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CONSTRUCTION OF TWO-SIDED APPROXIMATIONS TO POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SEMILINEAR ELLIPTIC SYSTEMS

M. V. Sidorov

Резюме. Розглядається однорідна задача Діріхле для системи напівлінійних еліптичних рівнянь. Для побудови двобічних наближень до додатного розв'язку цієї системи використовуються методи теорії напівупорядкованих просторів, зокрема, результати В. І. Опойцева про розв'язність операторних рівнянь з гетеротонним оператором. Можливості і ефективність розробленого метода продемонстрована обчислювальним експериментом для системи Лане-Емдена.

Abstract. A homogeneous Dirichlet problem for a system of semilinear elliptic equations is considered. To construct two-sided approximations to a positive solution of this system, methods of the theory of semiordered spaces, in particular, the results of V.I. Opoĭcev on the solvability of operator equations with a heterotone operator are used. The possibilities and effectiveness of the developed method is demonstrated by a numerical experiment for the Lane-Emden system.

1. Introduction

Let us consider a homogeneous Dirichlet problem for a system of semilinear elliptic equations:

$$-\Delta u_i = f_i(\mathbf{x}, u_1, \ldots, u_n) \quad \text{in} \quad \Omega \subset \mathbb{R}^m, \tag{1}$$

$$u_i|_{\partial\Omega} = 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

or in a vector form

$$-\Delta \mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad \text{in} \quad \Omega \subset \mathbb{R}^m,$$

$$\mathbf{u}|_{\partial\Omega} = \boldsymbol{\theta},$$

where $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{u} = (u_1, \ldots, u_n)$, $-\Delta\mathbf{u} = (-\Delta u_1, \ldots, -\Delta u_n)$, $\mathbf{f} = (f_1, \ldots, f_n)$, $\boldsymbol{\theta} = (0, \ldots, 0)$, Δ is the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_m^2}.$$

Let us assume that $\Omega \subset \mathbb{R}^m$ is bounded domain with a piecewise smooth boundary $\partial\Omega$, functions $f_i(\mathbf{x}, \mathbf{u})$, $i = 1, 2, \ldots, n$, are non negative and continuous on the set of variables $\mathbf{x}$, $\mathbf{u}$, if $\mathbf{x} \in \Omega$, $u_i > 0$, $i = 1, 2, \ldots, n$.

The problem (1), (2) is a mathematical model of many stationary processes that are considered in chemical kinetics, biology, combustion theory etc. [12]. Many works [1,2,6,9,10,12,16, etc.] are devoted to the investigation of problem

Key words. Positive solution; semilinear elliptic systems; heterotone operator; two-sided approach; Lane-Emden system.

(1), (2). But the focus in these works was mainly on clarifying the conditions of existence and uniqueness of the positive solution of the problem or on the conditions of having a solution with radial symmetry for a case where Ω is a unit sphere, and an effective algorithm for numerical finding the solution was not proposed.

The purpose of this work is to develop the iterative methods for solving the boundary value problem (1), (2), which have a two-sided nature of convergence to the desired solution. Two-sided approximate methods of solving the nonlinear operator equations based on the theory of nonlinear operators in semiordered spaces were developed in [4, 5, 7, 8, 13, 14]. This work continues the research begun in [5] and distributes it to systems of nonlinear equations.

2. Some information from the theory of nonlinear operators in spaces with cones

Let us consider some concepts and facts from the theory of nonlinear operators in semiordered spaces that will be used further [7, 13, 14].

Let E be a real Banach space, and θ is a zero element of space E. A closed convex set $\mathcal{K} \subset E$ is called a cone, if from the fact that $x \in \mathcal{K}$, $x \neq \theta$, follows $\alpha x \in \mathcal{K}$ with $\alpha \geq 0$ and $-x \notin \mathcal{K}$.

Any cone $\mathcal{K} \subset E$ allows to enter in space E a semiordering by rule: $x \leqslant y$, if $y - x \in \mathcal{K}$. Elements $x \geqslant \theta$ (i.e. $x \in \mathcal{K}$) are called positive. The set of elements $\langle y, z\rangle$ of a semiordered space, which consists of those $x \in E$ for which $y \leqslant x \leqslant z$, is called a cone segment.

Normal cones are important class of cones for application of the theory of semiordered spaces in computational mathematics. A cone $\mathcal{K}$ is called normal if there exists a number $N(\mathcal{K}) > 0$, that from $\theta \leqslant x \leqslant y$ follows $\|x\| \leq N(\mathcal{K})\,\|y\|$. In this case, it is said that the norm is semimonotonic. If $N(\mathcal{K}) = 1$, then the cone is called acute and it is said that the norm is monotonous.

Let us consider the definitions of some classes of operators in spaces with cone.

The operator $T : E \to E$ is called positive if it leaves invariant the cone $\mathcal{K}$, i.e. $T(x) \in \mathcal{K}$ for anyone $x \in \mathcal{K}$.

The operator $T : E \to E$ is called heterotone (or mixed monotone [3, 13, etc.]), if it allows a diagonal representation $T(x) \equiv \hat{T}(x, x)$, where the companion operator $\hat{T} : E \times E \to E$ monotonically increases with respect to the first argument and decreases with respect to the second one, i.e.

a) if $y_1 \leqslant y_2$, then $\hat{T}(y_1, z) \leqslant \hat{T}(y_2, z)$ for all $z \in E$;

b) if $z_1 \leqslant z_2$, then $\hat{T}(y, z_1) \geqslant \hat{T}(y, z_2)$ for all $y \in E$.

A cone segment $\langle y_0, z_0\rangle$ is called strongly invariant for a heterotone operator T, if

$$\hat{T}(y_0, z_0) \geq y_0, \quad \hat{T}(z_0, y_0) \leq z_0.$$

Let us fixate some nonzero element $u_0 \in \mathcal{K}$ and denote by $K(u_0)$ a set of such elements $x \in \mathcal{K}$, for which we can specify such $\alpha, \beta > 0$, that

$$\alpha u_0 \leqslant x \leqslant \beta u_0.$$

A positive heterotone operator T is called pseudoconcave, if $\hat{T}(y,z) \in K(u_0)$ for any $y, z \in \mathcal{K}$, $y \neq \theta$, $z \neq \theta$, and for any $v, w \in K(u_0)$ i $\tau \in (0;1)$

$$\hat{T}\left(\tau v, \frac{1}{\tau} w\right) \geqslant \tau \hat{T}(v,w),$$

and the sign of equality is impossible here.

A pseudoconcave operator T is called u_0-pseudoconcave, if for any $v, w \in K(u_0)$ and $\tau \in (0;1)$ you can find such $\eta(v,w,\tau) > 0$, that

$$\hat{T}\left(\tau v, \frac{1}{\tau} w\right) \geqslant \tau[1 + \eta(v,w,\tau)]\hat{T}(v,w).$$

Properties and the problem of constructing approximate solutions of operator equations with a heterotone operator have been considered in [3, 4, 11, 13, 14]. In particular, the following assertion holds [13, 14]: if the cone $\mathcal{K}$ is normal, the operator $\hat{T}$ is completely continuous, for T there is a strongly invariant cone segment $\langle y_0, z_0 \rangle$, and the system $\hat{T}(y,z) = y$, $\hat{T}(z,y) = z$ on $\langle y_0, z_0 \rangle$ has no solutions such that $y \neq z$, then the iterative process, which is formed by the rule $y_{n+1} = \hat{T}(y_n, z_n)$, $z_{n+1} = \hat{T}(z_n, y_n)$, $n = 0, 1, 2, \ldots$, starting from the point (y_0, z_0), two-sided converges to the unique on $\langle y_0, z_0 \rangle$ fixed point x^* of the operator T:

$$y_0 \leqslant y_1 \leqslant \ldots \leqslant y_n \leqslant \ldots \leqslant x^* \leqslant \ldots \leqslant z_n \leqslant \ldots \leqslant z_1 \leqslant z_0.$$

It is known [13, 14], that the system $\hat{T}(y,z) = y$, $\hat{T}(z,y) = z$ on $\langle y_0, z_0 \rangle$ has no solutions such that $y \neq z$, if T – u_0-pseudoconcave operator.

3. Construction of two-sided approximations

To analyze the problem (1), (2) and construct two-sided approximations to its positive solution, we will use the methods of the theory of nonlinear operators in semiordered spaces [7, 13, 14].

Let $\mathbf{C}^n(\bar{\Omega}) = \{\mathbf{u} = (u_1, \ldots, u_n) : u_i \in C(\bar{\Omega}),\ i = 1, \ldots, n\}$ be a Banach space of continuous in $\bar{\Omega} = \Omega \cup \partial\Omega$ vector-valued functions with a norm $\|\mathbf{u}\|_n = \max\{\|u_1\|, \ldots, \|u_n\|\}$, where $\|u_i\| = \max\limits_{\mathbf{x} \in \bar{\Omega}} |u_i(\mathbf{x})|$. Let us define in $\mathbf{C}^n(\bar{\Omega})$ a cone

$$\boldsymbol{\mathcal{K}}_+ = \{\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{C}^n(\bar{\Omega}) : u_i(\mathbf{x}) \geq 0,\ \mathbf{x} \in \bar{\Omega},\ i = 1, \ldots, n\}$$

of vector-valued functions with non negative coordinates. Notice that the cone $\boldsymbol{\mathcal{K}}_+$ in $\mathbf{C}^n(\bar{\Omega})$ is normal (and even acute) [7, 13, 14].

Using cone $\boldsymbol{\mathcal{K}}_+$ in space $\mathbf{C}^n(\bar{\Omega})$ we introduce a semiordering by the rule: for $\mathbf{u}, \mathbf{v} \in \mathbf{C}^n(\bar{\Omega})$ $\mathbf{u} \leqslant \mathbf{v}$, if $\mathbf{v} - \mathbf{u} \in \boldsymbol{\mathcal{K}}_+$, i.e.

$$\mathbf{u} \leqslant \mathbf{v}, \text{ if } u_i(\mathbf{x}) \leq v_i(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega} \text{ and for } i = 1, \ldots, n.$$

From the problem (1), (2) we go over to the system of integral equations of Hammerstein

$$u_i(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}) f_i(\boldsymbol{\xi}, u_1(\boldsymbol{\xi}), \ldots, u_n(\boldsymbol{\xi}))\, d\boldsymbol{\xi}, \quad i = 1, \ldots, n, \tag{3}$$

or in a vector form

$$\mathbf{u}(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\mathbf{f}(\boldsymbol{\xi},\mathbf{u}(\boldsymbol{\xi}))d\boldsymbol{\xi},$$

where $G(\mathbf{x},\boldsymbol{\xi})$ is Green's function of the first boundary value problem for the operator $-\Delta$ in the domain Ω, $\mathbf{x} = (x_1,\dots,x_m)$, $\boldsymbol{\xi} = (\xi_1,\dots,\xi_m)$.

The solution (generalized) of the problem (1), (2) will be called the vector-valued function $\mathbf{u}^* \in \mathbf{C}^n(\bar{\Omega})$, which is the solution of the system (3).

Let us introduce a nonlinear integral operator $\mathbf{T}$ acting in $\mathbf{C}^n(\bar{\Omega})$ by the rule defined by the right-hand side of the system of equations (3):

$$\mathbf{T}(\mathbf{u}) = \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\mathbf{f}(\boldsymbol{\xi},\mathbf{u}(\boldsymbol{\xi}))d\boldsymbol{\xi} = \left(\int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})f_1(\boldsymbol{\xi},u_1(\boldsymbol{\xi}),\dots,u_n(\boldsymbol{\xi}))d\boldsymbol{\xi},\dots, \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})f_n(\boldsymbol{\xi},u_1(\boldsymbol{\xi}),\dots,u_n(\boldsymbol{\xi}))d\boldsymbol{\xi}\right). \tag{4}$$

Since $f_i(\mathbf{x},u_1,\dots,u_n) \geq 0$, if $\mathbf{x}\in\Omega$, $i=1,\dots,n$, and $G(\mathbf{x},\boldsymbol{\xi}) \geq 0$, $\mathbf{x},\boldsymbol{\xi}\in\Omega$, $\mathbf{x}\neq\boldsymbol{\xi}$, then the operator $\mathbf{T}$ is positive, that is, it leaves invariant a cone $\boldsymbol{\mathcal{K}}_+$: $\mathbf{T}(\boldsymbol{\mathcal{K}}_+) \subset \boldsymbol{\mathcal{K}}_+$.

Let us assume that the vector-valued function $\mathbf{f}(\mathbf{x},\mathbf{u})$ allows a diagonal representation $\mathbf{f}(\mathbf{x},\mathbf{u}) = \hat{\mathbf{f}}(\mathbf{x},\mathbf{u},\mathbf{u})$, where continuous on the set of variables $\mathbf{x}$, $\mathbf{v}$, $\mathbf{w}$ the functions $\hat{f}_i(\mathbf{x},\mathbf{v},\mathbf{w}) = \hat{f}_i(\mathbf{x},v_1,\dots,v_n,w_1,\dots,w_n)$ monotonically increases with respect to all v_i and monotonically decreases with respect to all w_i, $i=1,\dots,n$, for all $\mathbf{x}\in\Omega$. Then the operator $\mathbf{T}$ of the form (4) will be heterotone with the companion operator

$$\hat{\mathbf{T}}(\mathbf{v},\mathbf{w}) = \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{\mathbf{f}}(\boldsymbol{\xi},\mathbf{v}(\boldsymbol{\xi}),\mathbf{w}(\boldsymbol{\xi}))d\boldsymbol{\xi} = \left(\int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_1(\boldsymbol{\xi},v_1(\boldsymbol{\xi}),\dots,v_n(\boldsymbol{\xi}),w_1(\boldsymbol{\xi}),\dots,w_n(\boldsymbol{\xi}))d\boldsymbol{\xi},\dots, \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_n(\boldsymbol{\xi},v_1(\boldsymbol{\xi}),\dots,v_n(\boldsymbol{\xi}),w_1(\boldsymbol{\xi}),\dots,w_n(\boldsymbol{\xi}))d\boldsymbol{\xi}\right). \tag{5}$$

Operators $\mathbf{T}$ and $\hat{\mathbf{T}}$ are completely continuous [7,13,14].

In a cone $\boldsymbol{\mathcal{K}}_+$ we will define a strongly invariant cone segment $\langle\mathbf{v}^0,\mathbf{w}^0\rangle$ by conditions

$$\hat{\mathbf{T}}(\mathbf{v}^0,\mathbf{w}^0) \geqslant \mathbf{v}^0, \quad \hat{\mathbf{T}}(\mathbf{w}^0,\mathbf{v}^0) \leqslant \mathbf{w}^0,$$

i.e.

$$\int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, v_1^0(\boldsymbol{\xi}),\dots,v_n^0(\boldsymbol{\xi}), w_1^0(\boldsymbol{\xi}),\dots,w_n^0(\boldsymbol{\xi}))d\boldsymbol{\xi} \geq v_i^0(\mathbf{x}) \text{ for all } \mathbf{x}\in\bar{\Omega},$$

$$\int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, w_1^0(\boldsymbol{\xi}),\dots,w_n^0(\boldsymbol{\xi}), v_1^0(\boldsymbol{\xi}),\dots,v_n^0(\boldsymbol{\xi}))d\boldsymbol{\xi} \leq w_i^0(\mathbf{x}) \text{ for all } \mathbf{x}\in\bar{\Omega},$$

$i = 1,\dots,n.$

If the boundary $\partial\Omega$ of the domain Ω consists of a finite number of pieces of lines $\sigma_i(\mathbf{x}) = 0$, $i = 1, 2, \dots, s$, where each $\sigma_i(\mathbf{x})$ is an elementary function, then using the R-functions method [15] one can construct in the form of a single analytic expression an elementary function $\omega(\mathbf{x})$ such that:

a) $\omega(\mathbf{x}) > 0$ in Ω;
b) $\omega(\mathbf{x}) = 0$ on $\partial\Omega$;
c) $|\nabla\omega(\mathbf{x})| \neq 0$ on $\partial\Omega$.

Then a strongly invariant cone segment can be searched in the form

$$\left\langle \mathbf{v}^0, \mathbf{w}^0 \right\rangle == \left\langle \alpha\omega(\mathbf{x}), \beta\omega(\mathbf{x}) \right\rangle,$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n)$, $0 \leq \alpha_i < \beta_i$, satisfy the system of inequalities

$$\int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, \alpha_1\omega(\boldsymbol{\xi}),\dots,\alpha_n\omega(\boldsymbol{\xi}), \beta_1\omega(\boldsymbol{\xi}),\dots,\beta_n\omega(\boldsymbol{\xi}))d\boldsymbol{\xi} \geq \alpha_i\omega(\mathbf{x})$$

for all $\mathbf{x}\in\bar{\Omega}$,

$$\int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, \beta_1\omega(\boldsymbol{\xi}),\dots,\beta_n\omega(\boldsymbol{\xi}), \alpha_1\omega(\boldsymbol{\xi}),\dots,\alpha_n\omega(\boldsymbol{\xi}))d\boldsymbol{\xi} \leq \beta_i\omega(\mathbf{x})$$

for all $\mathbf{x}\in\bar{\Omega}$, $\quad i = 1,\dots,n$.

Let us create an iterative process according to the scheme

$$\mathbf{v}^{(k+1)} = \hat{\mathbf{T}}(\mathbf{v}^{(k)}, \mathbf{w}^{(k)}), \quad \mathbf{w}^{(k+1)} = \hat{\mathbf{T}}(\mathbf{w}^{(k)}, \mathbf{v}^{(k)}), \quad k = 0, 1, 2, \dots,$$
$$\mathbf{v}^{(0)} = \mathbf{v}^0, \quad \mathbf{w}^{(0)} = \mathbf{w}^0,$$

i.e.

$$v_i^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, v_1^{(k)}(\boldsymbol{\xi}),\dots,v_n^{(k)}(\boldsymbol{\xi}), w_1^{(k)}(\boldsymbol{\xi}),\dots,w_n^{(k)}(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad (6)$$

$$w_i^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, w_1^{(k)}(\boldsymbol{\xi}),\dots,w_n^{(k)}(\boldsymbol{\xi}), v_1^{(k)}(\boldsymbol{\xi}),\dots,v_n^{(k)}(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad (7)$$

$k = 0, 1, 2, \dots,$

$$v_i^{(0)}(\mathbf{x}) = v_i^0(\mathbf{x}), \quad w_i^{(0)}(\mathbf{x}) = w_i^0(\mathbf{x}), \quad i = 1,\dots,n. \quad (8)$$

Given that the strong invariance of the constructed cone segment and heterotony of the operator $\mathbf{T}$, for which operator $\hat{\mathbf{T}}$ is an companion one, we can conclude that the sequence $\{\mathbf{v}^{(k)}(\mathbf{x})\}$ does not decrease behind the cone $\boldsymbol{\mathcal{K}}_+$, and the sequence $\{\mathbf{w}^{(k)}(\mathbf{x})\}$ does not increase behind the cone $\mathcal{K}_+$. In addition, from the normality of the cone $\mathcal{K}_+$ and completely continuity of the operator $\hat{\mathbf{T}}$ implies the existence of limits $\mathbf{v}^*(\mathbf{x})$ and $\mathbf{w}^*(\mathbf{x})$ of these sequences. Thus, the following inequalities hold:

$$\mathbf{v}^0 = \mathbf{v}^{(0)} \leqslant \mathbf{v}^{(1)} \leqslant \ldots \leqslant \mathbf{v}^{(k)} \leqslant \ldots \leqslant \mathbf{v}^* \leqslant \\ \leqslant \mathbf{w}^* \leqslant \ldots \leqslant \mathbf{w}^{(k)} \leqslant \ldots \leqslant \mathbf{w}^{(1)} \leqslant \mathbf{w}^{(0)} = \mathbf{w}^0.$$

The vector-valued functions $\mathbf{v}^* = (v_1^*, \ldots, v_n^*)$ and $\mathbf{w}^* = (w_1^*, \ldots, w_n^*)$ are a solution of the system of equations

$$\mathbf{v}^* = \hat{\mathbf{T}}(\mathbf{v}^*, \mathbf{w}^*), \quad \mathbf{w}^* = \hat{\mathbf{T}}(\mathbf{w}^*, \mathbf{v}^*),$$

i.e. the systems

$$v_i^*(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}) \hat{f}_i(\boldsymbol{\xi}, v_1^*(\boldsymbol{\xi}), \ldots, v_n^*(\boldsymbol{\xi}), w_1^*(\boldsymbol{\xi}), \ldots, w_n^*(\boldsymbol{\xi})) d\boldsymbol{\xi},$$

$$w_i^*(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}) \hat{f}_i(\boldsymbol{\xi}, w_1^*(\boldsymbol{\xi}), \ldots, w_n^*(\boldsymbol{\xi}), v_1^*(\boldsymbol{\xi}), \ldots, v_n^*(\boldsymbol{\xi})) d\boldsymbol{\xi}, \ i = 1, \ldots, n.$$

If we have received that $\mathbf{v}^* = \mathbf{w}^* = \mathbf{u}^*$, then $\mathbf{u}^*$ is the unique on the cone segment $\langle \mathbf{v}^0, \mathbf{w}^0 \rangle$ fixed point of the operator $\mathbf{T}$, and hence, $\mathbf{u}^*$ is the unique on $\langle \mathbf{v}^0, \mathbf{w}^0 \rangle$ solution of the boundary value problem (1), (2).

Sufficient condition for the implementation of equality $\mathbf{v}^* = \mathbf{w}^*$ is the condition [3] of the existence of such $\alpha \in (0; 1)$, that

$$\left\| \hat{\mathbf{T}}(\mathbf{v}, \mathbf{w}) - \hat{\mathbf{T}}(\mathbf{w}, \mathbf{v}) \right\|_n \le \alpha \|\mathbf{v} - \mathbf{w}\|_n \text{ for all } \mathbf{v}, \mathbf{w} \in \langle \mathbf{v}^0, \mathbf{w}^0 \rangle.$$

Let the functions $\hat{f}_i(\mathbf{x}, v_1(\mathbf{x}), \ldots, v_n(\mathbf{x}), w_1(\mathbf{x}), \ldots, w_n(\mathbf{x}))$, $i = 1, \ldots, n$, for all positive numbers $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ and for all $\mathbf{x} \in \Omega$ satisfy the inequality

$$\left| \hat{f}_i(\mathbf{x}, \mathbf{v}, \mathbf{w}) - \hat{f}_i(\mathbf{x}, \mathbf{w}, \mathbf{v}) \right| \le L_i \max\{|v_1 - w_1|, \ldots, |v_n - w_n|\}, \\ i = 1, \ldots, n, \tag{9}$$

where $L_i > 0$, $i = 1, \ldots, n$.

Then there will be an estimate

$$\left\| \hat{\mathbf{T}}(\mathbf{v}, \mathbf{w}) - \hat{\mathbf{T}}(\mathbf{w}, \mathbf{v}) \right\|_n \le LM \|\mathbf{v} - \mathbf{w}\|_n, \tag{10}$$

where $L = \max\{L_1, \ldots, L_n\}$, $M = \max\limits_{x \in \bar{\Omega}} \int\limits_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}) d\boldsymbol{\xi}$.

In addition, on the basis of estimate (10) we obtain that

$$\left\| \mathbf{w}^{(k)} - \mathbf{v}^{(k)} \right\|_n = \left\| \hat{\mathbf{T}}(\mathbf{w}^{(k-1)}, \mathbf{v}^{(k-1)}) - \hat{\mathbf{T}}(\mathbf{v}^{(k-1)}, \mathbf{w}^{(k-1)}) \right\|_n \le \\ \le LM \left\| \mathbf{w}^{(k-1)} - \mathbf{v}^{(k-1)} \right\|_n \le \ldots \le (LM)^k \left\| \mathbf{w}^{(0)} - \mathbf{v}^{(0)} \right\|_n.$$

Hence, the following theorem holds.

Theorem 1. *Let a heterotone operator* $\mathbf{T}$ *of the form (4) for which operator* $\hat{\mathbf{T}}$ *of the form (5) is an companion one, has a strongly invariant cone segment* $\langle \mathbf{v}^0, \mathbf{w}^0 \rangle$ *and the inequalities (9) are executed, moreover* $LM < 1$*. Then the iteration process (6–(8) converges to the unique on* $\langle \mathbf{v}^0, \mathbf{w}^0 \rangle$ *solution* $\mathbf{u}^*$ *of the boundary value problem (1), (2), and the following inequalities*

$$\mathbf{v}^0 = \mathbf{v}^{(0)} \leqslant \mathbf{v}^{(1)} \leqslant \ldots \leqslant \mathbf{v}^{(k)} \leqslant \ldots \leqslant \mathbf{u}^* \leqslant$$
$$\leqslant \ldots \leqslant \mathbf{w}^{(k)} \leqslant \ldots \leqslant \mathbf{w}^{(1)} \leqslant \mathbf{w}^{(0)} = \mathbf{w}^0$$

are satisfied and

$$\left\|\mathbf{w}^{(k)} - \mathbf{v}^{(k)}\right\|_n \leq (LM)^k \left\|\mathbf{w}^{(0)} - \mathbf{v}^{(0)}\right\|_n. \tag{11}$$

Another condition that ensures the uniqueness of the positive solution of the boundary value problem (1), (2) is u_0-pseudoconcavity of the operator $\mathbf{T}$ of the form (4) [13, 14].

Suppose that for all positive numbers $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ and any $\tau \in (0, 1)$ the inequalities

$$\hat{f}_i\left(\mathbf{x}, \tau\mathbf{v}, \frac{1}{\tau}\mathbf{w}\right) > \tau \hat{f}_i(\mathbf{x}, \mathbf{v}, \mathbf{w}), \quad \mathbf{x} \in \Omega, \quad i = 1, \ldots, n, \tag{12}$$

are performed.

Let us denote $u_0(\mathbf{x}) = \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi})d\boldsymbol{\xi}$. Then [7, 13, 14] for any $\mathbf{v}, \mathbf{w} \in \mathcal{K}_+$ there are such $\alpha_i(\mathbf{v}, \mathbf{w}) > 0$, $\beta_i(\mathbf{v}, \mathbf{w}) > 0$, $\tilde{\alpha}_i(\mathbf{v}, \mathbf{w}) > 0$, $\tilde{\beta}_i(\mathbf{v}, \mathbf{w}) > 0$, $i = 1, \ldots, n$, that

$$\alpha_i(\mathbf{v}, \mathbf{w})u_0(\mathbf{x}) \leq \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi})\hat{f}_i\boldsymbol{\xi}, \mathbf{v}(\boldsymbol{\xi}), \mathbf{w}(\boldsymbol{\xi}))d\boldsymbol{\xi} \leq \beta_i(\mathbf{v}, \mathbf{w})u_0(\mathbf{x}), \ i = 1, \ldots, n,$$

$$\tilde{\alpha}_i(\mathbf{v}, \mathbf{w})u_0(\mathbf{x}) \leq \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi})\left[\hat{f}_i\left(\mathbf{x}, \tau\mathbf{v}, \frac{1}{\tau}\mathbf{w}\right) - \tau\hat{f}_i(\mathbf{x}, \mathbf{v}, \mathbf{w})\right]d\boldsymbol{\xi} \leq$$
$$\leq \tilde{\beta}_i(\mathbf{v}, \mathbf{w})u_0(\mathbf{x}), \ i = 1, \ldots, n.$$

Hence we will have that

$$\int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi})\hat{f}_i\left(\mathbf{x}, \tau\mathbf{v}, \frac{1}{\tau}\mathbf{w}\right)d\boldsymbol{\xi} \geq \tilde{\alpha}_i(\mathbf{v}, \mathbf{w})u_0(\mathbf{x}) + \tau\int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi})\hat{f}_i(\mathbf{x}, \mathbf{v}, \mathbf{w})d\boldsymbol{\xi} \geq$$
$$\geq \tau\left(1 + \frac{\tilde{\alpha}_i(\mathbf{v}, \mathbf{w})}{\tau\beta_i(\mathbf{v}, \mathbf{w})}\right)\int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi})\hat{f}_i(\mathbf{x}, \mathbf{v}, \mathbf{w})d\boldsymbol{\xi}, \ i = 1, \ldots, n,$$

i.e.

$$\hat{\mathbf{T}}\left(\tau\mathbf{v}, \frac{1}{\tau}\mathbf{w}\right) \geq \tau[1 + \eta(\mathbf{v}, \mathbf{w}, \tau)]\hat{\mathbf{T}}(\mathbf{v}, \mathbf{w}), \tag{13}$$

where $\eta(\mathbf{v}, \mathbf{w}, \tau) = \min\left\{\dfrac{\tilde{\alpha}_1(\mathbf{v}, \mathbf{w})}{\tau\beta_1(\mathbf{v}, \mathbf{w})}, \ldots, \dfrac{\tilde{\alpha}_n(\mathbf{v}, \mathbf{w})}{\tau\beta_n(\mathbf{v}, \mathbf{w})}\right\}$.

Inequality (13) means u_0-pseudoconcavity of the operator $\mathbf{T}$.

Hence, the following theorem holds.

Theorem 2. *Let a heterotone operator* $\mathbf{T}$ *of the form (4), for which the operator* $\hat{\mathbf{T}}$ *of the form (5) is an companion one, has a strongly invariant cone segment* $\langle \mathbf{v}^0, \mathbf{w}^0 \rangle$ *and the inequalities (12) are performed. Then the iteration process (6) – (8) converges to the unique positive solution* $\mathbf{u}^* \in \langle \mathbf{v}^0, \mathbf{w}^0 \rangle$ *of the boundary value problem (1), (2), and the following inequalities*

$$\mathbf{v}^0 = \mathbf{v}^{(0)} \leqslant \mathbf{v}^{(1)} \leqslant \ldots \leqslant \mathbf{v}^{(k)} \leqslant \ldots \leqslant \mathbf{u}^* \leqslant$$
$$\leqslant \ldots \leqslant \mathbf{w}^{(k)} \leqslant \ldots \leqslant \mathbf{w}^{(1)} \leqslant \mathbf{w}^{(0)} = \mathbf{w}^0$$

are satisfied.

Note that the advantage of constructed two-sided iterative processes is that at each k iteration we have a convenient a posteriori estimation of the error for an approximate solution $\mathbf{u}^{(k)}(\mathbf{x}) = \frac{1}{2}(\mathbf{w}^{(k)}(\mathbf{x}) + \mathbf{v}^{(k)}(\mathbf{x}))$:

$$\left\| \mathbf{u}^* - \mathbf{u}^{(k)} \right\|_n \leq \frac{1}{2} \left\| \mathbf{w}^{(k)} - \mathbf{v}^{(k)} \right\|_n .$$

Then, if accuracy $\varepsilon > 0$ is given, then the iterative process should be carried out before the inequality $\max\{\max_{\mathbf{x}\in\bar{\Omega}}(w_1^{(k)}(\mathbf{x}) - v_1^{(k)}(\mathbf{x})), \ldots, \max_{\mathbf{x}\in\bar{\Omega}}(w_n^{(k)}(\mathbf{x}) - v_n^{(k)}(\mathbf{x}))\} < 2\varepsilon$ will be performed and with accuracy ε it can be assumed that $\mathbf{u}^*(\mathbf{x}) \approx \mathbf{u}^{(k)}(\mathbf{x})$.

Also, based on the inequality (11) we can obtain an estimate for the number of iterations required to achieve the given accuracy. Indeed, from the inequalities

$$\left\| \mathbf{u}^* - \mathbf{u}^{(k)} \right\|_n \leq \frac{1}{2} \left\| \mathbf{w}^{(k)} - \mathbf{v}^{(k)} \right\|_n \leq \frac{(LM)^k}{2} \left\| \mathbf{w}^{(0)} - \mathbf{v}^{(0)} \right\|_n < \varepsilon$$

we find that to achieve accuracy ε

$$k_0(\varepsilon) = \left[\frac{\ln \dfrac{\left\| \mathbf{w}^{(0)} - \mathbf{v}^{(0)} \right\|_n}{2\varepsilon}}{\ln \dfrac{1}{LM}} \right] + 1$$

iterations must be done, where the square brackets denote an integer part of the number.

4. Numerical experiment

The construction of the two-sided approximations to the positive solution of the boundary value problem (1), (2) will be demonstrated on the system of two Lane-Emden equations with a homogeneous Dirichlet condition:

$$-\Delta u_1 = u_2^{p_1}, \quad -\Delta u_2 = u_1^{-p_2} \quad \text{in} \quad \Omega, \tag{14}$$

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0, \tag{15}$$

where $p_1 > 0$, $p_2 > 0$.

The construction of two-sided approximations to the positive solution of the Lane-Emden equation $-\Delta u = u^p$ for $p = \frac{1}{2}$ was made in [5].

The questions of the existence and uniqueness of the solution of problem (14), (15) in the case when Ω is a sphere of radius R, $p_1 > 0$, $p_2 < 0$ were investigated in [2].

The functions $f_1(\mathbf{x}, u_1, u_2) = u_2^{p_1}$, $f_2(\mathbf{x}, u_1, u_2) = u_1^{-p_2}$ are positive and continuous on a set of variables, if $u_1, u_2 > 0$, and allow a diagonal representation by using the functions

$$\hat{f}_1(\mathbf{x}, v_1, v_2, w_1, w_2) = v_2^{p_1}, \quad \hat{f}_2(\mathbf{x}, v_1, v_2, w_1, w_2) = w_1^{-p_2}. \tag{16}$$

The problem (14), (15) is replaced by the equivalent system of Hammerstein integral equations

$$u_1(\mathbf{x}) = \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi}) u_2^{p_1}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad u_2(\mathbf{x}) = \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi}) u_1^{-p_2}(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{17}$$

With the system (17) we will associate a heterotone operator

$$\mathbf{T}(u_1, u_2) = \left(\int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi}) u_2^{p_1}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi}) u_1^{-p_2}(\boldsymbol{\xi}) d\boldsymbol{\xi} \right), \tag{18}$$

for which the companion operator has the form

$$\hat{\mathbf{T}}(v_1, v_2, w_1, w_2) = \left(\int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi}) v_2^{p_1}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \int\limits_\Omega G(\mathbf{x}, \boldsymbol{\xi}) w_1^{-p_2}(\boldsymbol{\xi}) d\boldsymbol{\xi} \right).$$

Condition (12) for functions (16) leads to inequalities

$$\hat{f}_1 \left(\mathbf{x}, \tau v_1, \tau v_2, \frac{1}{\tau} w_1, \frac{1}{\tau} w_2 \right) = (\tau v_2)^{p_1} > \tau \hat{f}_1(\mathbf{x}, v_1, v_2, w_1, w_2) = \tau v_2^{p_1},$$
$$\hat{f}_2 \left(\mathbf{x}, \tau v_1, \tau v_2, \frac{1}{\tau} w_1, \frac{1}{\tau} w_2 \right) = \left(\frac{1}{\tau} w_1 \right)^{-p_2} > \tau \hat{f}_2(\mathbf{x}, v_1, v_2, w_1, w_2) = \tau w_1^{-p_2},$$

whereof $\tau^{p_1 - 1} > 1$, $\tau^{p_2 - 1} > 1$, i.e. $0 < p_1 < 1$, $0 < p_2 < 1$.

For the operator (18) a strongly invariant cone segment will be search in the form $\langle \mathbf{v}^0, \mathbf{w}^0 \rangle$, where

$$\begin{aligned} &\mathbf{v}^0(\mathbf{x}) = (v_1^0(\mathbf{x}), v_2^0(\mathbf{x})) = (\alpha_1 \omega(\mathbf{x}), \alpha_2 \omega(\mathbf{x})), \\ &\mathbf{w}^0(\mathbf{x}) = (w_1^0(\mathbf{x}), w_2^0(\mathbf{x})) = (\beta_1 \omega(\mathbf{x}), \beta_2 \omega(\mathbf{x})), \\ &0 \le \alpha_1 < \beta_1, \quad 0 \le \alpha_2 < \beta_2, \end{aligned}$$

and the function $\omega(\mathbf{x})$ satisfies the conditions a) - c) of section 3.

The system of inequalities for the determination α_1, α_2, β_1, β_2 has the form:

$$
\begin{aligned}
&\alpha_2^{p_1} \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\omega^{p_1}(\boldsymbol{\xi})d\boldsymbol{\xi} \geq \alpha_1\omega(\mathbf{x}),\\
&\beta_1^{-p_2} \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\omega^{-p_2}(\boldsymbol{\xi})d\boldsymbol{\xi} \geq \alpha_2\omega(\mathbf{x}),\\
&\beta_2^{p_1} \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\omega^{p_1}(\boldsymbol{\xi})d\boldsymbol{\xi} \leq \beta_1\omega(\mathbf{x}),\\
&\alpha_1^{-p_2} \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\omega^{-p_2}(\boldsymbol{\xi})d\boldsymbol{\xi} \leq \beta_2\omega(\mathbf{x}) \text{ for all } \mathbf{x}\in\Omega.
\end{aligned}
\tag{19}
$$

Hence, the following theorem holds.

Theorem 3. *Let $0 < p_1 < 1$, $0 < p_2 < 1$ and the system (19) has a solution $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ such that $0 \leq \alpha_1 < \beta_1$, $0 \leq \alpha_2 < \beta_2$. Then the iterative process*

$$
v_1^{(k+1)}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\big(v_2^{(k)}(\boldsymbol{\xi})\big)^{p_1} d\boldsymbol{\xi},\ v_2^{(k+1)}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\big(w_1^{(k)}(\boldsymbol{\xi})\big)^{-p_2} d\boldsymbol{\xi},
$$

$$
w_1^{(k+1)}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\big(w_2^{(k)}(\boldsymbol{\xi})\big)^{p_1} d\boldsymbol{\xi},\ w_2^{(k+1)}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\boldsymbol{\xi})\big(v_1^{(k)}(\boldsymbol{\xi})\big)^{-p_2} d\boldsymbol{\xi},
$$

$k = 0, 1, 2, \ldots,$

where $v_1^{(0)}(\mathbf{x}) = \alpha_1\omega(\mathbf{x})$, $v_2^{(0)}(\mathbf{x}) = \alpha_2\omega(\mathbf{x})$, $w_1^{(0)}(\mathbf{x}) = \beta_1\omega(\mathbf{x})$, $w_2^{(0)}(\mathbf{x}) = \beta_2\omega(\mathbf{x})$, converges to the unique positive solution $(u_1^*(\mathbf{x}), u_2^*(\mathbf{x}))$ of system (14), (15), and besides, for all $\mathbf{x} \in \bar{\Omega}$ the following inequalities

$$
\alpha_1\omega(\mathbf{x}) = v_1^{(0)}(\mathbf{x}) \leq v_1^{(1)}(\mathbf{x}) \leq \ldots \leq u_1^*(\mathbf{x}) \leq \ldots \leq w_1^{(1)}(\mathbf{x}) \leq w_1^{(0)}(\mathbf{x}) = \beta_1\omega(\mathbf{x}),
$$

$$
\alpha_2\omega(\mathbf{x}) = v_2^{(0)}(\mathbf{x}) \leq v_2^{(1)}(\mathbf{x}) \leq \ldots \leq u_2^*(\mathbf{x}) \leq \ldots \leq w_2^{(1)}(\mathbf{x}) \leq w_2^{(0)}(\mathbf{x}) = \beta_2\omega(\mathbf{x})
$$

are satisfied.

A computational experiment was carried out for the values $p_1 = \dfrac{1}{4}$, $p_2 = \dfrac{1}{3}$, if $m = 2$ and $\Omega = \{\mathbf{x} = (x_1, x_2) : |\mathbf{x}| < 1\}$ is unit circle. For this domain we have $\omega(\mathbf{x}) == \dfrac{1}{2}(1 - x_1^2 - x_2^2)$, $G(\mathbf{x},\boldsymbol{\xi}) = \dfrac{1}{2\pi}\ln\dfrac{\rho r_{\mathbf{x}\boldsymbol{\xi}^1}}{r_{\mathbf{x}\boldsymbol{\xi}}}$, where $\rho = \sqrt{\xi_1^2 + \xi_2^2}$, points $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^1$ are symmetric with respect to the circle of the unit radius, $r_{\mathbf{x}\boldsymbol{\xi}}$, $r_{\mathbf{x}\boldsymbol{\xi}^1}$ are distances between points $\mathbf{x}$, $\boldsymbol{\xi}$ and $\mathbf{x}$, $\boldsymbol{\xi}^1$ accordingly. The solution of the system of inequalities (19) is, for example, numbers $\alpha_1 = 0.332$, $\alpha_2 = 0.959$, $\beta_1 = 0.418$, $\beta_2 = 1.364$. Accuracy $\varepsilon = 10^{-4}$ was reached on the sixth iteration.

An obtained approximate solution

$$
u_1^{(6)}(\mathbf{x}) = \frac{v_1^{(6)}(\mathbf{x}) + w_1^{(6)}(\mathbf{x})}{2}, \quad u_2^{(6)}(\mathbf{x}) = \frac{v_2^{(6)}(\mathbf{x}) + w_2^{(6)}(\mathbf{x})}{2}
$$

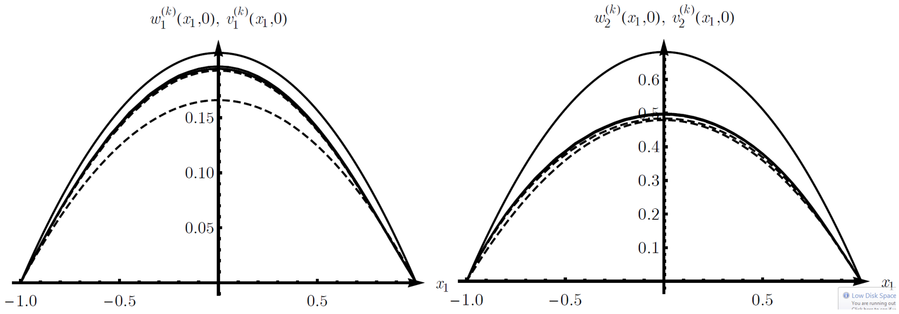
has a radial symmetry.

TABL. 1. The values of the error estimation of the approximate solution

Number of iteration k	$\varepsilon_1^{(k)}$	$\varepsilon_2^{(k)}$
0	$0.22 \cdot 10^{-1}$	$0.10 \cdot 10^{0}$
1	$0.88 \cdot 10^{-2}$	$0.19 \cdot 10^{-1}$
2	$0.19 \cdot 10^{-2}$	$0.72 \cdot 10^{-2}$
3	$0.71 \cdot 10^{-3}$	$0.16 \cdot 10^{-2}$
4	$0.16 \cdot 10^{-3}$	$0.61 \cdot 10^{-3}$
5	$0.60 \cdot 10^{-4}$	$0.13 \cdot 10^{-3}$
6	$0.13 \cdot 10^{-4}$	$0.51 \cdot 10^{-4}$

TABL. 2. The values of the approximate solution at the points $\mathbf{x}_i = (0.25i, 0)$, $i = 0, 1, 2, 3$

$\mathbf{x}_i = (0.25i, 0)$	(0,0)	(0.25,0)	(0.5,0)	(0.75,0)
$u_1^{(6)}(\mathbf{x}_i)$	0.1946	0.1806	0.1397	0.0752
$u_2^{(6)}(\mathbf{x}_i)$	0.4960	0.4674	0.3781	0.2192



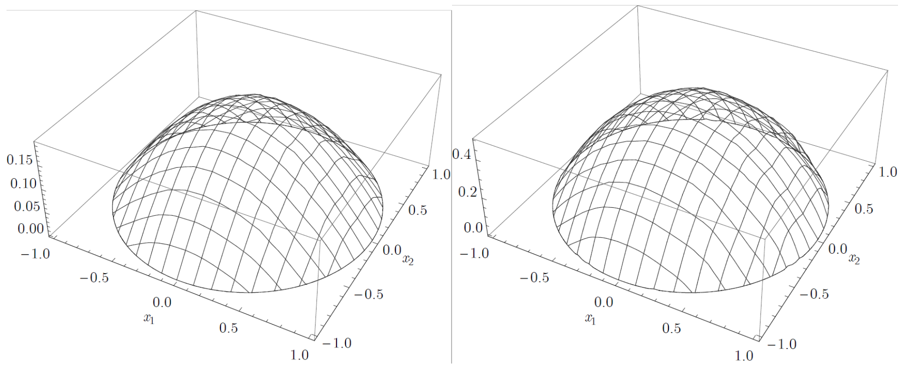
(A) (B)

FIG. 1. Graphs of cross-sections of upper and lower approximations $w_1^{(k)}(x_1, 0)$, $v_1^{(k)}(x_1, 0)$ (a) and $w_2^{(k)}(x_1, 0)$, $v_2^{(k)}(x_1, 0)$ (b), $k = 0, 2, 4, 6$

Table 1 gives the data on how the estimate

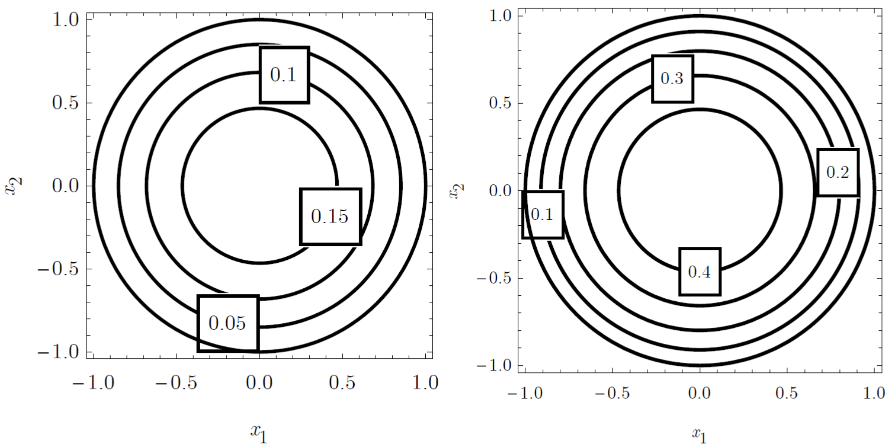
$$\varepsilon_i^{(k)} = \max_{\mathbf{x}\in\bar{\Omega}} \frac{1}{2} \left| w_i^{(k)}(\mathbf{x}) - v_i^{(k)}(\mathbf{x}) \right|$$

the norm of error $\left\| u_i^* - u_i^{(k)} \right\|$ of an approximate solution $u_i^{(k)}(\mathbf{x})$, $i = 1, 2$, is changed, depending on the iteration number k, $k = 0, 1, \ldots, 6$. Table 2 shows the values, found with accuracy $\varepsilon = 10^{-4}$ of the approximate solution $u_1^{(6)}(\mathbf{x})$, $u_2^{(6)}(\mathbf{x})$ at points located on the ray $\varphi = 0$. It was found that $\left\| u_1^{(6)} \right\| = 0.1946$, $\left\| u_2^{(6)} \right\| = 0.4960$.



(A) (B)

Fig. 2. Surfaces of approximate solutions $u_1^{(6)}(\mathbf{x})$ (a) and $u_2^{(6)}(\mathbf{x})$ (b)



(A) (B)

Fig. 3. Contour lines of approximate solutions $u_1^{(6)}(\mathbf{x})$ (a) and $u_2^{(6)}(\mathbf{x})$ (b)

Fig. 1 shows the graphs of the cross-sections of the upper $w_1^{(k)}(\mathbf{x})$, $w_2^{(k)}(\mathbf{x})$ and the lower $v_1^{(k)}(\mathbf{x})$, $v_2^{(k)}(\mathbf{x})$ approximations at $x_2 = 0$ for $k = 0, 2, 4, 6$. Fig. 2, 3 show the surfaces of the approximate solutions $u_1^{(6)}(\mathbf{x})$, $u_2^{(6)}(\mathbf{x})$ and their contour lines respectively.

5. Conclusions

The paper proposed a method of constructing the two-sided approximations to a positive solution of the homogeneous Dirichlet problem for a system of semilinear elliptic equations. The numerical experiment, conducted for the Lane-Emden system, demonstrated the possibilities and effectiveness of the method. The proposed approach to numerical solution of semilinear systems can be used in solving various applications, the mathematical models of which is the problem (1), (2).

The limitation of using the proposed method may be due to the fact that the Green's function of the first boundary value problem for an operator $-\Delta$ is known only for a certain number of classical domains. When considering the problem (1), (2) in the domains of non classical geometry or in domains for which the Green's function is known, but has a complex analytic expression, to construct the corresponding (1), (2) system of integral equations, can be used an approach based on the corresponding Green's quasi-function [15].

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Maksim Sidorov,
Kharkiv National University of Radio Electronics,
14, Nauki Ave., Kharkiv, 61166, Ukraine.

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