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DETERMINATION THE QUANTITY OF EIGENVALUE FOR TWO-PARAMETER EIGENVALUE PROBLEMS IN THE PRESCRIBED REGION

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Резюме. Запропоновано алгоритм знаходження кількості власних значень двопараметричних спектральних задач у деякій заданій області. В основі алгоритму лежить принцип аргумента аналітичної функції однієї змінної. Наведено чисельні результати для нелінійної двопараметричної задачі на власні значення.

Abstract. An algorithm for finding the number of eigenvalues of two-parameter spectral problems in a given region is proposed. At the heart of the algorithm lies the principle of the argument of the analytic function of one variable. Numerical results for a nonlinear two-parameter eigenvalue problems are given.

1. Introduction

The multiparameter eigenvalue problems $T(\lambda)x = 0$ with operator-valued functions $T(\lambda) : \ R^m \to L(H)$ ($L(H)$ – the set of linear bounded operators operating in a finite-dimensional Hilbert space$H$), which depends on several spectral parameters λ, have a classical analysis of their source. In particular, they arise in solving boundary value problems for differential equations with partial derivatives by separating the variables.

In abstract formulation, they are written in the form of a system of equations

$$T(\lambda)u \equiv \left(A_k - \sum_{i=1}^{m} \lambda_i B_{ki}\right) u_k = 0, \quad k = 1, 2, ..., m, \tag{1}$$

if the operator-function $T(\lambda)$ linearly depends on the spectral parameters $\lambda_i \in R$, $i = 1, 2, \dots, m$, $A, B_i, A_k, B_{ki} \in L(H)$, $k, i = 1, 2, \dots, m$.

An algebraic two-parameter eigenvalue problem as a partial case of a spectral problem (1) is written in the form of a system of two homogeneous linear equations

$$\begin{aligned} T_1(\lambda, \mu) &\equiv (A_1 + \lambda B_1 + \mu C_1)x = 0, \\ T_2(\lambda, \mu) &\equiv (A_2 + \lambda B_2 + \mu C_2)y = 0, \end{aligned} \tag{2}$$

where $A_i, \ B_i, \ C_i$ are the square matrices of the nth order. We will define our eigenvalue sets (in our case that are eigen pairs $(\lambda, \ \mu)$) such that the system (2) has non-trivial solutions $x \neq 0$ and $y \neq 0$.

Key words. Two-parameter eigenvalue problem, number of eigenvalues, principle of argument.

It is obvious that own pairs are solutions of the system of two nonlinear algebraic equations

$$\begin{aligned} f(\lambda,\mu) &\equiv \det(A_1+\lambda B_1+\mu C_1)=0. \\ g(\lambda,\mu) &\equiv \det(A_2+\lambda B_2+\mu C_2)=0. \end{aligned} \tag{3}$$

In this work the problem of finding the number of real roots of the system (3), which are in a certain region of the change of spectral parameters (λ, μ), is considered.

2. Preliminaries

An algorithm for finding the number of zeros of an analytic function in a given region, as well as some approximations to each of them, which can then be specified using iterative methods, in particular by the Newton method or its two-way analogues (see, for example, [5, 9]), is based on the ratio, which implies, in particular, the principle of the argument of the analytic function (see, for example, [2]):

Integral $\frac{1}{2\pi i}\int\limits_{\Gamma}\varphi(\lambda)\frac{f'(\lambda)}{f(\lambda)}\,d\lambda$ *is equal to the difference between the sum of values that takes the function* $\varphi(\lambda)$ *in the zeros of the function* $f(\lambda)$ *lying in inside the domain* G*, bounded by the curve* Γ *and the sum of the values that takes the same function* $\varphi(\lambda)$ *in the poles of the function* $f(\lambda)$ *that lying in inside of* Γ*, that is,*

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\varphi(\lambda)\frac{f'(\lambda)}{f(\lambda)}\,d\lambda \;=\sum_{j=1}^{m}\nu_j\varphi(\alpha_j)-\sum_{j=1}^{n}\mu_j\varphi(\beta_j). \tag{4}$$

Here $\varphi(\lambda)$ is an analytic function in the domain G; $f(\lambda)$ is analytic in G everywhere, except for the finite number of poles $\beta_j\in G$, $j=1,\ 2,\ \ldots,\ n$, and $f(\lambda)\neq 0$ in G everywhere except for the finite number of zeros $\alpha_j\in G$, $j=1,\ 2,\ \ldots,\ m$; ν_j and μ_j is the multiplicity of zero and the order of the pole, respectively.

In particular, if we take $\varphi(\lambda)\equiv 1$, then we get that

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f'(\lambda)}{f(\lambda)}\,d\lambda \;=\sum_{j=1}^{m}\nu_j-\sum_{j=1}^{n}\mu_j, \tag{5}$$

that is, the integral is equal to the difference between the number of zeros and the poles of function $f(\lambda)$ lying inside of Γ, taking into account their multiplicities (the so-called principle of the argument).

If the analytic function $f(\lambda)$ does not have poles in G, then the principle of argument (5) allows us to determine the number of all its zeros that lie in the domain G. However, this does not allow you to localize each of them.

To locate the zeros we use again the relation (4). Taking now $\varphi(\lambda)=\lambda^k$, $k=1,2,\ \ldots$, we get the following statement.

Suppose that the analytic function $f(\lambda)$ *does not have poles in* G*, but has in* G*, taking into account the multiplicity, the* m *zeros* $\lambda_1,\lambda_2,\ldots,\lambda_m$ *and has no zeros on the boundary* Γ *of the domain* G*, then the number* m *is determined in* *accordance with the principle of the argument*

$$m = s_0 = \frac{1}{2\pi\, i}\int\limits_{\Gamma} \frac{f'(\lambda)}{f(\lambda)}\, d\lambda \tag{6}$$

and the relationship is true

$$\sum_{j=1}^{m} (\lambda_j)^k = s_k\,, \quad k = 1,\ \dots\ , m, \tag{7}$$

where

$$s_k = \frac{1}{2\pi\, i}\int\limits_{\Gamma} \lambda^k \frac{f'(\lambda)}{f(\lambda)}\, d\lambda, \qquad k = 1,\ 2,\ \dots \quad . \tag{8}$$

The right-hand side of (7) is nothing but symmetrical functions of the roots $\lambda_1, \lambda_2, \dots, \lambda_m$ inside of Γ, from which, in principle, roots can be found, for example:

If $m = 1$ than

$$\lambda_1 = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\lambda f'(\lambda)}{f(\lambda)} d\lambda.$$

If $m = 2$

$$s_1 \equiv \lambda_1 + \lambda_2 = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\lambda f'(\lambda)}{f(\lambda)} d\lambda$$

and

$$s_2 \equiv \lambda_1^2 + \lambda_2^2 = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\lambda^2 f'(\lambda)}{f(\lambda)} d\lambda.$$

This will give us $\lambda_1\lambda_2 = \frac{1}{2}(\lambda_1 + \lambda_2)^2 - \frac{1}{2}(\lambda_1^2 + \lambda_2^2)$ and, consequently, we find λ_1 and λ_2 by solving a square equation. This procedure can be continued in an obvious way for $m = k$. Another approach, when the system (7) is solved directly, it was considered in the work [6, 7].

For the functions of one variable or one-parameter spectral problems, the principle of the argument (6) and the formulae of the principle of argument (7) and (8) have been repeatedly used for solving various problems (see, for example, [1, 3, 4, 6-8, 10]).

In this paper, based on the principle of the argument of the function of one variable, the algorithm for finding the number of eigenvalues of a two-parameter spectral problem in a given region of changing of the spectral parameters is proposed.

3. Number roots of a system of two real equations with two real variables

Let us consider a two-parameter spectral problem (2), whose eigenvalues λ, μ we will seek as the roots of the system of nonlinear equations (3), where the functions $f(\lambda, \mu)$ and $g(\lambda, \mu)$ are real functions of real variables.

For this purpose we will construct the function $u = f + ig$ and we will require that it be analytic and have no poles inside a certain region G. Then, as is known, the number m of roots $\nu = \lambda + i\mu$ of a function u in the region G, which is bounded by a curve Γ, that is, common solutions $(\lambda,\ \mu)$ of equations $f(\lambda,\mu) = 0$, $g(\lambda,\mu) = 0$, follows from the principele of argument of the analytic function (6), that is,

$$m = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{u'(\nu)}{u(\nu)} d\nu.$$

Taking into account that

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{u'(\nu)}{u(\nu)} d\nu = \frac{1}{2\pi i}\int\limits_{\Gamma} d\log u(\nu) = \frac{1}{2\pi}\int\limits_{\Gamma} d\phi,$$

where

$$\phi = \arg\log u(\nu) = \arctan\frac{g}{f} + \pi n, \tag{9}$$

we obtaine

$$m = \frac{1}{2\pi}\int\limits_{\Gamma} d(\arctan\frac{g}{f} + n\pi).$$

Consider the curve Γ with its parametric representation $\lambda = \lambda(t)$; $\mu = \mu(t)$; $0 \leqslant t \leqslant 1$. From (9) we have

$$d\phi = \frac{gdf - fdg}{f^2 + g^2}$$

If ϕ we replace the differentiation by t, we obtain

$$\frac{1}{2\pi}\int\limits_{\Gamma} d\phi = \frac{1}{2\pi}\int\limits_{\Gamma} \frac{d\phi}{dt} dt$$

Moreover, if we consider our expression $d\phi$ along the curve, we will have:

$$\frac{1}{2\pi}\int\limits_{\Gamma} \frac{d\phi}{dt} dt = \int\limits_{0}^{1} \frac{g\left(\frac{df}{d\lambda}\frac{d\lambda}{dt} + \frac{df}{d\mu}\frac{d\mu}{dt}\right) - f\left(\frac{dg}{d\lambda}\frac{d\lambda}{dt} + \frac{dg}{d\mu}\frac{d\mu}{dt}\right)}{f^2 + g^2} dt \tag{10}$$

Consequently, the number of eigenvalues m of the system of equations (3) is calculated by formula (10), in which the integral is replaced by some quadrature formula, for example, rectangles.

4. Numerical example

Let us consider a nonlinear two-parameter spectral problem

$$\begin{aligned} T_1(\lambda,\mu)\mathbf{x} &\equiv \begin{pmatrix} \lambda^2 - \mu^2 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x} = 0, \ \ \mathbf{x} \in R^2, \\ T_2(\lambda,\mu)\mathbf{y} &\equiv \begin{pmatrix} 2\lambda & 2 \\ 1 & 1 \end{pmatrix}\mathbf{y} = 0, \ \ \mathbf{y} \in R^2, \end{aligned} \tag{11}$$

and calculate the number of eigenvalues lying in different areas.

As was noted above, the eigenvalues of the problem (11) are solutions of the system of two nonlinear algebraic equations

$$\begin{aligned} f(\lambda,\mu) &= \det T_1(\lambda,\mu) = \lambda^2 - \mu^2 - 1 = 0, \\ g(\lambda,\mu) &= \det T_2(\lambda,\mu) = 2\lambda\mu - 2 = 0. \end{aligned} \tag{12}$$

It is easy to verify that the system (12) has two solutions:

$$(\lambda,\mu)_{1,2} = (\pm 1,272; 0,786).$$

The number of solutions m of the system (12) was calculated by the formula (10), in which the integral was replaced by the quadrature formula of rectangles, and the circle with center (λ^*,μ^*) and radius ρ^* was chosen as the boundary of Γ. The value of the functions (determinant) f and g and their derivatives on the boundary of the region (circle) were calculated on the basis of the LU-decomposition of the matrices $T_1(\lambda,\mu)$ and $T_2(\lambda,\mu)$ [6, 7].

Numerical calculations are carried out for different choices of the radius of circle and its center. The results are presented in Table 1. The first column of the table shows the coordinates of the center (λ^*,μ^*) of the circle, in the second column is its radius ρ^*, and in the third the number m of eigenvalues lying in that circle.

TABL. 1. Number eigenvalues of the problem (4.1)

(λ^*,μ^*)	ρ^*	m	(λ^*,μ^*)	ρ^*	m
(0.0, 0.0)	1.0	0	(0.0, 1.0)	2.0	2
(0.0, 1.0)	1.0	0	(0.0, 1.0)	2.0	1
(1.0, 1.0)	1.0	1	(-1.0, 0.0)	2.0	1

5. Conclusion

In this paper, based on the principle of the argument of the analytic function of one variable, an algorithm for finding the number of real eigenvalues of the system of two determinantal equations, that is, the real eigenvalues of a two-parameter spectral problem in a given region of changing of the spectral parameters, is proposed.

The numerical experiments performed for various problems have shown the effectiveness of the algorithm in the sense that for calculating the number of eigenvalues in a given region, there is no need for great accuracy in the calculation of the integral, and this does not require, in turn, a large partition of the integration boundary. This significantly reduces the calculation time, but, at the same time, it is sensitive to the choice of the boundary of the area. The algorithm ceases to work when the eigenvalues (though one) falls on the boundary that we preset. In this case, it is necessary to correct the boundary.

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