overline{A}$ and $\overline{M}$ The lemma is proved. The following theorem holds true.

Theorem 1. *Let the condition holds true:*

$$h_j |k_i^j| \leq \sqrt{3}, \ j \in J_2, \tag{25}$$

Then the bilinear forms of the problem (20)-(23) *are continuous and elliptic, assuming uniform third-order boundary condition.*

Proof. First we prove, that the theorem holds true in the case $j = 1,\ j \in J_2$; $k = 1,\ k \in J_3$. Then the indices can be neglected.

The proof of the ellipticity and continuity of bilinear forms $A(u, v), M(u, v)$ is is described in [6]. In order to show that the bilinear forms $a(u, v), m(u, v)$ have the property of ellipticity, one must prove that the inequalities are true

$$m(u, u) \geq c_1\|u\|, a(u, u) \geq c_2\|u\|, \text{ where } c_1 > 0, c_2 > 0.$$

We show that the matrices $\overline{A}$ and$\overline{M}$ are positively defined. Let's find eigen values of matrices $\overline{M}$, which are the roots of the algebraic equation. They are

$$\eta_{1,2} = \frac{c_j \rho_j h_j}{3}(2 \pm \sqrt{1 + \frac{h_j^2(k_1^j + k_2^j)^2}{4}})$$

From the fact that $c(\alpha_1^j, \alpha_2^j, 0), c(\alpha_1^j, \alpha_2^j, 0)$-positive functions

$$h_j|k_i^j| \leq \sqrt{3}, i = 1, 2, \tag{26}$$

that the eigen values η_1, η_2 are positive. Then for a quadratic form $m(u, u)$ the following estimation is valid

$$\int_\Omega u^T \overline{A} u d\Omega \geq$$
$$\geq (\frac{\overline{c\rho h}}{3}(2 - \sqrt{1 + \frac{h^2 \min_\Omega (k_1 + k_2)^2}{4}}))^2 \int_\Omega (u_1^2 + u_2^2) d\Omega = \gamma_1^2 \|u\|_{L_2(\Omega)}, \tag{27}$$

where

$$\bar{c} = \min_{(\alpha_1, \alpha_2) \in \Omega}(c), \bar{\rho} = \min_{(\alpha_1, \alpha_2) \in \Omega}(\rho),$$
$$\gamma_1^2 = \frac{\overline{c\rho h}}{3}(2 - \sqrt{1 + \frac{h^2 \max_\Omega (k_1 + k_2)^2}{4}}))^2.$$

Thus, $m(u, v)$ is elliptic in space $L_2(\Omega)$ To prove continuity we use the Cauchy-Bunyakovskii inequality.

$$|m(t, u)| = |\int_\Omega t^T \overline{M} u d\Omega| = |\int_\Omega c\rho h t_1 u_1 d\Omega + \int_\Omega \frac{1}{6} c\rho(k_1 + k_2) h^2 t_2 u_1 d\Omega +$$
$$+ \int_\Omega \frac{1}{6} c\rho(k_1 + k_2) h^2 t_1 u_2 d\Omega + \int_\Omega \frac{1}{3} c\rho h t_2 u_2 d\Omega| \leq$$
$$\leq \overline{c\rho} h \|t_1\|_V \|u_1\|_V + \frac{1}{6} \overline{c\rho} h^2 |\overline{k_1} + \overline{k_2}| \|t_2\|_V \|u_1\|_V +$$
$$+ \frac{1}{6} \overline{c\rho} h^2 |\overline{k_1} + \overline{k_2}| \|t_1\|_V \|u_2\|_V + \frac{1}{3} \overline{c\rho} h \|t_2\|_V \|u_2\|_V \leq$$
$$\leq C_1^2 (\|t_1\|_V \|u_1\|_V + \|t_1\|_V \|u_2\|_V + \|t_2\|_V \|u_1\|_V + \|t_2\|_V \|u_2\|_V) =$$
$$= C_1^2 (\|t_1\|_V + \|t_2\|_V)(\|u_1\|_V + \|u_2\|_V) = C_1^2 \|t\| \|u\|,$$
$$\bar{c} = \max_\Omega |c|, \bar{\rho} = \max_\Omega |\rho|, C_1^2 = \overline{c\rho} \max\{h, \frac{h(\overline{k_1} + \overline{k_2})}{6}\},$$
$$\overline{k_1} = \max_\Omega |k_1|, \overline{k_2} = \max_\Omega |k_2|,$$

$$\|u\| = \|u\|_{W_2^1(\Omega)}.$$

To prove the ellipticity of the bilinear form $a(u,v)$, we use the inequality (25). We first find the eigenvalues of a matrix of bilinear form

$$Q(\xi,\eta) = \lambda h\xi^2 + \frac{1}{3}h^2(k_{3-i} - k_i)\xi\eta + \frac{\lambda h}{3}\eta^2, \tag{28}$$

solving the equation for this $u^{(k)} \in U_k, u_1^{(j)}, u_2^{(j)} \in \overline{U}_j$ We obtain the following eigen values

$$y_{1,2} = \frac{2}{3}\lambda h \pm \frac{1}{3}\lambda h\sqrt{(1 + \frac{h^2(k_{3-i} - k_i)^2}{4})}. \tag{29}$$

Referencing 25,the eigen values are positive. So,

$$Q(\xi,\eta) \geq \lambda h(\frac{2}{3} - \frac{1}{3}\sqrt{(1 + \frac{h^2(k_{3-i} - k_i)^2}{4})})(\xi^2 + \eta^2). \tag{30}$$

Similarly, after finding the eigenvalues of a matrix of bilinear form

$$N(\xi,\eta) = ah\xi^2 + ah^2\frac{k_\Gamma}{3}\xi\eta + a\frac{h}{3}\eta^2 \tag{31}$$

we obtain the inequality

$$N(\xi,\eta) \geq a(\frac{2}{3} - \frac{1}{3}\sqrt{1 + \frac{h^2k_\Gamma^2}{4}})(\xi^2 + \eta^2). \tag{32}$$

Taking into account (25), (30), (32) and the Friedrichs inequality, we obtain

$$a(u,u) \geq \mu_1^2\int_\Omega((\frac{\partial u_1}{\partial\alpha_i})^2 +$$

$$+(\frac{\partial u_2}{\partial\alpha_i})^2)d\Omega + \mu_2^2\int_\Omega(u_1^2 + u_2^2)d\Omega + \mu_3^2\int_\Gamma(u_1^2 + u_2^2)d\Gamma \geq$$

$$\geq \gamma_2^2\Big(\int_\Omega((\frac{\partial u_1}{\partial\alpha_i})^2 + (\frac{\partial u_2}{\partial\alpha_i})^2)d\Omega + \int_\Omega(u_1^2 + u_2^2)d\Omega\Big) = \gamma_2^2\|u\|^2,$$

$$\mu_1^2 = \overline{\lambda}h\Big(\frac{2}{3} - \frac{1}{3}\sqrt{1 + \frac{h^2\max_\Omega(k_{3-i} - k_i)^2}{4}}\Big),$$

$$\mu_2^2 = \max\{g_{21}, g_{22}\}\,\frac{(2-\sqrt{3})^2}{4},$$

$$\mu_3^2 = \overline{a}h\Big(\frac{2}{3} - \frac{1}{3}\sqrt{1 + \frac{h^2\max_\Omega(k_\Gamma^2)}{4}}\Big),$$

$$\overline{\lambda} = \min_{(\alpha_1,\alpha_2)\in\Omega}(\lambda),\ \overline{a} = \min_{(\alpha_1,\alpha_2)\in\Gamma}(a)$$

$$\gamma_2^2 = \begin{cases}\min\{\mu_1^2, \mu_2^2\},\ \mu_2 \neq 0,\\ \min\{\frac{1}{2}\mu_1^2, \frac{1}{2}\mu_1^2\mu_4^2\},\ \mu_2 = 0\end{cases}$$

and μ_4^2 is a constant obtained from Friedrichs's inequality.

The continuity of the bilinear form follows from the following inequality

$$|a(t,u)| \leq |\int_\Omega^T \overline{A}ud\Omega| + |\int_\Gamma t^TGud\Gamma| \leq \sigma_1|\int_\Omega\frac{\partial t_1}{\partial\alpha_i}\frac{\partial u_1}{\partial\alpha_i}d\Omega| +$$

$$+\sigma_{2i}|\int_\Omega \frac{\partial t_2}{\partial \alpha_i}\frac{\partial u_1}{\partial \alpha_i}d\Omega| + \sigma_{2i}|\int_\Omega \frac{\partial t_1}{\partial \alpha_i}\frac{\partial u_2}{\partial \alpha_i}d\Omega|+$$
$$+\sigma_3|\int_\Omega \frac{\partial t_2}{\partial \alpha_i}\frac{\partial u_2}{\partial \alpha_i}d\Omega| + \sigma_4|\int_\Omega t_2u_2d\Omega|$$
$$+\sigma_5|\int_\Omega t_1u_1d\Omega| + \sigma_6|\int_\Omega t_2u_1d\Omega| + \sigma_6|\int_\Omega t_1u_2d\Omega| + \sigma_5|\int_\Omega t_2u_2d\Omega|+$$
$$+\sigma_7|\int_\Gamma t_1u_1d\Gamma| + \sigma_8|\int_\Gamma t_2u_1d\Gamma| + \sigma_8|\int_\Gamma t_1u_2d\Gamma| + \sigma_7|\int_\Gamma t_2u_2d\Gamma| \le$$
$$\le C_2(\|t_1\|_V\|u_1\|_V + \|t_1\|_V\|u_2\|_V + \|t_2\|_V\|u_1\|_V + \|t_2\|_V\|u_2\|_V) \le C_2\|t\|\|u\|,$$

where

$$\sigma_1 = \max_\Omega|\frac{\lambda h}{A_i^2}|, \quad \sigma_{2i} = \frac{1}{6}\max_\Omega|\frac{\lambda h^2}{A_i^2}(k_{3-i}-k_i)|,$$
$$\sigma_3 = \frac{1}{3}\max_\Omega|\frac{\lambda h}{A_i^2}|, \quad \sigma_4 = \frac{1}{3}\max_\Omega|\frac{4\lambda}{h}|,$$
$$\sigma_5 = \max_\Omega(1+k_1k_2\frac{h^2}{4}), \sigma_6 = \frac{h}{2}\max_\Omega(k_1+k_2), \sigma_7 = \max_\Gamma ah, \sigma_8 = \max_\Gamma|\frac{ah^2k_\Gamma^2\lambda}{6}|.$$

We have proved the properties of the bilinear forms for any thin and ordinary layer. However, these propositions hold true for the operator of a problem for a multilayered medium, since they are executed for a single term, which derives that they will be executed for the sum in the formula. Note that the inequality (25) can be considered as a criterion for the thin layer. The theorem is proved.

6. Approximation of the solution

To solve the beforementioned variational problem, we discretize the solution in spatial variables. In this case, for sampling functions $T_j(\alpha_1^j, \alpha_2^j, \alpha_3^j, \tau)$ we apply the approximations of the semi-analytic finite elements method and for functions $t_1^{(j)}(\alpha_1^j, \alpha_2^j, \tau), t_2^{(j)}(\alpha_1^j, \alpha_2^j, \tau)$ are the approximations of the finite elements method. According to these methods, we choose the approximation spaces $\{V_h\}$ from space V so that

$$dimV_h \longrightarrow \infty, h \longrightarrow 0,$$
$$\bigcup V_h\text{-tightly enclosed in V.}$$

We will present the unknown functions in the

$$T_j(\alpha_1^j, \alpha_2^j, \alpha_3^j, \tau) = \sum_{k=1}^{M}\sum_{i=1}^{N} T_{ki}^j(\tau)\widetilde{\psi}_k(\alpha_3^j)\widetilde{\phi}_i(\alpha_1^j, \alpha_2^j), \tag{33}$$

$$t_1^{jh}(\alpha_1^j, \alpha_2^j, \tau) = \sum_{i=1}^{N} t_{1i}^j(\tau)\widetilde{\phi}_i(\alpha_1^j, \alpha_2^j), \tag{34}$$

$$t_2^{jh}(\alpha_1^j, \alpha_2^j, \tau) = \sum_{i=1}^{N} t_{2i}^j(\tau)\widetilde{\phi}_i(\alpha_1^j, \alpha_2^j), \tag{35}$$

where $\widetilde{\psi_k}, \widetilde{\phi}_i(\alpha_1^j, \alpha_2^j)$ are basic functions, $T_{ki}^j, t_{1i}^j, t_{2i}^j$—— are unknown coefficients. To approximate the desired solution for the third spatial coordinate, we use the expansion of the desired function in a series of functions-"bubbles". These functions on the interval [-1,1] are given by the relations

$$\widetilde{\psi_1}(\xi) = \frac{1+\xi}{2},\ \widetilde{\psi_2}(\xi) = \frac{1-\xi}{2},\ \widetilde{\psi_i}(\xi) = \Phi_{i-1}(\xi), i = 3, 4...;$$

$$\Phi_{i-1}(\xi) = \sqrt{\frac{2i-1}{2}} \int_{-1}^{\xi} P_{i-1}(t)dt. \qquad (36)$$

Here $P_i(t)$ are known Legendre polynomials. It is convenient to use the recurrence formula for calculations

$$\Phi_j(\xi) = \frac{1}{\sqrt{2}(2j-1)}(P_j(\xi) - P_{j-2}(\xi)). \qquad (37)$$

The property of the orthogonality of the Legendre polynomial follows an important property of internal forms

$$\psi_i(-1) = \psi_i(-1) = 0,\ i = 3, 4, ... \qquad (38)$$

External forms allow to calculate solutions at the borders. It is essentially used for convenient and easy implementation of junction conditions with other areas. In addition, this system of functions has favorable properties in terms of numerical stability. In order to approximate the time-domain solution, we propose to use the well-known Crank-Nicholson difference scheme [6].

7. Numerical example

Based on the constructed heterogeneous mathematical model and the proposed numerical approximations, a program complex was created in the language C# that implements this approach. A series of computational experiments was conducted using it.

Let us consider the problem of stationary heat conductivity in an axisymmetric infinite hollow cylinder with a thin outer coating. The problem is to find the distribution of the function of temperature, if it is known that on the outer and inner parts of the cylinder surface there is a heat exchange according to Newton's law with different values of the temperature of the medium. Coefficients of thermal conductivity of the coating are $\lambda_1 = const$, massive part – $\lambda_2 = const$. For the analysis of stationary heat conductivity in a cylinder, a stationary analogue of the proposed mathematical model is used. Since boundary conditions and geometry do not depend on spatial coordinates the solution of the problem will depend only on one coordinate. This allows us to get rid of the dependence on the relationship between the parameters of the finite-element grid along two coordinate axes, to carry out only the P-adaptive refinement in the radial direction and to investigate its influence on the resulting solution. The mathematical model of the described problem is a system of ordinary differential equations. To analyze its numerical solution, we first find the analytic solution of the classical mathematical model without taking into account the small thickness of the layer.

$$\varepsilon = \frac{\max_V |T_i - T_a n|}{\max_V |T_a n|} 100. \qquad (39)$$

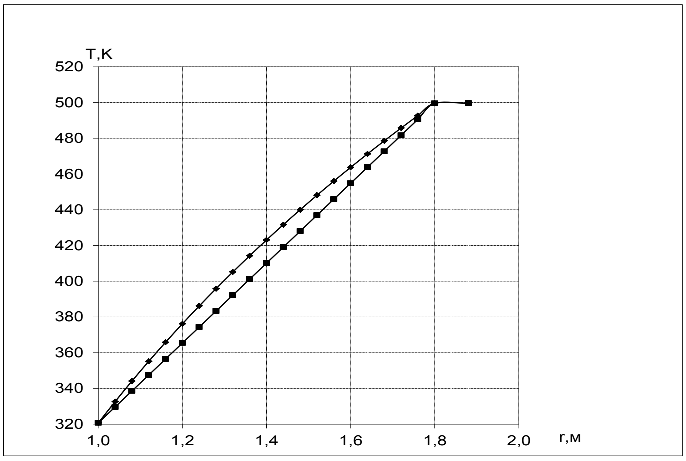

FIG. 2. Graphs of the function of temperature distribution using different numbers of members in expansion in thickness. (The curve with diamonds – analitical solution T_{an}, the curve with squares – numerical solution T_i for $p = 2$)

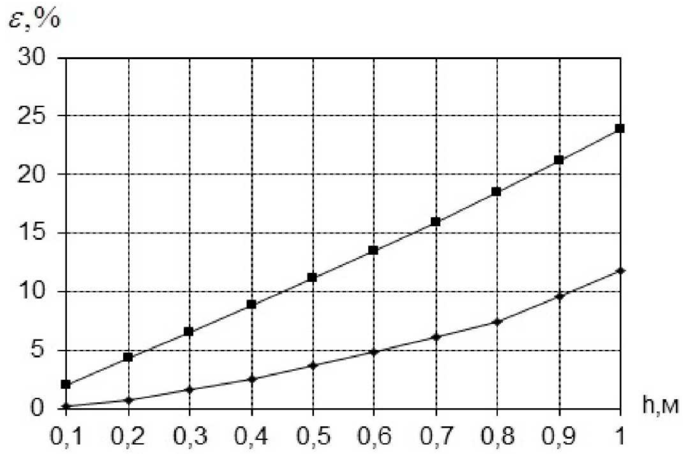

FIG. 3. Charts of the absolute error dependence on the thickness of the thin cylindrical layer (1. $\lambda = 385\frac{Dg}{Kms}$, 2. $\lambda = 3.85\frac{Dg}{Kms}$)

This solution is used to compare the results calculated using the algorithm proposed in the work. In the computational experiment, the effect of the content of a different number of members in the sum (33) was investigated to approximate the solution in thickness. Experiment results are shown in Fig.2.

The "Analytical solution" curve of this figure corresponds to the analytical solution, and the curve "Numerical solution" shows results, obtained with the

TABL. 1. The dependence of the relative error on the content of a different number of basic functions over the thickness of the layer

Number of polynomials	Relative error
2	3,1931
3	0,3321
4	0,0333
5	0,0034
6	0,0040
7	0,0001

algorithm using 2 members of the expansion for the thickness of the lower layer of the cylinder. In a numerical experiment, the solution of the model was also studied in the case of preserving 3, ..., 7 members of the decomposition. The graphs of the obtained solutions in the current scale almost coincide. As it should be expected, with increasing order of approximation, the graph of the numerical solution goes to the analytic solution, which confirms the theoretical conclusion about the convergence of the proposed algorithm. To confirm this, as a criterion for the analysis of approximate solutions (Fig. 3), the relative error rate is used

Here T_{an} is the analytical solution of the problem, T_i is the numerical solution. Table 1 it shows its decline, depending on the increase in the members of the schedule.

8. Conclusion

The sugested heterogeneous model allows to effectively analyze the process of thermal conductivity in multilayer environments, since it avoids the difficulties associated with the application of numerical methods.

Bibliography

1. Dejneka V. S. Models and methods for solving problems with conjugation conditions / I. V. Sergienko, V. S. Dejneka, V. S. Skopetsij. – V. M. Glushkov Institute of Cybernetics of NAS of Ukraine. – Kyiv: Naukova Dumka. – 1998. – 614 p.
2. Savula Ya. The investigation of the variational problem of heat-transfer in multilayered environment with thin inclusions / Ya. Savula, L. Diakoniuk // Visnyk of the Lviv University. — Series Applied Mathematics and Computer Science. – Lviv. -–2001. – Iss. 3. – P. 125-130. (in Ukrainian).
3. Savula Ya. Numerical analysis of advection-diffusion in the continuum with thin canal. / Ya. Savula, V. Koukharskyi, Ye. Chaplia // Numerical Heat Transfer. – Part A: Applications: An International Journal of Computation and Methodology. – 1998. – Vol. 33, Iss. 3. – P. 341-351.
4. Gavrysh V. Nonlineary boundary-value problem of heat conduction for a layered plate with inclusion / V. Gavrysh // Material science. – 2015. – Vol. 51, № 3. – P. 331-339.
5. Pidstrygach Ya. Selected Works // Ya. Pidstrygach. – Kiev: Naukova Dumka, 1995. – 450 p.

6. Savula Ya. Numerical analysis of the problems of mathematical physics by variational methods / Ya. Savula. – Lviv: Publishing Center of I.Franko National University of Lviv, 2004. – 221 p.
7. Quarteroni R. Multifields modelling in numerical simulation of partial differential equations / R. Quarteroni // Multifields modelling in numerical simulation of partial differential equations. – GAMM-Mitteilungen. – 1996. – Vol. 1. – P. 45-63.

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