Журнал обчислювальної та прикладної математики | 2017 № 3 (126) | Journal of Numerical & Applied Mathematics

UDC 519.6

APPLICATION OF (G'/G) – EXPANSION METHOD TO TWO KORTEWEG – DE VRIES TYPE DYNAMIC SYSTEMS

I. S. Mykhailiuk, M. M. Prytula

Резюме. Метод G'/G розвинення [12] застосовано до двох нелінійних динамічних ситем типу Кортевега – де Фріза [20]. Для обох систем побудовано розв'язки типу біжучих хвиль у формі гіперболічних, раціональних і тригонометричних функцій. Отримані результати порівняно з результатами, отриманими $tanh$- методом [4] і графічно проаналізовано.

Abstract. The (G'/G) – expansion method [15] is applied to two Korteweg – de Vries type nonlinear dynamic systems [1]. For both systems the traveling wave solutions in the form of hyperbolic, rational and trigonometric functions are constructed. The obtained results are compared to ones derived by means of the tanh – method [6] and graphically analyzed.

1. Introduction

Solutions to nonlinear evolution equations (NEE) play a crucial role in mathematical physics, therefore more and more scientists from all over the world dedicate their studies to investigate such equations. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solidstate physics, chemical kinematics, chemical physics and geochemistry.

With the advent of computers many effective numeric methods for finding approximate solutions to partial differential equations (PDEs) appeared. On the other hand, the creation of modern powerful computer algebra systems, such as MATLAB, MATHEMATICA and MAPLE, simplified the analytical investigation of NEEs, assisting mathematicians in their tiny computations. Hence during the past five decades a wide variety of analytical methods for finding exact solutions to NEEs was developed.

Recently, the (G'/G) – expansion method, firstly introduced by Wang et al. [15], has become widely used for many PDEs. It turned out that the method just mentioned provides solutions in a more general form compared to other analytical methods (e.g. the tanh – method [6]). What is more, with a certain choice of arbitrary parameters in the (G'/G) – expansion method some well-known solutions to PDEs can be rediscovered.

In paper [14], the authors constructed soliton solutions for two Korteweg–de Vries (KdV) type nonlinear dynamic systems [1, 3] by means of the tanh –

Key words. (G'/G) – expansion method, Korteweg – de Vries type dynamic system, soliton solution.

method [6]. In this work, we investigate these systems using the (G'/G) – expansion method and construct solutions in more general form. The rest of the paper is organized as follows. In Section 2, we describe the (G'/G) – expansion method [15] for finding traveling wave solutions to nonlinear evolution equations. In Section 3, we provide a brief overview of the main generalizations of the method being discussed. In Sections 4 and 5, we apply the method to two nonlinear KdV type dynamic systems [1,3], and analyze the obtained solutions. Finally, in Section 6, we summarize our results.

2. Description of the $\left(\frac{G'}{G}\right)$ – expansion method

Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...\right) = 0, \tag{1}$$

where $u = u\left(x, t\right)$ is an unknown function, P is a polynomial in $u = u\left(x, t\right)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G'/G) – expansion method [15].

Step 1. Combining independent variables x and t into one variable

$$\xi = x - Vt, \tag{2}$$

we suppose that $u\left(x, t\right) = u\left(\xi\right)$. Traveling wave variable (2) permits us to reduce Eq. (1) to an ordinary differential equation (ODE) for $u\left(x, t\right) = u\left(\xi\right)$

$$P\left(u, -Vu', u', V^2u'', -Vu'', u'', ...\right) = 0. \tag{3}$$

Step 2. Suppose that the solution to ODE (3) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G = G\left(\xi\right)$ satisfies the second order linear ODE in the form of

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

$\alpha_i \left(i = \overline{0, m}\right), \lambda, \mu$ are constants to be determined later, $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

Step 3. By substituting (4) into Eq. (3) and using the second order LODE (5), collecting all terms with the same order of (G'/G) together, the left-hand side of Eq. (3) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_i \left(i = \overline{0, m}\right), \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_i \left(i = \overline{0, m}\right), \lambda, \mu$ and V can be obtained by solving the algebraic equations in Step 3, since the general solutions to the second order linear ODE (5) have been well known for us, then substituting $\alpha_i \left(i = \overline{0, m}\right), \lambda, \mu, V$ and the general solutions to Eq. (5) into (4) we obtain traveling wave solutions to the original nonlinear evolution equation (1).

As it was already mentioned, the solution to Eq. (5) is well-known for us and can be easily derived by the Euler method:

$$G(\xi)=\begin{cases}\left(A_1\sinh\frac{\xi\sqrt{\lambda^2-4\mu}}{2}+A_2\cosh\frac{\xi\sqrt{\lambda^2-4\mu}}{2}\right)e^{-\frac{1}{2}\lambda\xi},\\ \qquad\qquad if\ \lambda^2-4\mu>0,\\ (A_1+A_2\xi)\,e^{-\frac{1}{2}\lambda\xi},\quad if\ \lambda^2-4\mu=0,\\ \left(A_1\sin\frac{\xi\sqrt{4\mu-\lambda^2}}{2}+A_2\cos\frac{\xi\sqrt{4\mu-\lambda^2}}{2}\right)e^{-\frac{1}{2}\lambda\xi},\\ \qquad\qquad if\ \lambda^2-4\mu<0.\end{cases}\tag{6}$$

3. Main generalizations of the (G'/G) – expansion method

Since 2008, when the (G'/G) – expansion method was introduced by Wang et al. [15], many modifications and generalizations of the algorithm have been developed, each of which concerned different aspect of the method. Therefore, it is worth classifying them by that aspect.

3.1. **Homogeneous balance value.**

The classical method [15] assumed that the homogeneous balance value, which determines a degree of polynomial (4), is a positive integer. In paper [4] the authors used a transform to handle the equations with negative or fractional homogeneous balance value. Let m be a value of balance for a certain equation. If $m=\frac{p}{q}$ is a fraction in the lowest terms, then we set the solution

$$u(\xi)=v^{\frac{p}{q}}(\xi),$$

and when m is a negative integer, then we set

$$u(\xi)=v^{m}(\xi),$$

then substitute the new expression for $u(\xi)$ into (3) and recompute the balance value for a new equation, which is now guaranteed to be a positive integer [4].

3.2. **Representation of the solution to NEE.**

Another way to modify the original method is to replace the polynomial in $\left(\frac{G'}{G}\right)$ with a more general structure.

In works [2] and [16] the solution was suggested to be found in the following form:

$$u(\xi)=a_0+\sum_{i=1}^{n}\left[a_i\left(\frac{G'}{G}\right)^i+b_i\left(\frac{G'}{G}\right)^{i-1}\sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}\right],$$

and, moreover, the function $G=G(\xi)$ was found as a solution to simplified equation

$$G''+\mu G=0,$$

where μ is a constant to be determined.

Yet another form of the solution representation was introduced in papers [21], [17] and [13], namely the solution was supposed to have the following form:

$$u(\xi) = \sum_{i=-n}^{n} \alpha_i \left(\frac{G'}{G}\right)^i ,$$

i. e. the expansion included the terms with negative degrees.

As it is shown in the corresponding works, both mentioned representations of function $u = u(\xi)$ yield more general solutions to certain NEEs [2,13,16,17,21].

3.3. **Auxiliary equation for function $G = G(\xi)$.** Other modifications of the method affected the form of the auxiliary equation, which in the classical $\left(\frac{G'}{G}\right)$ – expansion method is of the form (5). One of the most frequently used equations was the nonlinear one of the following form:

$$GG'' = AG^2 + BGG' + C\left(G'\right)^2 ,$$

where the prime denotes the derivative with respect to ξ; A, B, C are all real parameters.

This improvement of the method was firstly introduced by Liu et al. in [5] to obtain more general solutions to NEEs in comparison with the classical method. It was successfully applied to some well-known equations of mathematical physics, among other, in works [5, 7–12].

3.4. **Coefficient of the polynomial in $\left(\frac{G'}{G}\right)$.** One more generalization of the original method was the idea to find a solution to NEEs as a polynomial in $\left(\frac{G'}{G}\right)$ with variable coefficients [20], namely

$$u(\xi) = \sum_{i=1}^{n} \alpha_i(X) \left(\frac{G'}{G}\right)^i + \alpha_0(X) ,$$

where $\alpha_i = \alpha_i(X)\left(i = \overline{0, n}\right), \xi = \xi(X)$ are functions to be determined. As in the classical method, function $G = G(\xi)$ satisfies Eq. (5). The rest of the algorithm remains the same, except that at the third step one need to solve a system of ordinary differential equations rather than algebraic ones.

The described idea was successfully used to solve some NEEs in papers [18–20].

4. Application: Example 1

Consider the following Korteweg – de Vries (KdV) type nonlinear dynamic system [1, 3]

$$\begin{cases} u_t = u_{xxx} - v_x, \\ v_t = -2v_{xxx} - uv_x. \end{cases} \tag{7}$$

Let us solve system (7) by use of the (G'/G) – expansion method.

Step 1. Introducing traveling wave variable $\xi = x - Vt$, we reduce system (7) to a system of ODE for $u = u(\xi)$ and $v = v(\xi)$

$$\begin{cases} -Vu' = u''' - v', \\ -Vv' = -2v''' - uv'. \end{cases} \tag{8}$$

Suppose that the solution to system (8) can be expressed by polynomials in (G'/G) as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \quad v(\xi) = \sum_{i=0}^{n} \beta_i \left(\frac{G'}{G}\right)^i. \tag{9}$$

Considering the homogeneous balance between u''' and v', v''' and uv' in the first and the second equations of system (8) correspondingly, we obtain a simple system of algebraic equations

$$\begin{cases} m + 3 = n + 1, \\ n + 3 = m + n + 1, \end{cases} \tag{10}$$

from which it can be easily found that $m = 2$ and $n = 4$.

Step 2. Considering (9) and (10), we find the solution to system (8) in the following form:

$$\begin{cases} u(\xi) = \alpha_2 \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \\ v(\xi) = \beta_4 \left(\frac{G'}{G}\right)^4 + \beta_3 \left(\frac{G'}{G}\right)^3 + \beta_2 \left(\frac{G'}{G}\right)^2 + \beta_1 \left(\frac{G'}{G}\right) + \beta_0, \end{cases} \tag{11}$$

where function $G = G(\xi)$ satisfies the second order linear ODE (5), λ, μ, V, α_i $\left(i = \overline{0,2}\right)$, β_j $\left(j = \overline{0,4}\right)$ are all constants to be determined later, $\alpha_2 \neq 0,\ \beta_4 \neq 0$.

Step 3. Substituting (11) into system (8) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand sides of equations (8) are converted into another polynomials in $\left(\frac{G'}{G}\right)$. Equating each coefficient of these polynomials to zero yields a set of simultaneous algebraic equations for $\lambda, \mu, V, \alpha_i \left(i = \overline{0,2}\right), \beta_j \left(j = \overline{0,4}\right)$ as follows:

- from *the first equation* in (8):

$$\begin{aligned}
0:&\quad \alpha_1\lambda^2\mu + 6\alpha_2\lambda\mu^2 + 2\alpha_1\mu^2 - \beta_1\mu + \alpha_1\mu V = 0\\
1:&\quad \alpha_1\lambda^3 + 6\alpha_2\lambda^2\mu + 8\alpha_2\mu\left(\lambda^2 + 2\mu\right) + 8\alpha_1\lambda\mu - \beta_1\lambda - 2\beta_2\mu+\\
&\quad +V\left(\alpha_1\lambda + 2\alpha_2\mu\right) = 0\\
2:&\quad 8\alpha_2\lambda\left(\lambda^2 + 2\mu\right) + 7\alpha_1\lambda^2 + 36\alpha_2\lambda\mu + 8\alpha_1\mu - 2\beta_2\lambda - 3\beta_3\mu-\\
&\quad -\beta_1 + V\left(2\alpha_2\lambda + \alpha_1\right) = 0\\
3:&\quad 8\alpha_2\left(\lambda^2 + 2\mu\right) + 30\alpha_2\lambda^2 + 12\alpha_1\lambda + 24\alpha_2\mu - 3\beta_3\lambda - 4\beta_4\mu-\\
&\quad -2\beta_2 + 2\alpha_2 V = 0\\
4:&\quad 54\alpha_2\lambda + 6\alpha_1 - 4\beta_4\lambda - 3\beta_3 = 0\\
5:&\quad 24\alpha_2 - 4\beta_4 = 0;
\end{aligned}$$

– from *the second equation* in (8):

$$
\begin{aligned}
0:\quad & -\alpha_0\beta_1\mu - 2\beta_1\lambda^2\mu - 12\beta_2\lambda\mu^2 - 12\beta_3\mu^3 - 4\beta_1\mu^2 + \beta_1\mu V = 0\\
1:\quad & -\alpha_0\beta_1\lambda - \alpha_1\beta_1\mu - 2\alpha_0\beta_2\mu - 2\beta_1\lambda^3 - 28\beta_2\lambda^2\mu - 72\beta_3\lambda\mu^2 -\\
& -16\beta_1\lambda\mu - 48\beta_4\mu^3 - 32\beta_2\mu^2 + \beta_1\lambda V + 2\beta_2\mu V = 0\\
2:\quad & -\alpha_1\beta_1\lambda - 2\alpha_0\beta_2\lambda - \alpha_2\beta_1\mu - 2\alpha_1\beta_2\mu - 3\alpha_0\beta_3\mu - \alpha_0\beta_1 - 16\beta_2\lambda^3 -\\
& -114\beta_3\lambda^2\mu - 14\beta_1\lambda^2 - 216\beta_4\lambda\mu^2 - 104\beta_2\lambda\mu - 120\beta_3\mu^2 - 16\beta_1\mu +\\
& +2\beta_2\lambda V + 3\beta_3\mu V + \beta_1 V = 0\\
3:\quad & -\alpha_2\beta_1\lambda - 2\alpha_1\beta_2\lambda - 3\alpha_0\beta_3\lambda - 2\alpha_2\beta_2\mu - 3\alpha_1\beta_3\mu - 4\alpha_0\beta_4\mu -\\
& -\alpha_1\beta_1 - 2\alpha_0\beta_2 - 54\beta_3\lambda^3 - 296\beta_4\lambda^2\mu - 76\beta_2\lambda^2 - 336\beta_3\lambda\mu -\\
& -24\beta_1\lambda - 304\beta_4\mu^2 - 80\beta_2\mu + 3\beta_3\lambda V + 4\beta_4\mu V + 2\beta_2 V = 0\\
4:\quad & -2\alpha_2\beta_2\lambda - 3\alpha_1\beta_3\lambda - 4\alpha_0\beta_4\lambda - 3\alpha_2\beta_3\mu - 4\alpha_1\beta_4\mu - \alpha_2\beta_1 -\\
& -2\alpha_1\beta_2 - 3\alpha_0\beta_3 - 128\beta_4\lambda^3 - 222\beta_3\lambda^2 - 784\beta_4\lambda\mu - 108\beta_2\lambda -\\
& -228\beta_3\mu - 12\beta_1 + 4\beta_4\lambda V + 3\beta_3 V = 0\\
5:\quad & -3\alpha_2\beta_3\lambda - 4\alpha_1\beta_4\lambda - 4\alpha_2\beta_4\mu - 2\alpha_2\beta_2 - 3\alpha_1\beta_3 - 4\alpha_0\beta_4 - 488\beta_4\lambda^2 -\\
& -288\beta_3\lambda - 496\beta_4\mu - 48\beta_2 + 4\beta_4 V = 0\\
6:\quad & -\alpha_2\left(4\beta_4\lambda + 3\beta_3\right) - 4\alpha_1\beta_4 + 2\left(-120\beta_4\lambda - 60\left(3\beta_4\lambda + \beta_3\right)\right) = 0\\
7:\quad & -4\alpha_2\beta_4 - 240\beta_4 = 0.
\end{aligned}
$$

In addition to this, the highest order coefficients in (11) are supposed to be nonzero:

$$\alpha_2 \neq 0,\ \beta_4 \neq 0. \tag{12}$$

Step 4. Solving the system of algebraic equations from the previous step with conditions (12) with the aid of MATHEMATICA yields *four sets of solutions*:

– Set 1.

$$
\begin{gathered}
V = \lambda^2 - 4\mu, \quad \alpha_0 = -\lambda^2 - 56\mu, \quad \alpha_1 = -60\lambda, \quad \alpha_2 = -60,\\
\beta_1 = -120\left(\lambda^3 + 2\lambda\mu\right), \quad \beta_2 = -240\left(2\lambda^2 + \mu\right),\\
\beta_3 = -720\lambda, \quad \beta_4 = -360,
\end{gathered} \tag{13}
$$

where λ, μ and β_0 are arbitrary constants.

– Set 2.

$$
\begin{gathered}
V = 4\mu - \lambda^2, \quad \alpha_0 = -3\left(3\lambda^2 + 8\mu\right), \quad \alpha_1 = -60\lambda, \quad \alpha_2 = -60,\\
\beta_1 = -720\lambda\mu, \quad \beta_2 = -360\left(\lambda^2 + 2\mu\right),\\
\beta_3 = -720\lambda, \quad \beta_4 = -360,
\end{gathered} \tag{14}
$$

where λ, μ and β_0 are arbitrary constants.

– Set 3.

$$
\begin{gathered}
V = \lambda^2, \quad \mu = 0, \quad \alpha_0 = -\lambda^2, \quad \alpha_1 = -60\lambda, \quad \alpha_2 = -60,\\
\beta_1 = -120\lambda^3, \quad \beta_2 = -480\lambda^2, \quad \beta_3 = -720\lambda, \quad \beta_4 = -360,
\end{gathered} \tag{15}
$$

where λ and β_0 are arbitrary constants.

– Set 4.

$$
\begin{gathered}
V = -\lambda^2, \quad \mu = 0, \quad \alpha_0 = -9\lambda^2, \quad \alpha_1 = -60\lambda, \quad \alpha_2 = -60,\\
\beta_1 = 0, \quad \beta_2 = -360\lambda^2, \quad \beta_3 = -720\lambda, \quad \beta_4 = -360,
\end{gathered} \tag{16}
$$

where λ and β_0 are arbitrary constants.

Finally, substituting solutions (13)–(16) with the general solution to linear ODE (5) into representation (11) we obtain *four separate sets* of traveling wave solutions to the KdV type dynamic system (7) as follows.

Solutions set 1. Constants set (13) yields three families of solutions:

– when $\lambda^2 - 4\mu > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & -\frac{15\left(A_1^2-A_2^2\right)\sigma}{\left(A_1 \sinh\frac{\xi\sqrt{\sigma}}{2}+A_2\cosh\frac{\xi\sqrt{\sigma}}{2}\right)^2} - \sigma, \\ v(\xi) = & -\frac{15\left(A_1^2-A_2^2\right)\sigma^2\left(4A_2A_1\sinh\xi\sqrt{\sigma}+2\left(A_1^2+A_2^2\right)\cosh\xi\sqrt{\sigma}+A_1^2-A_2^2\right)}{2\left(A_1\sinh\frac{\xi\sqrt{\sigma}}{2}+A_2\cosh\frac{\xi\sqrt{\sigma}}{2}\right)^4} + \\ & +\beta_0 + 120\mu\left(\lambda^2-\mu\right), \end{cases} \tag{17}$$

where $\xi = x - \left(\lambda^2 - 4\mu\right)t,\ \sigma = \lambda^2 - 4\mu,\quad A_1, A_2, \beta_0$ are arbitrary constants; in particular, setting $\lambda = \pm\sqrt{\frac{8}{3}}|k_1|,\ \mu = -\frac{1}{3}k_1^2,\ A_1 = 0,\ \beta_0 = a_{20}$, we obtain exactly the soliton solution, found by means of the tanh – method in [14];

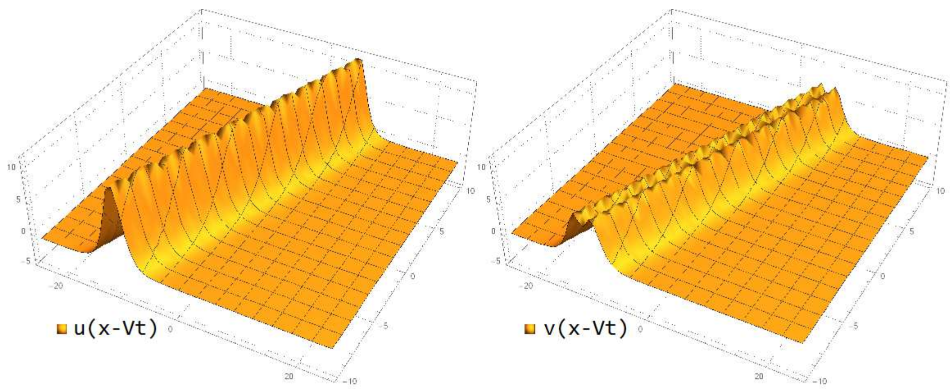

FIG. 1. Hyperbolic functions solution (17) when $A_1 = 1$, $A_2 = 1.2,\ \lambda = 2.2,\ \mu = 1,\ \beta_0 = -460.8$

– when $\lambda^2 - 4\mu = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & -\frac{60A_2^2}{(A_2\xi+A_1)^2}, \\ v(\xi) = & \frac{360\left(\mu^2(A_2\xi+A_1)^4-A_2^4\right)}{(A_2\xi+A_1)^4} + \beta_0, \end{cases} \tag{18}$$

where $\xi = x,\quad A_1, A_2, \beta_0$ are arbitrary constants;

– when $\lambda^2 - 4\mu < 0$, we get the family of trigonometric functions solutions

$$\begin{cases} u(\xi) = & -\frac{15\left(A_1^2+A_2^2\right)\sigma}{\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2}+A_2\cos\frac{\xi\sqrt{\sigma}}{2}\right)^2} + \sigma, \\ v(\xi) = & -\frac{15\left(A_1^2+A_2^2\right)\sigma^2\left(-4A_2A_1\sin\xi\sqrt{\sigma}+2\left(A_1^2-A_2^2\right)\cos\xi\sqrt{\sigma}+A_1^2+A_2^2\right)}{2\left(A_1\sin\frac{\xi\sqrt{\sigma}}{2}+A_2\cos\frac{\xi\sqrt{\sigma}}{2}\right)^4} + \\ & +\beta_0 + 120\mu\left(\lambda^2-\mu\right), \end{cases} \tag{19}$$

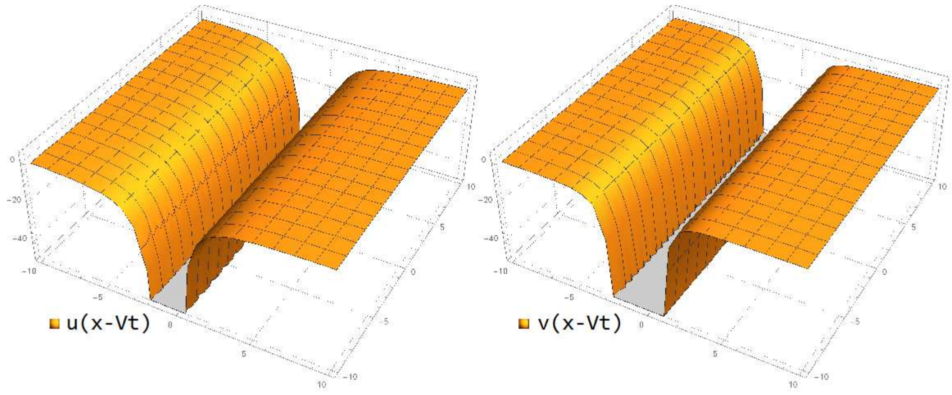

FIG. 2. Rational functions solution (18) when $A_1 = 1,\ A_2 = 1.2,\ \lambda = 1,\ \mu = 0.25,\ \beta_0 = -22.5$

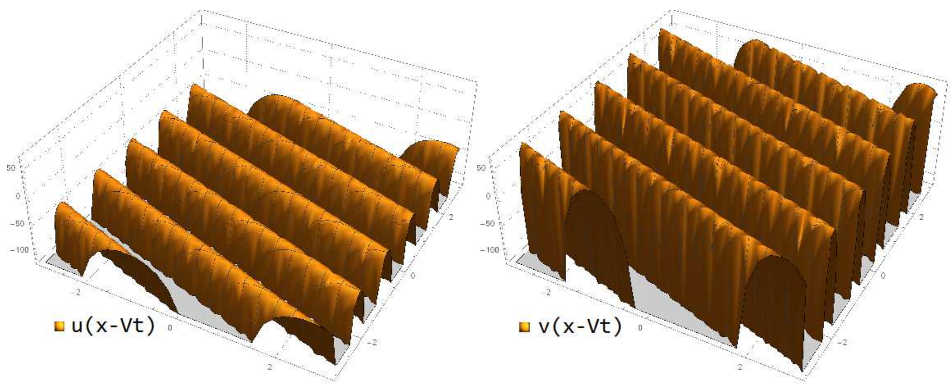

FIG. 3. Trigonometric functions solution (19) when $A_1 = 1,\ A_2 = 1.2,\ \lambda = 1,\ \mu = 1,\ \beta_0 = 0$

where $\xi = x - \left(\lambda^2 - 4\mu\right) t,\ \sigma = 4\mu - \lambda^2,\quad A_1, A_2, \beta_0$ are arbitrary constants.

Solutions set 2. Constants set (14) yields three families of solutions:

– when $\lambda^2 - 4\mu > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & -\frac{15\sigma\left(A_1^2 - A_2^2\right)}{\left(A_1 \sinh \frac{\xi\sqrt{\sigma}}{2} + A_2 \cosh \frac{\xi\sqrt{\sigma}}{2}\right)^2} - 9\sigma \\ v(\xi) = & -\frac{45\left(A_1^2 - A_2^2\right)^2\sigma^2}{2\left(A_1 \sinh \frac{\xi\sqrt{\sigma}}{2} + A_2 \cosh \frac{\xi\sqrt{\sigma}}{2}\right)^4} + \beta_0 + 360\mu^2, \end{cases} \tag{20}$$

where $\xi = x + \left(\lambda^2 - 4\mu\right) t,\ \sigma = \lambda^2 - 4\mu,\quad A_1, A_2, \beta_0$ are arbitrary constants;

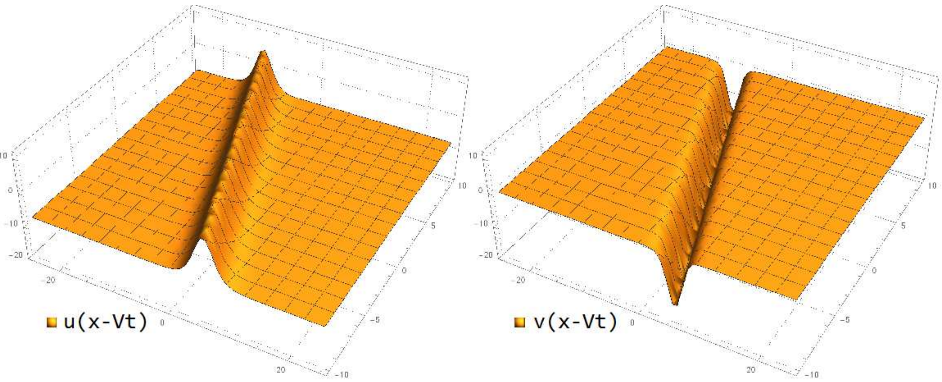

FIG. 4. Hyperbolic functions solution (20) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 2.2$, $\mu = 1$, $\beta_0 = -360$

– when $\lambda^2 - 4\mu = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & -\frac{60A_2^2}{(A_2\xi + A_1)^2}, \\ v(\xi) = & \frac{360\left(\mu^2 (A_2\xi + A_1)^4 - A_2^4\right)}{(A_2\xi + A_1)^4} + \beta_0, \end{cases} \tag{21}$$

where $\xi = x$, A_1, A_2, β_0 are arbitrary constants; note that solutions (21) coincide with corresponding family (18) from the first set.

– when $\lambda^2 - 4\mu < 0$, we get the family of trigonometric functions solutions

$$\begin{cases} u(\xi) = & \frac{-15\sigma\left(A_1^2 + A_2^2\right)}{\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2} + A_2 \cos\frac{\xi\sqrt{\sigma}}{2}\right)^2} + 9\sigma, \\ v(\xi) = & -\frac{45\left(A_1^2 + A_2^2\right)^2\sigma^2}{2\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2} + A_2 \cos\frac{\xi\sqrt{\sigma}}{2}\right)^4} + \beta_0 + 360\mu^2, \end{cases} \tag{22}$$

where $\xi = x + \left(\lambda^2 - 4\mu\right) t$, $\sigma = 4\mu - \lambda^2$, A_1, A_2, β_0 are arbitrary constants.

Solutions set 3. Constants set (15) yields two families of solutions:

– when $\lambda > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & \frac{15\lambda^2\left(A_2^2 - A_1^2\right)}{\left(A_1 \sinh\frac{\xi|\lambda|}{2} + A_2 \cosh\frac{\lambda\xi}{2}\right)^2} - \lambda^2, \\ v(\xi) = & \frac{2A_1A_2\left(A_2^2 - A_1^2\right)\left(\beta_0 + 30\lambda^4\right)\sinh\xi|\lambda| + A_1A_2\left(A_1^2 + A_2^2\right)\beta_0 \sinh 2\xi|\lambda|}{2\left(A_1 \sinh\frac{\xi|\lambda|}{2} + A_2 \cosh\frac{\lambda\xi}{2}\right)^4} + \\ & + \frac{-\left(A_1^4 - A_2^4\right)\left(\beta_0 + 30\lambda^4\right)\cosh\lambda\xi - \frac{3}{4}\left(A_1^2 - A_2^2\right)^2\left(20\lambda^4 - \beta_0\right)}{2\left(A_1 \sinh\frac{\xi|\lambda|}{2} + A_2 \cosh\frac{\lambda\xi}{2}\right)^4} + \\ & + \frac{\frac{1}{4}\left(A_1^4 + 6A_2^2A_1^2 + A_2^4\right)\beta_0 \cosh 2\lambda\xi}{2\left(A_1 \sinh\frac{\xi|\lambda|}{2} + A_2 \cosh\frac{\lambda\xi}{2}\right)^4}, \end{cases} \tag{23}$$

where $\xi = x - \lambda^2 t$, A_1, A_2, β_0 are arbitrary constants;

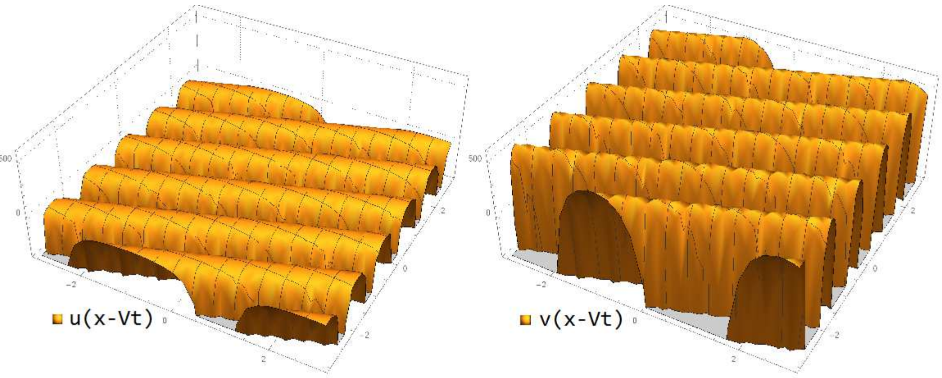

FIG. 5. Trigonometric functions solution (22) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 1$, $\mu = 1$, $\beta_0 = 360$

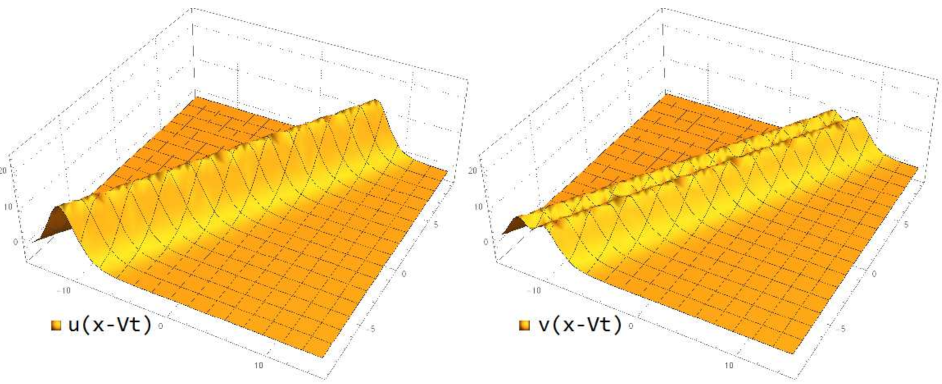

FIG. 6. Hyperbolic functions solution (23) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 1$, $\beta_0 = 0$

– when $\lambda = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & -\frac{60A_2^2}{(A_2\xi+A_1)^2}, \\ v(\xi) = & \beta_0 - \frac{360A_2^4}{(A_2\xi+A_1)^4}, \end{cases} \quad \% \tag{24}$$

where $\xi = x$, A_1, A_2, β_0 are arbitrary constants; note that solutions (24) coincide with corresponding family (18) from the first set.

Solutions set 4. Constants set (16) yields two families of solutions:

– when $\lambda > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & -\frac{3\lambda^2\left(6A_2A_1\sinh\xi|\lambda|+A_1^2(3\cosh\lambda\xi+7)+A_2^2(3\cosh\lambda\xi-7)\right)}{2\left(A_1\sinh\frac{\xi|\lambda|}{2}+A_2\cosh\frac{\lambda\xi}{2}\right)^2}, \\ v(\xi) = & \beta_0 - \frac{45\left(A_1^2-A_2^2\right)^2\lambda^4}{2\left(A_1\sinh\frac{\xi|\lambda|}{2}+A_2\cosh\frac{\lambda\xi}{2}\right)^4}, \end{cases} \tag{25}$$

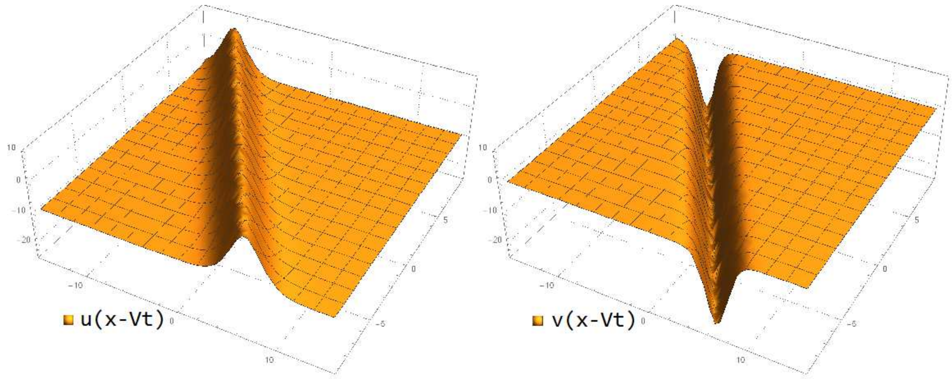

FIG. 7. Hyperbolic functions solution (25) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 1$, $\beta_0 = 0$

where $\xi = x + \lambda^2 t$, A_1, A_2, β_0 are arbitrary constants;

– when $\lambda = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & -\frac{60A_2^2}{(A_2\xi+A_1)^2}, \\ v(\xi) = & \beta_0 - \frac{360A_2^4}{(A_2\xi+A_1)^4}, \end{cases} \tag{26}$$

where $\xi = x$, A_1, A_2, β_0 are arbitrary constants; note that solutions (26) coincide with corresponding family (18) from the first set.

5. Application: Example 2

Consider the following Korteweg – de Vries (KdV) type nonlinear dynamic system [1]

$$\begin{cases} u_t = u_{xxx} + uu_x - vv_x, \\ v_t = -2v_{xxx} - uv_x. \end{cases} \tag{27}$$

Let us solve system (27) by use of the (G'/G) – expansion method.

Step 1. Introducing traveling wave variable $\xi = x - Vt$, we reduce system (27) to a system of ODE for $u = u(\xi)$ and $v = v(\xi)$

$$\begin{cases} -Vu' = u''' + uu' - vv', \\ -Vv' = -2v''' - uv'. \end{cases} \tag{28}$$

Suppose that the solution to system (28) can be expressed by polynomials in (G'/G) as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \quad v(\xi) = \sum_{i=0}^{n} \beta_i \left(\frac{G'}{G}\right)^i. \tag{29}$$

Considering the homogeneous balance between u''' and vv', v''' and uv' in the first and the second equations of system (28) correspondingly, we obtain a simple system of algebraic equations

$$\begin{cases} m+3=2n+1. \\ n+3=m+n+1, \end{cases} \tag{30}$$

from which it can be easily found that $m=2$ and $n=2$.

Step 2. Considering (29) and (30), we find the solution to system (28) in the following form:

$$\begin{cases} u\left(\xi\right)=\alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \\ v\left(\xi\right)=\beta_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)+\beta_0, \end{cases} \tag{31}$$

where function $G=G\left(\xi\right)$ satisfies the second order linear ODE (5), λ, μ, V, α_i $\left(i=\overline{0,2}\right)$, β_j $\left(j=\overline{0,2}\right)$ are all constants to be determined later, $\alpha_2\neq 0,\ \beta_2\neq 0$.

Step 3. Substituting (31) into system (28) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand sides of equations (28) are converted into another polynomials in $\left(\frac{G'}{G}\right)$. Equating each coefficient of these polynomials to zero yields a set of simultaneous algebraic equations for $\lambda,\mu,V,\alpha_i\left(i=\overline{0,2}\right),\beta_j\left(j=\overline{0,2}\right)$ as follows:

– from *the first equation* in (28):

$$\begin{aligned}
0:&\quad \alpha_1\lambda^2\mu+6\alpha_2\lambda\mu^2+2\alpha_1\mu^2+\alpha_0\alpha_1\mu-\beta_0\beta_1\mu+\alpha_1\mu V=0\\
1:&\quad \alpha_1\lambda^3+6\alpha_2\lambda^2\mu+8\alpha_2\mu\left(\lambda^2+2\mu\right)+8\alpha_1\lambda\mu+\alpha_0\left(\alpha_1\lambda+2\alpha_2\mu\right)+\\
&\quad +\alpha_1^2\mu-\beta_0\left(\beta_1\lambda+2\beta_2\mu\right)-\beta_1^2\mu+V\left(\alpha_1\lambda+2\alpha_2\mu\right)=0\\
2:&\quad 8\alpha_2\lambda\left(\lambda^2+2\mu\right)+7\alpha_1\lambda^2+36\alpha_2\lambda\mu+\alpha_1\left(\alpha_1\lambda+2\alpha_2\mu\right)+\\
&\quad +\alpha_0\left(2\alpha_2\lambda+\alpha_1\right)+8\alpha_1\mu+\alpha_1\alpha_2\mu-\beta_1\left(\beta_1\lambda+2\beta_2\mu\right)-\\
&\quad -\beta_0\left(2\beta_2\lambda+\beta_1\right)-\beta_1\beta_2\mu+V\left(2\alpha_2\lambda+\alpha_1\right)=0\\
3:&\quad 8\alpha_2\left(\lambda^2+2\mu\right)+30\alpha_2\lambda^2+\alpha_2\left(\alpha_1\lambda+2\alpha_2\mu\right)+12\alpha_1\lambda+\\
&\quad +\alpha_1\left(2\alpha_2\lambda+\alpha_1\right)+24\alpha_2\mu+2\alpha_0\alpha_2-\beta_2\left(\beta_1\lambda+2\beta_2\mu\right)-\\
&\quad -\beta_1\left(2\beta_2\lambda+\beta_1\right)-2\beta_0\beta_2+2\alpha_2V=0\\
4:&\quad 54\alpha_2\lambda+\alpha_2\left(2\alpha_2\lambda+\alpha_1\right)+2\alpha_2\alpha_1+6\alpha_1-\beta_2\left(2\beta_2\lambda+\beta_1\right)-\\
&\quad -2\beta_1\beta_2=0\\
5:&\quad 2\alpha_2^2+24\alpha_2-2\beta_2^2=0;
\end{aligned}$$

– from *the second equation* in (28):

$$\begin{aligned}
0:&\quad -\alpha_0\beta_1\mu-2\beta_1\lambda^2\mu-12\beta_2\lambda\mu^2-4\beta_1\mu^2+\beta_1\mu V=0\\
1:&\quad -2\left(\beta_1\lambda^3+6\beta_2\lambda^2\mu+8\beta_2\mu\left(\lambda^2+2\mu\right)+8\beta_1\lambda\mu\right)+\\
&\quad +V\left(\beta_1\lambda+2\beta_2\mu\right)-\alpha_0\left(\beta_1\lambda+2\beta_2\mu\right)+\alpha_1\beta_1(-\mu)=0\\
2:&\quad -\alpha_1\left(\beta_1\lambda+2\beta_2\mu\right)-\alpha_0\left(2\beta_2\lambda+\beta_1\right)+\alpha_2\beta_1(-\mu)-\\
&\quad -2\left(8\beta_2\lambda\left(\lambda^2+2\mu\right)+7\beta_1\lambda^2+36\beta_2\lambda\mu+8\beta_1\mu\right)+\\
&\quad +V\left(2\beta_2\lambda+\beta_1\right)=0\\
3:&\quad -\alpha_2\left(\beta_1\lambda+2\beta_2\mu\right)-\alpha_1\left(2\beta_2\lambda+\beta_1\right)-2\alpha_0\beta_2-\\
&\quad -2\left(8\beta_2\left(\lambda^2+2\mu\right)+30\beta_2\lambda^2+12\beta_1\lambda+24\beta_2\mu\right)+2\beta_2V=0\\
4:&\quad -\alpha_2\left(2\beta_2\lambda+\beta_1\right)-2\alpha_1\beta_2-2\left(54\beta_2\lambda+6\beta_1\right)=0\\
5:&\quad -2\alpha_2\beta_2-48\beta_2=0.
\end{aligned}$$

In addition to this, the highest order coefficients in (31) are supposed to be nonzero:

$$\alpha_2 \neq 0,\ \beta_2 \neq 0. \tag{32}$$

Step 4. Solving the system of algebraic equations from the previous step with conditions (32) with the aid of MATHEMATICA yields *four sets of solutions*:

- Set 1.

$$\begin{gathered}\alpha_0 = -2\lambda^2 - 16\mu + V, \quad \alpha_1 = -24\lambda, \quad \alpha_2 = -24, \\ \beta_0 = \sqrt{2}\left(-\lambda^2 - 8\mu + 2V\right), \quad \beta_1 = -12\sqrt{2}\lambda, \quad \beta_2 = -12\sqrt{2},\end{gathered} \tag{33}$$

  where $\lambda,\ \mu$ and V are arbitrary constants.
- Set 2.

$$\begin{gathered}\alpha_0 = -2\lambda^2 - 16\mu + V, \quad \alpha_1 = -24\lambda, \quad \alpha_2 = -24, \\ \beta_0 = \sqrt{2}\left(\lambda^2 + 8\mu - 2V\right), \quad \beta_1 = 12\sqrt{2}\lambda, \quad \beta_2 = 12\sqrt{2},\end{gathered} \tag{34}$$

  where $\lambda,\ \mu$ and V are arbitrary constants.
- Set 3.

$$\begin{gathered}\mu = 0, \quad \alpha_0 = V - 2\lambda^2, \quad \alpha_1 = -24\lambda, \quad \alpha_2 = -24, \\ \beta_0 = 2\sqrt{2}V - \sqrt{2}\lambda^2, \quad \beta_1 = -12\sqrt{2}\lambda, \quad \beta_2 = -12\sqrt{2},\end{gathered} \tag{35}$$

  where λ and V are arbitrary constants.
- Set 4.

$$\begin{gathered}\mu = 0, \quad \alpha_0 = V - 2\lambda^2, \quad \alpha_1 = -24\lambda, \quad \alpha_2 = -24, \\ \beta_0 = \sqrt{2}\lambda^2 - 2\sqrt{2}V, \quad \beta_1 = 12\sqrt{2}\lambda, \quad \beta_2 = 12\sqrt{2}\end{gathered} \tag{36}$$

  where λ and V are arbitrary constants.

Finally, substituting solutions (33)–(36) with the general solution to linear ODE (5) into representation (31) we obtain *four separate sets* of traveling wave solutions to the KdV type dynamic system (27) as follows.

Solutions set 1. Constants set (33) yields three families of solutions:

- when $\lambda^2 - 4\mu > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & -\frac{6\left(A_1^2 - A_2^2\right)\sigma}{\left(A_1 \sinh\frac{\xi\sqrt{\sigma}}{2} + A_2 \cosh\frac{\xi\sqrt{\sigma}}{2}\right)^2} - 2\lambda^2 + 8\mu + V, \\ v(\xi) = & -\frac{3\sqrt{2}\left(A_1^2 - A_2^2\right)\sigma}{\left(A_1 \sinh\frac{\xi\sqrt{\sigma}}{2} + A_2 \cosh\frac{\xi\sqrt{\sigma}}{2}\right)^2} - \sqrt{2}\sigma + 2\sqrt{2}V, \end{cases} \tag{37}$$

  where $\xi = x - Vt,\ \sigma = \lambda^2 - 4\mu,\ \ A_1, A_2, V$ are arbitrary constants;
- when $\lambda^2 - 4\mu = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & \frac{A_2^2\left(\xi^2 V - 24\right) + 2A_2A_1\xi V + A_1^2 V}{(A_2\xi + A_1)^2}, \\ v(\xi) = & 2\sqrt{2}\left(V - \frac{6A_2^2}{(A_2\xi + A_1)^2}\right), \end{cases} \tag{38}$$

  where $\xi = x - Vt,\ \ A_1, A_2, V$ are arbitrary constants;

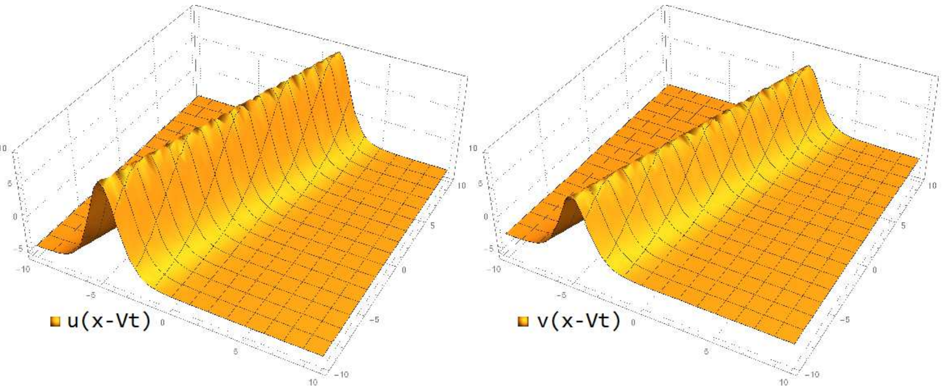

FIG. 8. Hyperbolic functions solution (37) when $A_1 = 1$, $A_2 = 1.2,\ \lambda = 2.5,\ \mu = 1,\ V = 0.3$

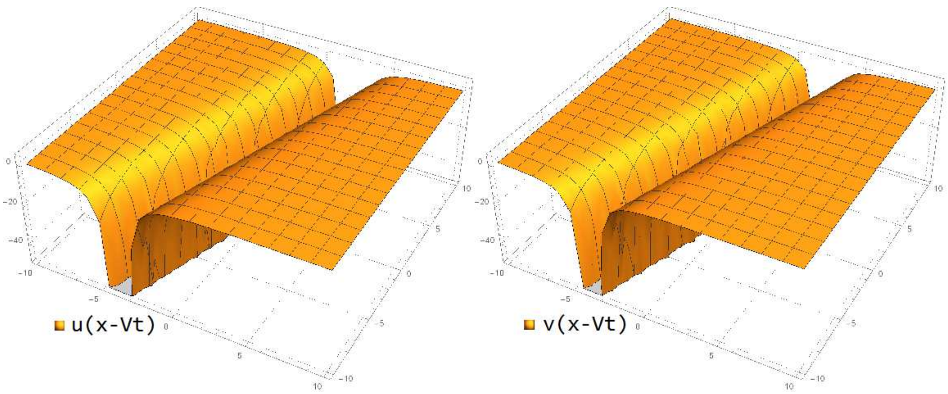

FIG. 9. Rational functions solution (38) when $A_1 = 1,\ A_2 = 1.2,\ \lambda = 2,\ \mu = 1,\ V = 0.3$

– when $\lambda^2 - 4\mu < 0$, we get the family of trigonometric functions solutions

$$\begin{cases} u(\xi) = & -\frac{6\sigma\left(A_1^2+A_2^2\right)}{\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2} + A_2 \cos\frac{\xi\sqrt{\sigma}}{2}\right)^2} - 2\lambda^2 + 8\mu + V, \\ v(\xi) = & -\frac{3\sqrt{2}\left(A_1^2+A_2^2\right)\sigma}{\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2} + A_2 \cos\frac{\xi\sqrt{\sigma}}{2}\right)^2} + \sqrt{2}\sigma + 2\sqrt{2}V, \end{cases} \tag{39}$$

where $\xi = x - Vt,\ \sigma = 4\mu - \lambda^2,\ \ A_1, A_2, V$ are arbitrary constants.

Solutions set 2. Constants set (34) yields three families of solutions:

– when $\lambda^2 - 4\mu > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & -\frac{6\left(A_1^2-A_2^2\right)\sigma}{\left(A_1 \sinh\frac{\xi\sqrt{\sigma}}{2} + A_2 \cosh\frac{\xi\sqrt{\sigma}}{2}\right)^2} - 2\lambda^2 + 8\mu + V, \\ v(\xi) = & \frac{3\sqrt{2}\left(A_1^2-A_2^2\right)\sigma}{\left(A_1 \sinh\frac{\xi\sqrt{\sigma}}{2} + A_2 \cosh\frac{\xi\sqrt{\sigma}}{2}\right)^2} + \sqrt{2}\sigma - 2\sqrt{2}V, \end{cases} \tag{40}$$

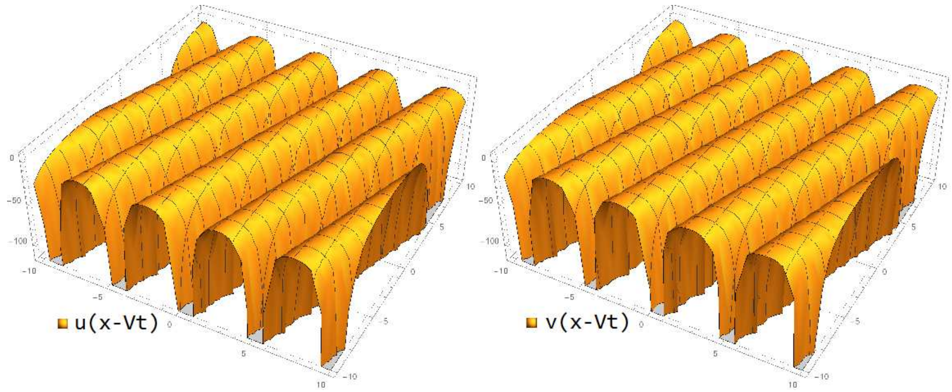

FIG. 10. Trigonometric functions solution (39) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 1.5$, $\mu = 1$, $V = 0.3$

where $\xi = x - Vt$, $\sigma = \lambda^2 - 4\mu$, A_1, A_2, V are arbitrary constants; in particular, setting $A_1 = 0$, $\sigma = 4k_1^2$, $V = a_{10} - 16k_1^2$, we obtain exactly the soliton solution, found by means of the tanh – method in [14].

– when $\lambda^2 - 4\mu = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & \frac{A_2^2\left(\xi^2 V - 24\right) + 2A_2A_1\xi V + A_1^2 V}{(A_2\xi + A_1)^2}, \\ v(\xi) = & 2\sqrt{2}\left(\frac{6A_2^2}{(A_2\xi + A_1)^2} - V\right), \end{cases} \tag{41}$$

where $\xi = x - Vt$, A_1, A_2, V are arbitrary constants;

– when $\lambda^2 - 4\mu < 0$, we get the family of trigonometric functions solutions

$$\begin{cases} u(\xi) = & -\frac{6\left(A_1^2 + A_2^2\right)\sigma}{\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2} + A_2 \cos\frac{\xi\sqrt{\sigma}}{2}\right)^2} - 2\lambda^2 + 8\mu + V, \\ v(\xi) = & \frac{3\sqrt{2}\left(A_1^2 + A_2^2\right)\sigma}{\left(A_1 \sin\frac{\xi\sqrt{\sigma}}{2} + A_2 \cos\frac{\xi\sqrt{\sigma}}{2}\right)^2} - \sqrt{2}\sigma - 2\sqrt{2}V, \end{cases} \tag{42}$$

where $\xi = x - Vt$, $\sigma = 4\mu - \lambda^2$, A_1, A_2, V are arbitrary constants.

Solutions set 3. Constants set (35) yields two families of solutions:

– when $\lambda > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & \frac{\left(V - 2\lambda^2\right)\left(2A_1A_2 \sinh\xi|\lambda| + \left(A_1^2 + A_2^2\right)\cosh\lambda\xi\right) - \left(A_1^2 - A_2^2\right)\left(10\lambda^2 + V\right)}{2\left(A_1 \sinh\frac{\xi|\lambda|}{2} + A_2 \cosh\frac{\lambda\xi}{2}\right)^2}, \\ v(\xi) = & \frac{\left(2V - \lambda^2\right)\left(2A_1A_2 \sinh\xi|\lambda| + \left(A_1^2 + A_2^2\right)\cosh\lambda\xi\right) - \left(A_1^2 - A_2^2\right)\left(5\lambda^2 + 2V\right)}{\sqrt{2}\left(A_1 \sinh\frac{\xi|\lambda|}{2} + A_2 \cosh\frac{\lambda\xi}{2}\right)^2}, \end{cases} \tag{43}$$

where $\xi = x - Vt$, A_1, A_2, V are arbitrary constants;

– when $\lambda = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & \frac{A_2^2\left(\xi^2 V - 24\right) + 2A_2A_1\xi V + A_1^2 V}{(A_2\xi + A_1)^2}, \\ v(\xi) = & 2\sqrt{2}\left(V - \frac{6A_2^2}{(A_2\xi + A_1)^2}\right), \end{cases} \tag{44}$$

where $\xi = x - Vt$, A_1, A_2, V are arbitrary constants.

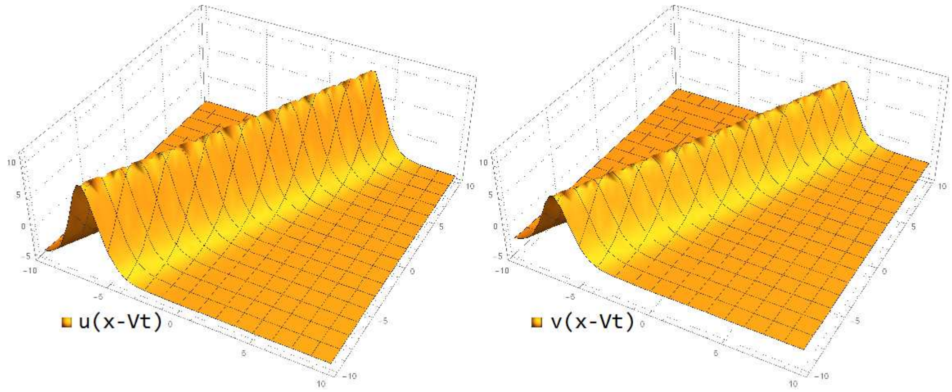

FIG. 11. Hyperbolic functions solution (43) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 1.5$, $V = 0.5$

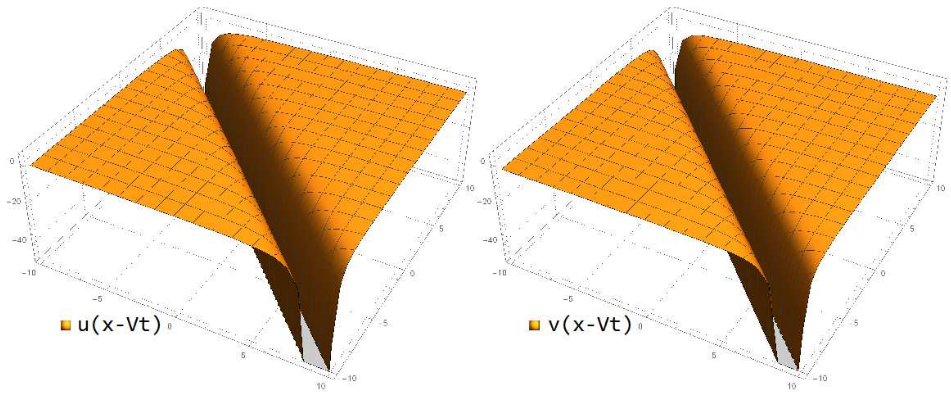

FIG. 12. Rational functions solution (44) when $A_1 = 1$, $A_2 = 1.2$, $\lambda = 0$, $V = -1$

Solutions set 4. Constants set (36) yields two families of solutions:

– when $\lambda > 0$, we get the family of hyperbolic functions solutions

$$\begin{cases} u(\xi) = & \frac{\left(V-2\lambda^2\right)\left(2A_1A_2\sinh\xi|\lambda|+\left(A_1^2+A_2^2\right)\cosh\lambda\xi\right)-\left(A_1^2-A_2^2\right)\left(10\lambda^2+V\right)}{2\left(A_1\sinh\frac{\xi|\lambda|}{2}+A_2\cosh\frac{\lambda\xi}{2}\right)^2}, \\ v(\xi) = & \frac{\left(\lambda^2-2V\right)\left(2A_1A_2\sinh\xi|\lambda|+\left(A_1^2+A_2^2\right)\cosh\lambda\xi\right)+\left(A_1^2-A_2^2\right)\left(5\lambda^2+2V\right)}{\sqrt{2}\left(A_1\sinh\frac{\xi|\lambda|}{2}+A_2\cosh\frac{\lambda\xi}{2}\right)^2}, \end{cases} \tag{45}$$

where $\xi = x - Vt$, A_1, A_2, V are arbitrary constants;

– when $\lambda = 0$, we get the family of rational functions solutions

$$\begin{cases} u(\xi) = & \frac{A_2^2\left(\xi^2V-24\right)+2A_2A_1\xi V+A_1^2V}{(A_2\xi+A_1)^2}, \\ v(\xi) = & 2\sqrt{2}\left(\frac{6A_2^2}{(A_2\xi+A_1)^2}-V\right), \end{cases} \tag{46}$$

where $\xi = x - Vt$, A_1, A_2, V are arbitrary constants.

6. Conclusion

The (G'/G) – expansion method was successfully used to derive exact traveling wave solutions to two KdV type nonlinear dynamic systems [1, 3].

The method was implemented in computer system MATHEMATICA, with the aid of which we obtained the solutions in the form of hyperbolic, rational and trigonometric functions for both systems. Moreover, it is shown that with a certain choice of arbitrary parameters in both systems it is possible to rediscover the soliton solutions, found by means of the tanh – method in [14], and hence the solutions obtained in the present paper are of more general forms.

The correctness of the obtained results was assured by putting them back into the original systems with the aid of MATHEMATICA. Most of the obtained solutions were graphically analyzed.

The main advantage of the method is that it provides solutions with relatively many arbitrary parameters, and thus these solutions are often more general compared to other analytical methods. As it was shown in Section 3, there exist certain modifications of the method to provide solutions in more general form in comparison with the classical $\left(\frac{G'}{G}\right)$ – expansion method [15], therefore the authors plan to use them for further investigations.

Finally, the method is confirmed to be suitable for implementation in modern computer algebra systems.

Bibliography

1. Drinfeld V. G. Lie algebras and equations of Korteweg–de Vries type / V. G. Drinfeld, V. V. Sokolov // Itogi Nauki i Tekhniki. – Ser. Sovrem. Probl. Mat. Nov. Dostizh. – M.: VINITI, 1984. – № 15. – P. 81-180. (in Russin).
2. Guo S. The extended (G'/G) – expansion method and its applications to the Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations / S. Guo, Y. Zhou // Applied Mathematics and Computation. – 2010. – № 215. – P. 3214-3221.
3. Hentosh O. Ye. Differential-Geometric and Lie-Algebraic Foundations of Investigating Nonlinear Dynamical Systems on Functional Manifolds / O. Ye. Hentosh, M. M. Prytula, A. K. Prykarpatskyi. – Lviv: Publishing center of the Ivan Franko National University of Lviv, 2006. – 408 p. (in Ukrainian).
4. Jabbari A. A generalization of (G'/G) – expansion method and its application to nonlinear reaction–diffusion equations arising in mathematical biology / A. Jabbari, J. Manafian Heris, H. Kheiri // International Journal of Biomathematics. – 2014. – Vol. 7. – № 3. – 10 p.
5. Liu X. Application of Improved (G'/G) – Expansion Method to Traveling Wave Solutions of Two Nonlinear Evolution Equations / X. Liu, W. Zhang, Z. Li // Advances in Applied Mathematics and Mechanics. – 2012. – Vol. 4. – № 1. – P. 122-130.
6. Malfliet W. Solitary wave solutions of nonlinear wave equation / W. Malfliet // Am. J. Phys. – 1992. – Vol. 60. – P. 650-654.
7. Naher H. Algorithms and Some New Traveling Wave Solutions via New Extension of (G'/G) – expansion Method of Coupled Nonlinear Equations / H. Naher // British Journal of Mathematics and Computer Science. – 2015. – № 11. – P. 1-13.
8. Naher H. New approach of (G'/G) – expansion method and new approach of generalized (G'/G) – expansion method for nonlinear evolution equation / H. Naher, F. Abdullah // AIP ADVANCES. – 2013. – № 3. – 15 p.
9. Naher H. New approach of (G'/G) – expansion method and new approach of generalized (G'/G) – expansion method for ZKBBM equation / H. Naher // Journal of Egyptian Mathematical Society. – 2015 – Vol. 23, № 1. – C. 42-48.

10. Naher H. New Approach of (G'/G) – expansion Method for RLW Equation / H. Naher, F. Abdullah // Research Journal of Applied Sciences, Engineering and Technology. – 2014. – № 7. – P. 4864-4871.
11. Nur Alam M. Application of New Generalized (G'/G) –Expansion Method to the (3+1)-Dimensional Kadomtsev-Petviashvili Equation / M. Nur Alam, M. Ali Akbar, M. Hafez, F. Belgacem // Italian journal of pure and applied mathematics. – 2016. – № 36. – P. 415-428.
12. Nur Alam M. Application of the Novel (G'/G) – Expansion Method to the Regularized Long Wave Equation / M. Nur Alam, F. Belgacem // Waves Wavelets Fractals Adv. Anal. – 2015. – № 1. – P. 51-67.
13. Shehata A. The traveling wave solutions of the perturbed nonlinear Schrodinger equation and the cubic–quintic Ginzburg Landau equation using the modified (G'/G) – expansion method / A. R. Shehata // Applied Mathem. and Computation. – 2010. – № 217. – P. 1-10.
14. Starchak M. Soliton solutions of nonlinear dynamical systems of KdV–type equations / M. Starchak, M. Prytula // Visnyk of the Lviv University. – Series Applied Mathematics and Computer Science. – 2009. – Vol. 15. – P. 71-80. (in Ukrainian).
15. Wang M. The (G'/G) – expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics / M. Wang, X. Li, J. Zhang // Physics Letters. – 2008. – № 372. – P. 417-423.
16. Zayed E. Applications of an Extended (G'/G) – Expansion Method to Find Exact Solutions of Nonlinear PDEs in Mathematical Physics / E. Zayed, S. Al-Joudi // Mathematical Problems in Engineering. – 2010. – Vol. 2010. – 19 p.
17. Zayed E. Applications of an Improved (G'/G) – expansion Method to Nonlinear PDEs in Mathematical Physics / E. Zayed, S. Al-Joudi // Numerical Analysis and Applied Mathematics, International Conference. – 2009. – Vol. 1. – P. 371-376.
18. Zhang J. A generalized (G'/G) – expansion method and its applications / J. Zhang, X. Wei, Y. Lu // Physics Letters A. – 2008. – № 372. – P. 3653-3658.
19. Zhang S. A generalized (G'/G) – expansion method and its application to the (2+1)-dimensional Broer–Kaup equations / S. Zhang, W. Wang, J. Tong // Applied Mathematics and Computation. – 2009. – № 209. – P. 399-404.
20. Zhang S. A generalized (G'/G) – expansion method for the mKdV equation with variable coefficients / S. Zhang, J. Tong, W. Wang // Physics Letters A. – 2008. – № 372. – P. 2254-2257.
21. Zhou Y. Application of Modified (G'/G) – Expansion Method to Traveling Wave Solutions for Whitham Broer Kaup-Like Equations / Y. Zhou, C. Li // Commun. Theor. Phys. Beijing, China. – 2009. – Vol. 51. – C. 664-670.

Ivan Mykhailiuk, Mykola Prytula,
Faculty of Applied Mathematics and Informatics,
Ivan Franko National University of Lviv,
1, Universytets'ka Str., Lviv, 79000, Ukraine.

Received 05.09.2017; revised 27.10.2017