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SINC APPROXIMATION OF ALGEBRAICALLY DECAYING FUNCTIONS

D. O. Sytnyk

Резюме. В роботі запропоновано узагальнення Sinc інтерполяційного методу, яке дозволяє наближати на $\mathbb{R}$ функції спадаючі алгебраїчно. Подібно до класичної Sinc інтерполяції ми формулюємо два типи оцінок похибки. Перший стосується загального класу функцій, що мають алгебраїчний порядок спадання на $\mathbb{R}$. Оцінки похибки другого типу є справедливими для випадку коли порядок спадання функції відомий у смузі комплексної площини навколо дійсної осі. Теоретичні викладки підкріплені чисельними експериментами.

Abstract. An extension of sinc interpolation on $\mathbb{R}$ to the class of algebraically decaying functions is developed in the paper. Similar to the classical sinc interpolation we establish two types of error estimates. First covers a wider class of functions with the algebraic order of decay on $\mathbb{R}$. The second type of error estimates governs the case when the order of function's decay can be estimated everywhere in the horizontal strip of complex plane around $\mathbb{R}$. The numerical examples are provided.

1. Introduction

We begin by introducing some necessary notation. Let

$$\mathrm{sinc}\,(x) = \frac{\sin \pi x}{\pi x},$$

$$S\{k,h\}(x) = \mathrm{sinc}\left(\frac{x}{h} - k\right), \quad h > 0,\ k \in \mathbb{Z}. \tag{1}$$

By $H^1(D_d)$ in the paper we denote the class of functions $f(x)$ analytic in the horizontal strip D_d

$$D_d = \{z = x + iy \quad x \in (-\infty, \infty),\ |y| \le d\}, \tag{2}$$

and such, that the quantity

$$N_1(f, D_d) \equiv \int_{\partial D_d} |f(z)|dz,$$

is bounded. Next, for some given $h > 0$ and integer $N > 0$ we define a sinc interpolation polynomial as

$$C_N\{f,h\}(x) = \sum_{k=-N}^{N} f(kh) S\{k,h\}(x). \tag{3}$$

Key words. Sinc methods, sinc interpolation, algebraically decaying functions, Lambert-W function, polynomial order of convergence, approximation on real-line.

The following classical result characterize the accuracy of interpolation of $f \in H^1(D_d)$ by $C_N\{f,h\}(x)$ for the case, when $f(s)$ is exponentially decaying.

Theorem (Stenger [6, p. 137]) *Assume that the function $f \in H^1(D_d)$ is bounded by*

$$|f(x)| \le L\mathrm{e}^{-\alpha|x|}, \quad \forall x \in \mathbb{R}, \tag{4}$$

with some $\alpha, L > 0$. Then the error of $2N+1$ term sinc interpolation of $f(x)$ by $C_N\{f,h\}(x)$, satisfies the following estimate

$$\sup_{x\in\mathbb{R}} |f(x) - C_N\{f,h\}(x)| \le c\mathcal{E}_N, \qquad \mathcal{E}_N = N^{1/2}\mathrm{e}^{-\sqrt{\pi d \alpha N}}, \tag{5}$$

provided that

$$h = \sqrt{\frac{\pi d}{\alpha N}}. \tag{6}$$

Here $c > 0$ is some constant dependent on f, d, α and independent on N. In this paper we extend the results of the above theorem to a class of algebraically decaying functions on $\mathbb{R}$. All theoretical considerations are given in sections 1,2. Section 3 is devoted to numerical examples and discussion.

2. Interpolation of functions with algebraic decay on real line

In this section we study the convergence of sinc interpolation for the class of algebraically decaying functions. Specifically, we consider the situation when a function $f(x)$ satisfies

$$|f(x)| \le \frac{L}{1+|x|^\alpha}, \quad \forall x \in \mathbb{R} \tag{7}$$

instead of inequality (4), convenient for the classical sinc methods [6].

Theorem 1. *Assume that the function $f \in H^1(D_d)$ has an algebraic decay defined by (7) with some $\alpha > 1$, $L > 0$. Then the error of $2N+1$-term sinc interpolation (3) satisfies the following estimate*

$$\sup_{x\in\mathbb{R}} |f(x) - C_N\{f,h\}(x)| \le c\mathcal{E}_N, \quad \forall x \in \mathbb{R}, \qquad \mathcal{E}_N = \frac{\alpha^\alpha (N+1)^{1-\alpha}}{(\alpha-1)(\pi d)^\alpha} \left(\mathbf{W}\left(\frac{\pi d}{\alpha}\left(\frac{\alpha-1}{\pi d}\right)^{\frac{1}{\alpha}} (N+1)^{\frac{\alpha-1}{\alpha}}\right)\right)^\alpha, \tag{8}$$

provided that h in (3) is chosen as

$$h = \frac{\pi d}{\alpha}\left(\mathbf{W}\left(\frac{\pi d}{\alpha}\left(\frac{\alpha-1}{\pi d}\right)^{\frac{1}{\alpha}} (N+1)^{\frac{\alpha-1}{\alpha}}\right)\right)^{-1}. \tag{9}$$

Here $\mathbf{W}[\cdot]$ denotes a positive branch of the Lambert-W function, $c = c_1 N_1(f, D_d) + 2L$ and $c_1 > 1$ is the constant independent of N:

$$c_1 = \frac{(\pi d)^{2(\alpha-1)}(\alpha-1)^2}{(\pi d)^{2(\alpha-1)}(\alpha-1)^2 - \alpha^{2\alpha}\mathbf{W}^{2\alpha}\left(\frac{\pi d}{\alpha}\sqrt[\alpha]{\frac{\alpha-1}{\pi d}}\right)}. \tag{10}$$

Proof. For any fixed h the error of sinc interpolation can be represented as follows [6, equation (3.1.29)]

$$|f(x) - C_N\{f,h\}(x)| \le |f(x) - C_\infty\{f,h\}(x)| + \sum_{|k|>N} |f(kh)| .$$

Bound of the first term on the right-hand side of this formula was obtained in Theorem 3.1.3 from [6]. For $x \in \mathbb{R}$ this term satisfies

$$|f(x) - C_\infty\{f,h\}(x)| \le \frac{N_1(f,D_d)}{2\pi d \sinh \frac{\pi d}{h}} \le \frac{c_1 N_1(f,D_d)}{\pi d} \mathrm{e}^{-\frac{\pi d}{h}}, \tag{11}$$

where $c_1 > 1$ is some constant to be determined later. For the second term we get

$$\begin{aligned} \sum_{|k|>N} |f(kh)| \le \le 2L \sum_{k=N+1}^{\infty} (kh)^{-\alpha} \le 2L \int\limits_{N+1}^{\infty} (th)^{-\alpha} dt \\ \le \frac{2L(N+1)^{1-\alpha}}{(\alpha-1)h^\alpha}. \end{aligned} \tag{12}$$

The above sequence of inequalities is justified as long as $f(x)$ satisfy (7) with some $\alpha > 1$. For such $f(x)$, truncation error (12) decays algebraically as $N \to \infty$. In order to balance it with exponentially decaying discretization error (11) one needs to solve for h the equation

$$\frac{\mathrm{e}^{-\frac{\pi d}{h}}}{c_2} = \frac{(N+1)^{1-\alpha}}{(\alpha-1)h^\alpha}. \tag{13}$$

Let $s = \frac{\pi d}{\alpha} h^{-1}$ and assume that $c_2 > 0$ is some fixed parameter. Then, equation (13) takes the form

$$\frac{\pi d}{\alpha} \left(\frac{\alpha - 1}{c_2} (N+1)^{\alpha-1} \right)^{\frac{1}{\alpha}} = s\mathrm{e}^s,$$

which has a unique solution

$$s = \mathbf{W}\left(\frac{\pi d}{\alpha} \left(\frac{\alpha-1}{c_2} (N+1)^{\alpha-1} \right)^{\frac{1}{\alpha}} \right).$$

Next, we set $c_2 = \pi d$ and substitute back the expression for s in terms of h to obtain (9). The proof of (8) is straightforward

$$\begin{aligned} |f(x) - C_N\{f,h\}(x)| &\le (c_1 N_1(f,D_d) + 2L) \frac{(N+1)^{1-\alpha}}{(\alpha-1)h^\alpha} \le \\ &\le c \frac{\alpha^\alpha (N+1)^{1-\alpha}}{(\alpha-1)(\pi d)^\alpha} \left(\mathbf{W}\left(\frac{\pi d}{\alpha} \left(\frac{\alpha-1}{\pi d} \right)^{\frac{1}{\alpha}} (N+1)^{\frac{\alpha-1}{\alpha}} \right) \right)^\alpha . \end{aligned}$$

Now, let us come back to the determination of c_1. The smallest c_1 suitable for (11) can be defined as follows

$$c_1 = \sup_{N \in \mathbb{Z}_+} \left\{ \frac{e^{\frac{\pi d}{h}}}{2 \sinh \frac{\pi d}{h}} \right\} = \max_{N \in \mathbb{Z}_+} \left(1 - e^{-\frac{2\pi d}{h}} \right)^{-1} .$$

Its not hard to see that the maximum is attained at $N = 0$. Therefore, the value of c_1:

$$c_1 = \left(1 - \exp\left(-2\alpha \mathbf{W}\left(\frac{\pi d}{\alpha}\sqrt[\alpha]{\frac{\alpha-1}{\pi d}}\right)\right)\right)^{-1}$$

is clearly greater than one, for any $\alpha > 1$, $d > 0$. To get (10) we apply the identity $\exp(-\mathbf{W}(x)) = \mathbf{W}(x)/x$ to the above formula for c_1 and rearrange the result accordingly

$$c_1 = \left(1 - \frac{\alpha^{2\alpha}}{(\pi d)^{2(\alpha-1)}(\alpha-1)^2}\left(\mathbf{W}\left(\frac{\pi d}{\alpha}\sqrt[\alpha]{\frac{\alpha-1}{\pi d}}\right)\right)^{2\alpha}\right)^{-1}$$
$$= \frac{(\pi d)^{2(\alpha-1)}(\alpha-1)^2}{(\pi d)^{2(\alpha-1)}(\alpha-1)^2 - \alpha^{2\alpha}\mathbf{W}^{2\alpha}\left(\frac{\pi d}{\alpha}\sqrt[\alpha]{\frac{\alpha-1}{\pi d}}\right)}.$$

The presence of $\mathbf{W}(x)$ in estimate (8) makes it harder to perceive the asymptotic behavior of the interpolation error intuitively. To fix that we recall a well-established result [5] on the asymptotic properties of $\mathbf{W}(x)$, valid for any $x > e$:

$$\ln x - \ln(\ln x) + \frac{\ln(\ln x)}{2\ln x} \leq \mathbf{W}(x) \leq \ln x - \ln(\ln x) + \frac{e\ln(\ln x)}{(e-1)\ln x}.$$

By using the above inequality along with the definition of $\mathbf{W}(x)$ and (13) we transform (8) in the following way

$$|f(x) - C_N\{f,h\}(x)| \leq \frac{c}{\mathrm{e}^{\alpha s}} \leq c\left(\frac{\ln\left(\frac{\pi d}{\alpha}\left(\frac{\alpha-1}{\pi d}\right)^{\frac{1}{\alpha}}(N+1)^{\frac{\alpha-1}{\alpha}}\right)}{\frac{\pi d}{\alpha}\left(\frac{\alpha-1}{\pi d}\right)^{\frac{1}{\alpha}}(N+1)^{\frac{\alpha-1}{\alpha}}}\right)^{\alpha} \leq$$
$$\leq \frac{c}{(\pi d)^{\alpha-1}}\left(\frac{N+1}{\alpha-1}\right)^{1-\alpha}\ln^{\alpha}\left(\pi d\left(\frac{\alpha-1}{\alpha^{\alpha}}\right)^{\frac{1}{\alpha-1}}(N+1)\right);$$

whence it is clear that the error of sinc interpolation provided by Theorem 1 is asymptotically equal to $(N+1)^{1-\alpha}\ln^{\alpha}(N+1)$ as $N \to \infty$. To analyze the error for small N we note that, in the view of (13), $\mathcal{E}_N$ is bounded by the exponent with a strictly decreasing negative argument. Consequently, for any $\alpha > 1$, $x \in \mathbb{R}$, the error $\sup_{x\in\mathbb{R}}|f(x) - C_N\{f,h\}(x)|$ lies within the interval $[0, c]$ and decreases as $N \to \infty$.

One might conclude from the foregoing analysis that a simple asymptotic formula $W(x) \approx \ln(x)$ can be used to redefine h (9) in terms of logarithms, which are computationally more favorable than the Lambert-W function. To explore this possibility we set

$$h = \frac{\pi d}{\alpha}\left(\ln\left(\frac{\pi d}{\alpha}\left(\frac{\alpha-1}{c_2}\right)^{\frac{1}{\alpha}}(N+1)^{\frac{\alpha-1}{\alpha}}\right)\right)^{-1},$$

and study the corresponding error terms of the approximation. Discretization error (11) is positive and monotonically decreasing in N for any $c_2 > 0$, since

h is monotonic. The principal part $\frac{(N+1)^{1-\alpha}}{(\alpha-1)h^{\alpha}}$ of truncation error (12) has one global maximum at $N = N_0$:

$$N_0 = \left(\frac{\alpha}{\pi\, d}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(\frac{\alpha}{\alpha-1}\right)\left(\frac{\alpha-1}{c_2}\right)^{-\frac{1}{(\alpha-1)}} - 1.$$

To guarantee a monotonous decrease of the truncation error for all $N \geq 0$ we must require $N_0 = 0$, which yields $c_2 = (\alpha-1)\left(\frac{\pi d}{\alpha e}\right)^{\alpha}$. The aforementioned formula for h is thereby reduced to

$$h = \frac{\pi d}{\alpha + (\alpha-1)\ln(N+1)}. \tag{14}$$

For such h, the error of sinc interpolation will be bounded by (8) with

$$\mathcal{E}_N = \frac{(N+1)^{1-\alpha}}{(\alpha-1)(\pi d)^{\alpha}}\left(\alpha + (\alpha-1)\ln(N+1)\right)^{\alpha}, \tag{15}$$

and $c = (\alpha-1)\left(\frac{\pi d}{\alpha e}\right)^{\alpha} N_1(f, D_d) + 2L$. The main concern with (15), is the presence of additional summand α when compared to (8).

Remark 1. *The definition of h from Theorem 1 can not be simplified by adopting $W(x) \approx \ln(x)$, since such simplification, as described by* (14), (15), *would make the approximation method ineffective for large α.*

With an additional a-priory knowledge about $f(x)$ we should be able to improve the convergence properties of $C_N\{f, h\}(x)$ described by Theorem 1. The following improvement of (8) offers a more realistic balance of discretization and truncation errors, presuming that both $N_1(f, D_d)$ and L are known.

Corollary 4. *Assume that the function $f(x)$ satisfies the conditions of Theorem 1. If*

$$h = \frac{\pi d}{\alpha}\left(\mathbf{W}\left(\frac{\pi d}{\alpha}\left(\frac{N_1(f, D_d)(\alpha-1)}{\pi d L}\right)^{\frac{1}{\alpha}}(N+1)^{\frac{\alpha-1}{\alpha}}\right)\right)^{-1}, \tag{16}$$

then the error of sinc interpolation fulfills estimate (8), *with $c = (c_1+2)L$ and $\mathcal{E}_N$ given by*

$$\mathcal{E}_N = \frac{(N+1)^{1-\alpha}}{(\alpha-1)} h^{-\alpha}.$$

Formula (16) was obtained in the same way as (9), except this time we set

$$c_2 = \frac{\pi d L}{N_1(f, D_d)}.$$

3. Interpolation of functions with algebraic decay in the strip

Corollary 4 is difficult to apply as it is, because the evaluation of $N_1(f, D_d)$ requires computation of the contour integral over ∂D_d. In order to make this result more applicable we note, that if $f \in H^1(D_r)$, for some $r > 0$, then $\lim\limits_{x\to\pm\infty} f(x+iy) = 0$ uniformly with respect to $y \in [d, d]$, for all $d \in (0, r)$ [2, Proposition 6]. Hence, for any $r > 0$ there exist a nonempty subspace of $H^1(D_r)$, such that its elements f satisfy

$$|f(z)| \le \frac{L}{1+|z|^\alpha}, \quad \forall z \in D_d, \tag{17}$$

with some $d \in (0, r)$.

Theorem 2. *Assume that the function $f(z)$ is analytic in the horizontal strip D_d, $d > 0$. If $f(z)$ is bounded by (17) with some $\alpha > 1$, $L > 0$, then the error of sinc interpolation (3) satisfies the following estimate*

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |f(x) - C_N\{f,h\}(x)| \le c\mathcal{E}_N, \\ &\mathcal{E}_N = \frac{\alpha^\alpha (N+1)^{1-\alpha}}{(\alpha-1)(\pi d)^\alpha} h^\alpha, \end{aligned} \tag{18}$$

provided that

$$h = \frac{\pi d}{\alpha}\left(\mathbf{W}\left(\frac{\pi d}{\alpha}\left(\frac{4\beta(\alpha-1)}{\pi d}\right)^{\frac{1}{\alpha}}(N+1)^{\frac{\alpha-1}{\alpha}}\right)\right)^{-1}, \tag{19}$$

with $\beta = \min\left\{\frac{1}{\operatorname{sinc}(\alpha^{-1})}, \left(\frac{2}{d}\right)^{\alpha-1} B\left(\frac{\alpha}{2}-\frac{1}{2}, \frac{\alpha}{2}+\frac{1}{2}\right)\right\}$. Here $B(\cdot,\cdot)$ is the beta function, $c = 2\left(c_1\beta + 1\right)L$ and c_1 is the constant dependent on α, d.

Proof.

$$\int_{-\infty}^{+\infty} |f(x+id)|\, dx \le \int_{-\infty}^{+\infty} \frac{L dx}{1+|x+id|^\alpha} = 2L\int_0^{+\infty} \frac{dx}{1+(x^2+d^2)^{\frac{\alpha}{2}}}, \tag{20}$$

$$\int_0^{+\infty} \frac{dx}{1+(x^2+d^2)^{\alpha/2}} \le \int_0^{+\infty} \frac{dx}{1+x^\alpha} = \lim_{x\to\infty} \frac{x\ \Phi\left(-x^\alpha, 1, \alpha^{-1}\right)}{\alpha} =$$

$$= \lim_{x\to+\infty}\left|\frac{x\ \Phi\left(-x^\alpha, 1, \alpha^{-1}\right)}{\alpha}\right| = \lim_{\substack{\Re z\to+\infty \\ \Im z\to 0}}\left|\frac{z\Phi\left(-z^\alpha, 1, \alpha^{-1}\right)}{\alpha}\right|.$$

Here $\Re z$ and $\Im z$ is real and imaginary part of z correspondingly. To evaluate the last limit we employ Corollary 1 from [3]. It offers a convergent expansion of Hurwitz-Lerch zeta function $\Phi(z, s, a)$ when its second parameter s has integer value

$$z\Phi\left(z^\alpha, 1, \frac{1}{\alpha}\right) = \pi\left(sgn\left\{\operatorname{Arg}(\alpha\ln(z))\right\} i + \cot\frac{\pi}{\alpha}\right) - \sum_{k=1}^{\infty} \frac{z^{1-\alpha k}}{1/\alpha - k}. \tag{21}$$

The expression on the right of (21) is bounded and uniformly convergent to the left-hand side for any $\alpha > 1$, $|z| > 1$, such that $z^\alpha \notin (-\infty, -1) \cup (1, \infty)$. Therefore

$$\lim_{\substack{\Re z\to+\infty \\ \Im z\to 0}}\left|\frac{z\Phi\left(-z^\alpha, 1, \alpha^{-1}\right)}{\alpha}\right| = \frac{\pi}{\alpha}\sqrt{1+\cot^2\frac{\pi}{\alpha}} - \frac{1}{\alpha}\sum_{k=1}^{\infty} \lim_{\substack{\Re z\to+\infty \\ \Im z\to 0}} \frac{z^{1-\alpha k}}{1/\alpha - k},$$

which leads us to the bound

$$\int_{-\infty}^{+\infty} |f(x+id)|\, dx \leq \frac{2\pi L}{\alpha}\sqrt{1+\cot^2\frac{\pi}{\alpha}} = 2L\,\mathrm{sinc}^{-1}\left(\frac{1}{\alpha}\right). \tag{22}$$

For large d, the integral from (20) can be estimated as follows

$$\int_{0}^{+\infty} \frac{1}{1+(x^2+d^2)^{\alpha/2}}dx \leq \int_{0}^{+\infty} \frac{1}{(x^2+d^2)^{\alpha/2}}dx = \\ = \frac{\sqrt{\pi}d^{1-\alpha}\Gamma\left((\alpha-1)/2\right)}{2\Gamma\left(\alpha/2\right)} = \\ = \frac{d^{1-\alpha}\Gamma\left((\alpha-1)/2\right)\Gamma\left((\alpha+1)/2\right)}{2^{2-\alpha}\Gamma(\alpha)} \leq \\ \leq \frac{1}{2}B\left(\frac{\alpha}{2}-\frac{1}{2},\frac{\alpha}{2}+\frac{1}{2}\right)\left(\frac{2}{d}\right)^{\alpha-1}.$$

To obtain the above estimate we used a well-known multiplication theorem [1, p. 4] for Gamma function $\Gamma(\cdot)$. The next bound is a direct consequence of the above formula and (20)

$$\int_{-\infty}^{+\infty} |f(x+id)|\, dx \leq 2LB\left(\frac{\alpha}{2}-\frac{1}{2},\frac{\alpha}{2}+\frac{1}{2}\right)\left(\frac{2}{d}\right)^{\alpha-1}. \tag{23}$$

By combining bounds (22), (23) and taking in to account the fact that the expression on the right of (17) is invariant with respect to $z \to \bar{z}$ we arrive at the following estimate

$$N_1(f, D_d) \leq 4L\min\left\{\frac{1}{\mathrm{sinc}\left(\frac{1}{\alpha}\right)}, \left(\frac{2}{d}\right)^{\alpha-1}B\left(\frac{\alpha}{2}-\frac{1}{2},\frac{\alpha}{2}+\frac{1}{2}\right)\right\}.$$

To finalize the proof, we evaluate (16) assuming that the value of $N_1(f, D_d)$ is equal to its estimate provided by the previous formula. This will get us (19).

4. Examples and discussion

In this section we consider several examples of the developed approximation method. As measure of experimental error we use a discrete norm

$$err = \max_{\forall x\in X}\left|f(x) - C_N\{f,h\}(x)\right|,$$

defined on a uniform grid $X = \left\{jh/2 \mid j = \overline{-2N, 2N}\right\}$. With such choice of X the specified discrete norm ought to capture the contribution from both the descretization and truncation parts of the error. To experimentally check the convergence of $C_N\{f,h\}(x)$ we repeat the approximation procedure on a sequence of grids determined by

$$N_i \in \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\},$$

and the corresponding h_i evaluated by one of the formulas (9), (16) or (19).

Example 1. Let
$$f(x) = \frac{4}{2 + x^{2a}},$$
where $a \geq 2$ is integer. Then, the largest possible value of d such that $f(x)$ remains analytic in D_d, is equal to $\sqrt[2\alpha]{2} \sin \frac{\pi}{a}$. To simplify the computation of $N_1(f, D_d)$ we set $d = \frac{\sqrt[2a]{2}}{2} \sin \frac{\pi}{a}$, $a = 2$, then $N_1(f, D_d) \approx 4.550125680$, $L \approx 4.5$, $\alpha = 4$. The behaviour of an error $err(x) = f(x) - C_{32}\{f, h\}(x)$ for three different values of h is depicted in Fig. 1.

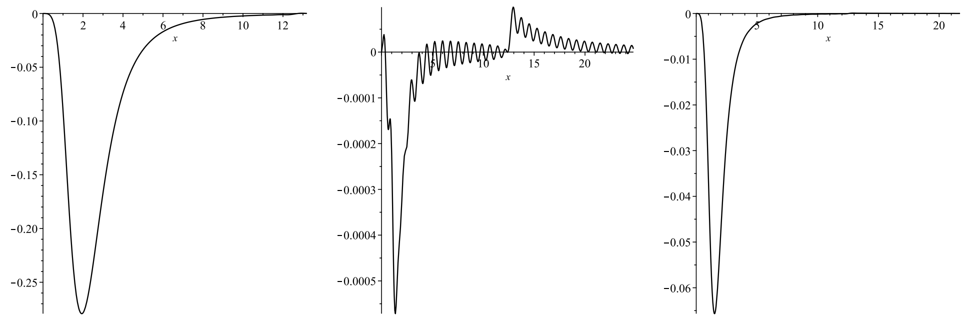

FIG. 1. Graphs of $err(x) = f(x) - C_{32}\{f, h\}(x)$ from Example 1 for h calculated by (9) – left, (16) – center, (19) – right

Predictably the value of h calculated by (16) is superior to those calculated by (19) and (9). One can see a discernible bump in the error function at $x_0 = N_6 h \approx 12.3792$. The values of $err(x)$ on the left of x_0 corresponds to the discretization error, whilst the values on the right of x_0 corresponds to the truncation error. The magnitude of those errors are almost match. This highlight the fact that the chosen h is really close to theoretically optimal value (16).

Example 2. In this example we set $f(x) \in H^1(D_d)$ as
$$f(x) = \frac{6 \cos 2x}{(5 + \cos^2 x)\,(1 + x^4)},$$
and choose formula (9) for the evaluation of h. The function $f(x)$ is meromorphic and bounded in D_d for any d smaller than the imaginary part of zeros of $\left(5 + \cos^2 x\right)\left(1 + x^4\right)$. The zeros of the polynomial part of this expression lie closer to the real line than any zero of $5 + \cos^2 x$, so $d \leq \Im \sqrt[4]{-1} = \frac{\sqrt{2}}{2} \approx .707106781186550$. Therefore it is safe to set $d = 0.7$. For given $f(x)$ we can also explicitly find the parameters of algebraic decay bound (7): $L = f(0) = 1$, $\alpha = 4$.

Note, that for a more general function $f(x)$ the corresponding L, α can be calculated numerically from a sequence of its values. For explicitly given $f(x)$ the possible values of d can be calculated numerically as well, for example using `Analytic` routine from Maple [4].

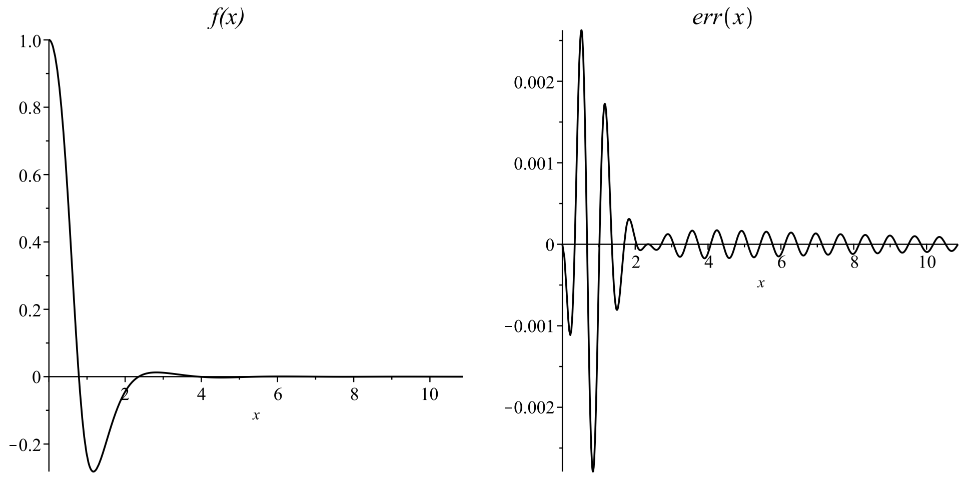

FIG. 2. Graphs of $f(x)$ and $err(x) = f(x) - C_{32}\{f,h\}(x)$ from Example 2

The graphs of the approximated function $f(x)$ and the error of its interpolation by $C_{32}\{f,h\}(x)$ are given in Fig. 2.

The precise values of err_i for $i = 1,\dots,11$ are presented in Table 1. Here we additionally supply the theoretical estimate $\mathcal{E}_{N_i}$ defined in Theorem 1 and the value of $c_i = err_i/\mathcal{E}_{N_i}$.

TABL. 1. Result of the numerical experiments for $f(x)$ from Example 2. The step size h is calculated by (9), the quantities $\mathcal{E}_N$ and c are evaluated with help of (8)

i	N_i	err_i	$\mathcal{E}_{N_i}$	c_i
1	1	0.164468448	0.04709645766	3.49216175
2	2	0.06868780928	0.02952007611	2.326816808
3	4	0.05758701686	0.01520376206	3.787682064
4	8	0.03584624921	0.006430513883	5.574398852
5	16	0.0096295153	0.002280722496	4.222133695
6	32	0.00277964663	0.0006985817398	3.978985524
7	64	0.001039781276	0.0001901179719	5.469137218
8	128	0.0001265620194	4.706647235E-05	2.689005848
9	256	6.005526369E-05	1.079496434E-05	5.563266519
10	512	5.048493593E-06	2.325942889E-06	2.170514855
11	1024	2.594213457E-06	4.758456168E-07	5.451796476

The data from in Table 1 demonstrates that the approximation method presented by Theorem 1 converges to $f(x)$. The of observed approximation error is consistent with the estimate provided by (8). Moreover the estimated value of c from (8) remains bounded by 5.6 for all $i = \overline{1,6}$. All this prove the effectiveness of the developed method.

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Dmytro Sytnyk,
Institute of Mathematics, NAS of Ukraine,
3, Tereschenkivska Str., Kyiv, 01004, Ukraine.

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