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ON THE APPLICATION OF THE ONE *HP*-ADAPTIVE FINITE ELEMENT STRATEGY FOR NONSYMMETRIC CONVECTION-DIFFUSION-REACTION PROBLEMS

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Резюме. Ми розглядаємо застосування однієї *hp*-адаптивної стратегії методу скінченних елементів до розв'язування несиметричних крайових задач конвекції-дифузії-реакції. В основі розглядуваної стратегії лежить процедура вибору на кожному скінчнному елементі між збільшенням його порядку чи поділом, що базується на порівнянні норм наближень до похибки для розглядуваних способів перебудови скінченного елемента. Ми розглядаємо алгоритм адаптування та наводимо обґрунтування ідеї алгоритму у випадку симетричної крайової задачі. Застосовність алгоритму до несиметричних задач ми аналізуємо шляхом розгляду результатів числових експериментів, а також доповнюємо наведені результати теоретичним аналізом можливості зведення вихідної варіаційної задачі до симетричної форми. Ми наводимо дві процедури, що дають змогу перейти від несиметричної задачі до еквівалентної симетричної, або до послідовності симетричних задач, послідовність розв'язків яких збігається до розв'язку вихідної несиметричної задачі. Отриманий результат врешті може бути використаний для побудови комбінованих алгоритмів на основі однієї із схем симетризаціїі та алгоритму *hp*-адаптування.

Abstract. We consider application of certain *hp*-adaptive strategy for finite element method for solving nonsymmetric convection-diffusion-reaction boundary value problems. In the base of described strategy lies refinement selection procedure which is used to choose on each finite element between degree increase or bisection. It uses special comparative criteria for norms of approximation to local errors on different refinement patterns. We present the adaptation algorithm itself and proof of idea behind it for symmetric problems. For the case when problem is nonsymmetric we provide corresponding analysis of numerical experiments and also we add pure theoretical analysis of the possibility of bringing given variational problem to symmetric form, taking into account that the algorithm is naturally applicable in the latter case. We describe two approaches that can provide transition from nonsymmetric variational problem to directly equivalent symmetric problem in the first approach or to sequence of symmetric problems, solutions of which forms sequence of functions that is convergent to the solution of initial nonsymmetric problem in the second approach. Obtained result can be used to build algorithms, based on a combination of one of the described symmetrization methods with *hp*-adaptive scheme.

Key words. Convection-diffusion-reaction problem, finite element method, a posteriori error estimator, adaptive strategy, *hp*-adaptivity, nonsymmetric problem.

1. Introduction

Space mesh adaptivity today is the major technique which is used to optimize the process of finding the approximate solution by finite element method in various free and commercial engineering simulation tools. Using it also is crucial, since in most cases the nature of considered boundary problem is characterized by highly nonuniform distribution of local errors in the case of uniform mesh. In the context of modeling of convection-diffusion-reaction phenomena, the reason of such error distribution lies in relatively large values of Péclet and Strouhal numbers for the given problem.

Special and natural attention is on so-called hp-adaptive methods [2, 4, 5, 8–10], since they provide most wide approximation capabilities by using both space mesh adaptivity (h-) and element polynomial degree adaptivity (p-). Despite that there are reasonable facts to believe that such algorithms (hp-) can be considered "exotic" in some sense, investigation in that field is still important, since it is proved [8] that there is possibility to obtain exponentially convergent sequence of approximations by using hp-refined meshes.

In this paper we study the possibility of application of hp-adaptive strategy, introduced in [5], to nonsymmetric variational problems. The fact is that the nature of introduced algorithm can be explained only for problems with self-adjoint operators. Despite this, in practice, it can be seen, that algorithm still can be used for nonsymmetric problems which is shown in provided numerical example. The goal is of this example is to demonstrate that algorithm can provide solid results, regardless of the used a posteriori error estimators or adaptation criteria. The second part of this work is the pure theoretical investigation of the possible methods of symmetrization of nonsymmetric variational problems.

The paper structure is the following: in section 2 we define model problem; in section 3 we construct variational formulation; in section 4 we present hp-adaptation algorithm and discuss the main idea behind it; in section 5 we extend algorithm with some specific error estimator; in section 6 we review adaptation criteria which we will use in numerical experiment; in section 7 we provide numerical results for direct application of described algorithm and in section 8 we study two methods of symmetrization of variational problem. Final conclusions are given in section 9.

2. Model boundary value problem

Let us consider the following boundary value problem:

Find function $u = u(x)$ such that

$$\begin{cases} -\left(\mu u'\right)' + \beta u' + \sigma u = f \text{ in } \Omega = (0, L) \\ \left(\mu u'\right)\big|_{x=0} = \alpha[u(0) - \bar{u}_0], \ -(\mu u')\big|_{x=L} = \gamma[u(L) - \bar{u}_L], \end{cases} \tag{1}$$

where

$$\alpha, \gamma \geq 0, \ \mu = \mu(x) \geq \mu_0 > 0, \ \beta(0) \leq 0, \ \beta(L) \geq 0, \ \sigma = \sigma(x) \geq 0,$$
$$\sigma(x) - \beta'(x)/2 \geq \sigma_0 > 0 \text{ almost everywhere in } (0, L), \tag{2}$$

$$\mu, \beta, \sigma \in L^\infty(0,L), \;\; f \in L^2(0,L).$$

Considered problem is used in analysis of ecologic phenomena, semiconductors, biology etc. Many real problems of such kind are singularly perturbed [3]. In the terms of differential equation parameters it means that coefficients near highest order derivatives are relatively small in comparison to others. So in this case a second order equation is almost degenerated to first order one. In combination with standard boundary conditions it causes existence of layers near domain's boundary with high solution gradient. Those boundary layers are making the solving of problem by using well-known uniform-mesh-based FEM quite difficult. Such conditions leads to large Péclet and Strouhal criteria and to nonuniform local error distribution.

3. Variational formulation

Using standard approach [1], we can simply define variational problem corresponding to (1): find solution $u \in V$, such that

$$a(u,v) = \langle l, v \rangle \quad \forall v \in V, \tag{3}$$

where

$$\begin{aligned} a(u,v) &:= \int_0^L [\mu u'v' + \beta u'v + \sigma uv]\,dx + \alpha u(0)v(0) + \gamma u(L)v(L), \\ \langle l, v \rangle &:= \int_0^L fv\,dx + \alpha \bar{u}_0 v(0) + \gamma \bar{u}_L v(L), \;\; \forall u, v \in V := H^1(0,L). \end{aligned} \tag{4}$$

Under conditions (2) problem data satisfies (for details see [6]) conditions of Lax-Milgram theorem [1] and therefore this variational problem is well-posed.

For further needs, let us define energy norm $\|v\|_E = \sqrt{a(v,v)}$.

To discretize obtained variational problem we use general finite element method with high-order polynomial basis functions. In other words, we define some space $V_h \subset V, \dim V_h < +\infty$, of piecewise-polynomial functions and find finite element approximation $u_h \in V_h$ as a solution of variational equation:

$$a(u_h, v_h) = \langle l, v_h \rangle \quad \forall v_h \in V_h. \tag{5}$$

Now if we construct finite basis $\{\varphi_i\}_{i=1}^n$ of space V_h then by expanding $u_h = \sum_{i=1}^n q_i \varphi_i$, where $q_i \in \mathbb{R}, i = \overline{1,n}$ we can clearly see, that (5) is equal to the following system of algebraic linear equations for $q_i, i = \overline{1,n}$:

$$\sum_{i=1}^n q_i a(\varphi_i, \varphi_j) = \langle l, \varphi_j \rangle \quad j = \overline{1,n}. \tag{6}$$

For general reference see [2, 9, 10].

4. *hp*-ADAPTATION ALGORITHM

In this section we briefly present discussion and review of algorithm from [5].

Let us consider finite element mesh $\tau_h = \{K = (x_{k-1}, x_k)\}_{k=1}^n$ where $0 = x_0 < x_1 < \cdots < x_n = L$. Let us define global error approximation space in the form:

$$E_h = \bigoplus_{K \in \tau_h} E_h^K, \tag{7}$$

where space of functions $E_h^K = \{v \in V \,|\, \operatorname{supp} v \subset K\}$ and $\dim E_h^K < +\infty$. Let us define the following variational problem for error approximation:

$$\begin{cases} \text{find } e_h \in E_h \text{ such that} \\ a(e_h, v_h) = \displaystyle\int_\Omega R[u_h] v_h dx \quad \forall v_h \in E_h, \end{cases} \tag{8}$$

where R is the residual:

$$R[u_h] := f - \left(\mu u_h{}'\right)' - \beta u_h{}' - \sigma u_h. \tag{9}$$

It is not hard to see that problem (8) can be decomposed per elements. For each element we have to solve a problem:

$$\begin{cases} \text{find } e_h^K \in E_h^K \text{ such that} \\ a(e_h^K, v_h^K) = \displaystyle\int_K R[u_h] v_h^K dx \quad \forall v_h^K \in E_h^K \end{cases} \tag{10}$$

and then $e_h = \sum_{K \in \tau_h} e_h^K$.

Consider now the case $\beta \equiv 0$, i.e. the problem has symmetric bilinear form. Then the following well-known equality holds:

$$\|u - u_h\|_E^2 = \|u\|_E^2 - \|u_h\|_E^2. \tag{11}$$

Since error estimation problem has the same bilinear form as the original, then for finite element error approximation e_h the equality above also holds:

$$\|e - e_h\|_E^2 = \|e\|_E^2 - \|e_h\|_E^2. \tag{12}$$

From this equality we see that if energy norm of error approximation increases than also increases accuracy of this approximation. Denote the finite element solution on the current mesh as $u_h \in V_h$ and corresponding error $e = u - u_h$. Then (12) we can rewrite as

$$\|u - (u_h + e_h)\|_E^2 = \|u - u_h\|_E^2 - \|e_h\|_E^2. \tag{13}$$

Let us find finite element solution $\tilde{u}_h$ in space $\tilde{V}_h = V_h + E_h \subset V$, where E_h is the error approximation space, defined in (7). For symmetric case we have well-known optimality inequality:

$$\|u - \tilde{u}_h\|_E \leq \|u - \tilde{v}_h\|_E, \ \forall \tilde{v}_h \in \tilde{V}_h. \tag{14}$$

Using now (13), and the fact that $u_h + e_h \in \tilde{V}_h$ we have:

$$\|u - \tilde{u}_h\|_E^2 \leq \|u - u_h\|_E^2 - \|e_h\|_E^2. \tag{15}$$

Decomposing the second term in the right part we obtain inequality:

$$\|u - \tilde{u}_h\|_E^2 \le \|u - u_h\|_E^2 - \sum_{K \in \tau_h} \left\|e_h^K\right\|_E^2 . \tag{16}$$

Consider now decomposition of approximation space V_h into local approximation spaces V_h^K, $K \in \tau_h$. Spaces $V_h^K + E_h^K$ are considered as refined local finite element spaces according to transition from current mesh to mesh defined by space $\tilde{V}_h$. In the case when E_h^K consists of piecewise-polynomial functions it directly defines some refinement pattern on element K. For each element K we can consider now several different choices of space E_h^K: $E_1, \ldots, E_S$ and taking into account (16) we see, that it is optimal to use refinement pattern defined by the space $E_h^K := E_{s_K}$, $s_K \in \{1, \ldots, S\}$ which gives a maximum to a value of $\left\|e_h^K\right\|_E$ in the right part of (16).

So, now we can review the entire algorithm, which consists of two phases:

Initialization:

Compute:

$$\mu_0 = \min_{x \in [0,L]} \mu(x),$$
$$\sigma_0 = \min_{x \in [0,L]} \left\{ \sigma(x) - \frac{\beta'(x)}{2} \right\}, \tag{17}$$
$$C = 2 \cdot \left[\min\{\mu_0,\, \sigma_0\}\right]^{-1/2}.$$

Set τ_h to some initial finite element mesh.

For each finite element $K = (x_{k-1}, x_k) \in \tau_h$ we define quadratic bubble function

$$\omega_K(x) := (x_k - x)(x - x_{k-1}). \tag{18}$$

TOL is acceptable relative error level in percent.

p_{max} is the maximum supported degree of polynomial basis function on finite element.

$\theta \in (0, 1)$ is fixed value.

Iteration:

Step 1: Find FEM solution u_h on the current mesh τ_h. Define u_h^K as restriction of u_h to the element K and $p_K := \deg(u_h^K)$.

Step 2: For all elements $K \in \tau_h$ compute

$$\eta_K = \frac{C}{\sqrt{p_K(p_K + 1)}} \|\sqrt{\omega_K} R[u_h]\|_{L^2(K)} . \tag{19}$$

Define $\eta := \sqrt{\sum_K \eta_K^2}$.

Then if $\frac{\eta}{\|u_h\|_E} \times 100\% < TOL$ we stop the algorithm, else:

Step 3: Choose elements for refinement.

Compute $\eta_{\max} = \max_K \eta_K$.

We will change those elements K, for which $\eta_K > (1 - \theta)\eta_{\max}$. The set of all selected elements we name as A_θ.

Step 4: Mesh modification. For all selected elements $K = (x_{k-1}, x_k) \in A_\theta$ choose between bisection and increasing of polynomial degree on it by 1.

Step 4a: If $p_K = p_{\max}$ then we divide element into two with orders (p_K, p_K), otherwise:
Step 4b: Define $X^p(a,b)$ as a space of all polynomials of order p on closed interval $[a,b]$.

Define spaces:

$$\begin{aligned} V^1_{hp}(K) &= \{v \in C(K) | v \in X^{p_K}(x_{k-1}, (x_{k-1}+x_k)/2), \\ &\qquad v \in X^{p_K}((x_{k-1}+x_k)/2, x_k),\ v|_{\partial K} = 0\} \\ V^2_{hp}(K) &= \{v \in X^{p_K+1}(K) |\ v|_{\partial K} = 0\}. \end{aligned} \tag{20}$$

Now we solve problem (10) for $E^K_h := V^1_{hp}(K)$ and $E^K_h := V^2_{hp}(K)$. Let us denote obtained solutions as e^1_h and e^2_h respectively.

Compute $r_m = \|e^m_h\|_E,\ \ m = 1, 2$
Step 5: Consider the difference $\Delta = r_2 - r_1$.
If $\Delta > \delta$ where δ is predefined value, then we increase element degree by 1, otherwise we bisect it into two elements with approximation polynomial degrees (p_K, p_K).
Step 6: Go to **Step 1**.

Idea of described algorithm is clear for symmetric problems. Some numerical experiments are available in [5,6]. Technically we can run algorithm on nonsymmetric problems too, without having any theoretical background in that case. We will try to perform some numerical experiments to show how described algorithm will work in practice for nonsymmetric problem. We describe additional error estimator in next section 5 and additional adaptation criteria in section 6. Using those we will provide corresponding comparative numerical results in section 7 to show that algorithm can provide solid results despite of which combination of estimator and adaptation criteria we use.

5. Error estimator based on fundamental solution

For error indicator η_K, introduced by (19) in section 4, instead of using explicit formula we can use implicit indicator in the form of problem (10) but with special approximation space $E^K_h = \text{span}\{\varphi_K\}$, where:

$$\varphi_K(x) = \begin{cases} c_{11}\varphi_{11}(x) + c_{12}\varphi_{12}(x) & \text{on } x \in [x_{k-1}, x_{k-1/2}], \\ & \varphi_1(x_{k-1}) = 0, \varphi_1(x_{k-1/2}) = 1, \\ c_{21}\varphi_{21}(x) + c_{22}\varphi_{22}(x) & \text{on } x \in [x_{k-1/2}, x_k], \\ & \varphi_2(x_{k-1/2}) = 1, \varphi_2(x_k) = 0, \end{cases} \tag{21}$$

and $\{\varphi_{1i}(x)\}, \{\varphi_{2i}(x)\}$ are the sets of fundamental solutions for equations

$$-\left(\tilde{\mu}_i w'\right)' + \tilde{\beta}_i w' + \tilde{\sigma}_i w = 0,\ i = \overline{1,2} \tag{22}$$

with constant coefficients (selected as mean values of corresponding functions) on corresponding intervals $[x_{k-1}, x_{k-1/2}]$ and $[x_{k-1/2}, x_k]$. Then we solve (10) and use the energy norm of obtained approximation as an error indicator η_K. To find fundamental solutions we solve corresponding quadratic equations

$$-\tilde{\mu}_i\lambda_i^2 + \tilde{\beta}_i\lambda_i + \tilde{\sigma}_i = 0,\ i = \overline{1,2}. \tag{23}$$

Here for each of two equations we have three cases possible:

i. if $\lambda_i^{(1)}, \lambda_i^{(2)} \in \mathbb{R}, \lambda_i^{(1)} \neq \lambda_i^{(2)}$ then
$\varphi_{i1}(x) = \exp(\lambda_i^{(1)} x)$, $\varphi_{i2}(x) = \exp(\lambda_i^{(2)} x)$;

ii. if $\lambda_i^{(1)}, \lambda_i^{(2)} \in \mathbb{R}, \lambda_i^{(1)} = \lambda_i^{(2)}$ then
$\varphi_{i1}(x) = \exp(\lambda_i^{(1)} x)$, $\varphi_{i2}(x) = x\exp(\lambda_i^{(1)} x)$;

iii. if $\lambda_i^{(1)}, \lambda_i^{(2)} \in \mathbb{C}\backslash\mathbb{R}, \lambda_i^{(1)} = \alpha + \beta i, \lambda_2 = \alpha - \beta i$ then
$\varphi_{i1}(x) = \exp(\alpha x)\sin(\beta x)$, $\varphi_{i2}(x) = \exp(\alpha x)\cos(\beta x)$.

6. Element selection criteria

In addition to adding new estimator in previous section, we also will try to run algorithm with different adaptation criteria, used in step 3 to choose elements for refinement procedure. So we will have two criteria:

i. (*"maximum"* criteria) element K is refined if

$$\eta_K > (1-\theta)\eta_{max}, \tag{24}$$

where $\eta_{max} = \max\limits_K \eta_K$ and $\theta \in (0,1)$ is fixed value;

ii. (*"average"* criteria) element K is refined if

$$\frac{\sqrt{N}\eta_K}{\sqrt{\|u_h\|_E^2 + \sum_{K'} \eta_{K'}^2}} 100\% > \varepsilon, \tag{25}$$

where ε is is acceptable tolerance in % for average error level over finite element, N is element count.

7. Numerical example

We consider boundary value problem (1) with the following data

$$\begin{aligned} &\mu = 0.01, \beta = 100.896(x-1)^3, \sigma = 84(2-(x-1)^2), f = 200, \\ &\alpha = \gamma = 10^{14}, \bar{u}_0 = \bar{u}_L = 0, L = 2. \end{aligned} \tag{26}$$

Algorithm parameters are: $TOL = 5\%, p_{max} = 3, \delta = -150, \theta = 0.6, \varepsilon = 20$.

Fig. 1 demonstrates approximation obtained by introduced algorithm using fundamental solution error indicator "maximum" adaptation criteria. Taking into account boundary conditions we can clearly see that we have two boundary layers in the both ends of interval (which we don't see directly in the plot according to very large gradient of approximation near those two points). In tables 1 and 2 we present convergence history for different combinations of introduced error estimators from sections 5 and 4 in combination with "maximum" criteria (24) and "average" criteria (25). Average convergence rate is found using least squares method.

In general we can see from provided numerical examples that:

i. the better choice in according to count of elements, iterations and d.o.f. reached is a combination of the explicit indicator and "maximum" criteria;

ii. there is no large difference between "maximum" and "average" selection criteria;

iii. if we need to have almost monotonic relative error decreasing we need to choose explicit indicator from 4.

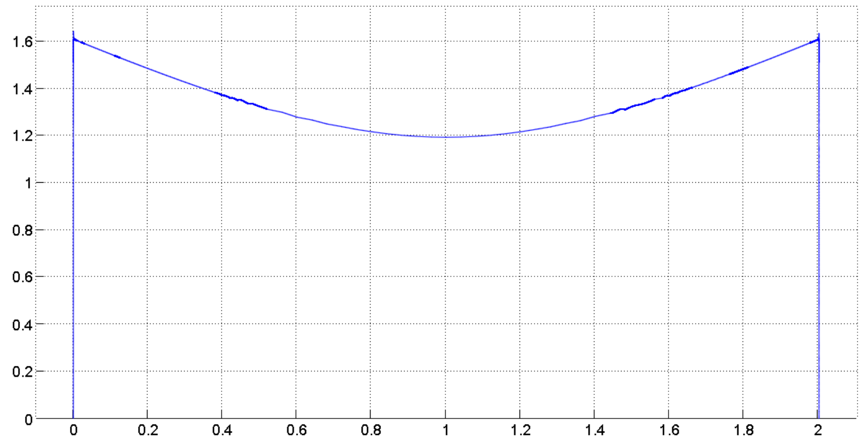

FIG. 1. Approximation to solution of problem with data (26) using implicit error indicator based on fundamental solution basis which was introduced in section 5 combined with the "maximum" criteria (24)

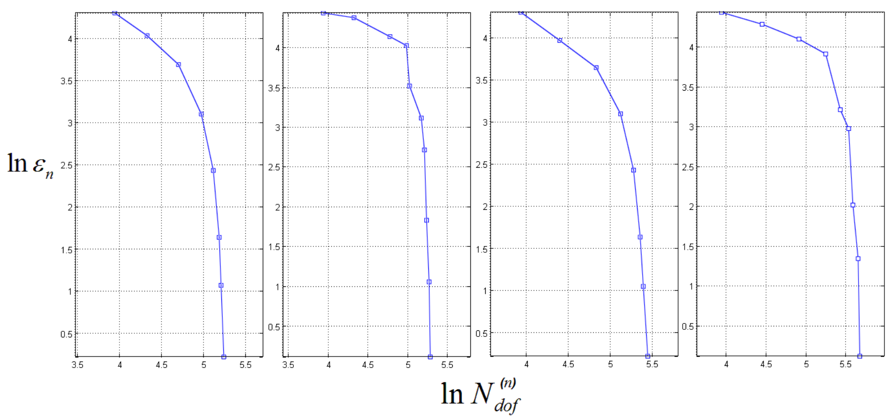

FIG. 2. Dependency between absolute error indicator ϵ_n and number of degrees of freedom $N_{dof}^{(n)}$ in log-log scale for previous results: a) for algorithm with explicit error indicator from section 4 and "maximum" criteria (24); b) for algorithm with indicator based on fundamental solution described in section 5 and "maximum" criteria (24); c) for algorithm with explicit error indicator from section 4 and "average" criteria (25); d) for algorithm with indicator based on fundamental solution described in section 5 and "average" criteria (25)

TABL. 1. Convergence history for problem with data (26) for the "maximum" criteria (24): n is an iteration number, N element count, $N_{dof}^{(n)}$ count of degrees of freedom, $\epsilon_n = \eta$ absolute error indicator, $r_n = \eta \|u_h\|_E^{-1} \times 100\%$ relative error, $p_n = -(\ln \epsilon_n - \ln \epsilon_{n-1}) \times \left(\ln N_{dof}^{(n)} - \ln N_{dof}^{(n-1)}\right)^{-1}$ rate of convergence

Explicit indicator						Fundamental solution indicator					
n	N	$N_{dof}^{(n)}$	ϵ_n	r_n	p_n	n	N	$N_{dof}^{(n)}$	ϵ_n	r_n	p_n
0	50	51	74.00	73.85		0	50	51	84.41	84.25	
1	72	75	56.51	50.94	0.69	1	69	75	79.52	67.86	0.15
2	106	109	40.08	32.25	0.91	2	102	118	62.69	52.61	0.52
3	136	143	22.26	21.22	2.16	3	124	145	56.07	49.96	0.54
4	144	165	11.39	15.61	4.68	4	130	151	33.69	33.22	12.56
5	144	177	5.14	17.90	11.33	5	142	175	22.55	37.16	2.72
6	144	181	2.90	8.39	25.48	6	142	182	15.10	43.72	10.21
7	146	187	1.24	4.57	26.08	7	143	187	6.24	19.58	32.59
						8	145	193	2.88	11.29	24.49
						9	146	196	1.12	4.72	61.11
average rate of convergence 2.66						average rate of convergence 2.38					

TABL. 2. Convergence history for problem with data (26) for the "average" criteria (25).

Explicit indicator						Fundamental solution indicator					
n	N	$N_{dof}^{(n)}$	ϵ_n	r_n	p_n	n	N	$N_{dof}^{(n)}$	ϵ_n	r_n	p_n
0	50	51	74.00	73.85		0	50	51	84.41	84.25	
1	72	81	52.93	52.15	0.72	1	72	85	72.67	71.90	0.29
2	106	125	38.34	32.55	0.74	2	106	135	60.36	50.78	0.40
3	134	167	22.12	21.43	1.89	3	136	189	50.07	48.36	0.55
4	142	195	11.33	15.90	4.31	4	144	227	24.87	35.20	3.81
5	142	211	5.11	18.80	10.08	5	144	252	19.63	70.70	2.26
6	142	219	2.84	8.02	15.76	6	144	266	7.52	21.09	17.73
7	146	231	1.24	4.57	15.56	7	150	284	3.84	14.18	10.26
						8	152	290	1.13	4.75	58.56
average rate of convergence 2.32						average rate of convergence 1.86					

Also, taking into account, that during preparation of this paper the algorithm was tested on several other problems, we can conclude from solid numerical results that the algorithm is applicable in practice in the case of nonsymmetric problems too, despite of which indicators or element selection criteria we use (without any theoretical background). In the next section we provide some pure theoretical analysis in that case.

8. Symmetrization methods

Instead of trying to generalize somehow (11) to nonsymmetric problems to bring similar argument as in remark in section 4, it is natural to try to construct equivalent (in some sense) to (3) but symmetric variational problem.

Here we present two pure theoretical results which can not be used in practice directly but can be considered as a starting point in further investigation in described direction.

8.1. **Equivalent symmetric problem approach.** Let us recall variational equation (1) in expanded form:

$$\begin{aligned} &\int_0^L [\mu u'v' + \beta u'v + \sigma uv]\, dx + \alpha u(0)v(0) + \gamma u(L)v(L) = \\ &= \int_0^L fv\, dx + \alpha \bar{u}_0 v(0) + \gamma \bar{u}_L v(L), \qquad \forall v \in V. \end{aligned} \tag{27}$$

We are free to choose arbitrary function v in (27) in the form: $v = zw$, where both functions z and w are arbitrary, but z is fixed. After substitution into (27) and small algebra we obtain equivalend equation:

$$\begin{aligned} &\int_0^L [\mu z u'w' + \underbrace{(\mu z' + \beta z)}\, u'w + \sigma zuw]\, dx + \\ &+ \alpha z(0)u(0)w(0) + \gamma z(L)u(L)w(L) = \\ &= \int_0^L fzw\, dx + \alpha \bar{u}_0 z(0)w(0) + \gamma \bar{u}_L z(L)w(L), \qquad \forall w \in V. \end{aligned} \tag{28}$$

Lets choose z as a solution of the ordinary differential equation $\mu z' + \beta z = 0$. It is not hard to find partial solution:

$$z(x) = \exp\left\{ -\int_0^x \frac{\beta(\xi)}{\mu(\xi)} d\xi \right\}. \tag{29}$$

Substituting (29) into (28) lead us to:

$$\begin{aligned} &\int_0^L [\mu z u'w' + \sigma zuw]\, dx + \alpha u(0)w(0) + \gamma z(L)u(L)w(L) = \\ &= \int_0^L fzw\, dx + \alpha \bar{u}_0 w(0) + \gamma \bar{u}_L z(L)w(L), \qquad \forall w \in V. \end{aligned} \tag{30}$$

It is not hard to see that (3) and (30) are equivalent and furthermore the bilinear form

$$b(u,v) := \int_0^L [\mu z u' w' + \sigma z u w]\, dx + \alpha u(0) w(0) + \gamma z(L) u(L) w(L), \tag{31}$$

in the left part of (30), is symmetric. Corresponding to (30) boundary value problem is:

$$\begin{cases} \textit{find function } u = u(x), \textit{ such that} \\ -\left(\mu z u'\right)' + \sigma z u = f z \text{ on } \Omega = (0, L) \\ (\mu z u')\big|_{x=0} = \alpha[u(0) - \bar{u}_0], -(\mu z u')\big|_{x=L} = \gamma z(L)[u(L) - \bar{u}_L]. \end{cases} \tag{32}$$

Visual simplicity of obtained symmetrization procedure and the problem (32), in practice lead us to problem which is technically hard to solve. The reason is in function z (29). Fraction $\frac{\beta(\xi)}{\mu(\xi)}$ is almost proportional to Péclet number for the given problem and in the latter is singular perturbed multiplier z will be the exponent with large negative power. In such conditions it is very problematically to calculate integrals from (30) when we use standard Galerkin discretization according to very large quadrature round-off errors. We investigated numerically the following approaches:

i. trapezoidal rule;
ii. interpolation-type quadrature based on L-splines;
iii. asymptotic formula at $Pe \to +\infty$;
iv. tanh – sinh quadratures;
v. adaptive quadratures using previous methods;
vi. implementation of adaptation algorithm using Wolfram Mathematica.

Those approaches even with combination with element-wise scaling of function z does not provide successful practical result.

8.2. **Iterative approach.** The second approach does not provide directly equivalent symmetric problem. Let us suppose that the bilinear form a and linear functional l from (1) satisfy conditions of Lax-Milgram theorem, i.e. a and l are bounded and moreover bilinear form a is V-elliptical. So, there are two positive constants $M > 0$ and $\alpha > 0$ such that:

$$\begin{aligned} a(u,v) &\le M\|u\|_V \|v\|_V, && \forall u, v \in V, \\ a(u,u) &\ge \alpha \|u\|_V^2, && \forall u \in V. \end{aligned} \tag{33}$$

By the way, where the conditions from (1) guarantees existence of such constants M and α.

Let us construct sequence $\{u^k\}_{k=0}^{\infty} \in V$. We select arbitrary $u^0 \in V$, u^k, $k > 0$ we find from the following *symmetric* variational problem:

$$\begin{cases} \textit{find function } u^k \in V, \textit{ such that} \\ a(u^k, v) + a(v, u^k) = \langle l, v\rangle + a(v, u^{k-1}), \quad \forall v \in V. \end{cases} \tag{34}$$

Under previous conditions for a and l it is not hard to conclude that the sequence is well-defined, i.e. the solution of (34) exists on each step.

Theorem 1. *If $M < 2\alpha$, than $u^k \xrightarrow[k\to\infty]{} u$ in V, where u is the solution of* (3), *moreover*

$$\|u - u^k\|_V \leq \left(\frac{M}{2\alpha}\right)^k \|u - u^0\|_V. \tag{35}$$

Proof. Let us define $e^k = u^k - u$. Then substitute $u^k = u + e^k$ into equation from (34). We get:

$$a(u + e^k, v) + a(v, u + e^k) = \langle l, v\rangle + a(v, u + e^{k-1}), \tag{36}$$

or after simplification:

$$a(e^k, v) + a(v, e^k) = a(v, e^{k-1}). \tag{37}$$

Taking $v = e^k$ and using (33) we obtain the following inequality chain:

$$2\alpha\|e^k\|_V^2 \leq 2a(e^k, e^k) = a(e^k, e^{k-1}) \leq M\|e^k\|_V\|e^{k-1}\|_V. \tag{38}$$

If there exist $k_0 : e^{k_0} = 0_V$ than it is obvious that $u^k = u, \forall k \geq k_0$, i.e. we have convergent sequence and the inequality from theorem statement holds. In other case $\forall k \in \mathbb{N}$ we can divide (38) by $\|e^k\|_V \neq 0$ and we obtain:

$$\|e^k\|_V \leq \frac{M}{2\alpha}\|e^{k-1}\|_V. \tag{39}$$

By combining the last recurrent formula we simply get the final estimate (35):

$$\|e^k\|_V \leq \left(\frac{M}{2\alpha}\right)^k \|e^0\|_V, \tag{40}$$

and convergence if $M < 2\alpha$.

9. Conclusion

In this paper we studied application of certain *hp*-adaptive algorithm to nonsymmetric problems. We combined this algorithm with different a posteriori error estimators and adaptation criteria to show by numerical experiment that algorithm can be directly applied to nonsymmetric problems. Also we construct several methods of symmetrization of given variational problem and provide corresponding theoretical analysis of those procedures. Two approaches are described. First can be used to build equivalent symmetric problem. In the second approach we built iterative procedure, where by solving symmetric variational problem on each step we can obtain sequence of elements that is convergent in the space of test functions to the solution of the original nonsymmetric problem. We still are working on the problem of theorem applicability to singular perturbed problems and schemes of combining this theorem with adaptive finite element algorithms. Also we are working on practical implementation of both symmetrization schemes which in practice involve building some ad hoc numerical quadratures.

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