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ON THE FINITE ELEMENT APPROXIMATION OF A SYSTEM OF ELLIPTIC QUASI-VARIATIONAL INEQUALITIES RELATED TO HAMILTON-JACOBI-BELLMAN EQUATIONS

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Резюме. В роботі розвинуто новий підхід, запропонований в [3], для вивчення скінченно-елементної апроксимації систем еліптичних квазі-варіаційних нерівностей, що пов'язані з рівняннями Гамільтона-Якобі-Бельтрана. Метод поєднує в собі підходи часткових розв'язків, дискретної регулярності для варіаційних нерівностей та геометричну збіжність ітераційної схеми, що наближає розв'язок.

Abstract. In this paper, we exploit a new approach, introduced in [3], to study the finite element approximation of a system of elliptic quasi-variational inequalities (Q.V.I.) related to Hamilton-Jacobi-Bellman (HJB) equations. The method combines the concepts of subsolutions, discrete regularity for variational inequalities, and the geometrical convergence of an iterative scheme approximating the solution.

1. Introduction

We are concerned with the standard finite element approximation of the system of elliptic quasi-variational inequalities (Q.V.I): Find $U = (u_1, ..., u_M) \in \left(H_0^1(\Omega)\right)^M$ such that

$$\begin{cases} a_i(u_i, v - u_i) \geq (f_i, v - u_i) \quad \forall v \in H_0^1(\Omega), \\ u_i \leq k + u_{i+1},\, v \leq k + u_{i+1}, \\ u_{M+1} = u_1, \end{cases} \tag{1}$$

where, Ω is a bounded convex domain of $\mathbb{R}^N$ with sufficiently smooth boundary Γ, $f \geq 0$ is a right hand in $L^\infty(\Omega)$, $k > 0$, $(.,.)$ is the inner product in $L^2(\Omega)$, $a(.,.)$ is the bilinear form defined by: $\forall u, v \in H^1(\Omega)$

$$a_i(u\,, v) = \int_\Omega \left(\sum_{j,k=1}^{N} a_{jk}^i(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} b_k^i(x) \frac{\partial u}{\partial x_k} v + a_0^i(x) uv \right) dx \tag{2}$$

such that

$$a_i(v, v) \geq \delta \left\| v \right\|_{H^1(\Omega)}^2 \ \ \forall v \in H^1(\Omega),$$

where the coefficients $a_{jk}^i(x)$, $b_k^i(x)$, $a_0^i(x)$, $(j, k = 1, ..., N)$, are sufficiently smooth such that

$$a_0^i(x) \geq c_0 > 0,\, \forall x \in \Omega \tag{3}$$

Key words. Quasi-variational inequalities, Iterative scheme, Finite element, Discrete regularity, Subsolutions, Error estimate.

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and
$$\sum_{1\leq\ j,k\leq\ N} a^i_{jk}(x)\xi_j\xi_k \geq \alpha\left|\xi\right|^2 ; (x\in\bar{\Omega},\ \xi\in R^N,\ \ \alpha>0). \tag{4}$$

Denoting by $\mathbb{V}_h$, the finite element space consisting of continuous piecewise linear functions vanishing at the boundary, r_h the usual interpolation operator, we define the discrete counterpart of (1) by: find $U_h=(u_{1,h},...,u_{M,h})\in(\mathbb{V}_h)^M$ such that
$$\begin{cases} a_i(u_{i,h},v-u_{i,h})\geq(f,v-u_{i,h}) \quad \forall v\in\mathbb{V}_h,\\ u_{i,h}\leq r_h\left(k+u_{i+1,h}\right), v\leq r_h\left(k+u_{i+1,h}\right),\\ u_{M+1,h}=u_{1,h}. \end{cases} \tag{5}$$

This system appears in stochastic control problems related to Hamilton-Jacobi-Bellman equations (HJB) (see [1], [2]). Its finite element approximation was studied in (cf.,e.g., [4], [5], [6], where different methods were employed.

In this paper, we exploit an idea developed in [3] to derive optimal convergence order for the system of Q.V.I (1).

This method consists, mainly, of combining, in both the continuous and discrete contexts, the concept of subsolutions for variational inequalities and a geometrical convergence of an iterative scheme approximating the solution. For a computational purpose, this method provides an interesting information as it permits to control the error between the continuous iterative scheme and its finite element counterpart.

A brief description of this method is as follows: Let $U^n=(u^n_1,...,u^n_M)$ be the nth iterate of the scheme approximating the solution U, and $U^n_h=(u^n_{1h},...,u^n_{Mh})$ its finite element counterpart, approximating U_h.We construct a sequence of continuous subsolutions $\beta^n=(\beta^n_1,...,\beta^n_M)$ such that
$$\beta^n\leq U^n$$
and
$$\left\|\beta^n-U^n_h\right\|_\infty\leq Ch^2\left|\ln h\right|^2$$
and a sequence of discrete subsolutions $\gamma^n=(\gamma^n_{1,h},...,\gamma^n_{M,h})$ such that:
$$\gamma^n_h\leq U^n_h$$
and
$$\left\|U^n-\gamma^n_h\right\|_\infty\leq Ch^2\left|\ln h\right|^2.$$

In this situation, using a concept of discrete regularity, we establish an optimal error estimate for the iterative scheme:
$$\left\|U^n-U^n_h\right\|_\infty\leq Ch^2\left|\ln h\right|^2 \tag{6}$$
and then, combining estimate (6) with the geometrical convergence of the iterative scheme (U^n) and (U^n_h) to the solutions U and U_h of systems (1) and (5), respectively, we also derive error estimate for the system of Q.V.I. (1):
$$\left\|U-U_h\right\|_\infty\leq Ch^2\left|\ln h\right|^2 \tag{7}$$
where
$$\left\|V\right\|_\infty=\max\left\|v_i\right\|_{L^\infty(\Omega)}, \ V=(v_1,...,v_M)$$

and, in all the above error estimates, C is a constant independent of both h and n.

It is worth pointing out that estimate (6) is new for the system (1).

The paper is organized as follows. In sections 2, we recall the construction and convergence of the continuous iterative scheme for system (1). In section 3, we also recall analog discrete results and detail discrete regularity for the discrete iterative scheme. In section 4, we discuss the new approximation approach and derive the main results of this paper. In section 5, we give a numerical example and, finally, in section 6, a short conclusion.

2. The Continuous Problem

2.1. **A Continuous Iterative Scheme.** Let $U^0 = \left(u_1^0, ..., u_M^0\right) \in \left(H^1(\Omega)\right)^M$ be such that u_i^0 solves the equation

$$a(u_i^0, v) = (f_i, v) \quad \forall v \in H_0^1(\Omega);\ \forall i = 1, ..., M. \tag{8}$$

Then, starting from U^0 solution of (8), we define the continuous sequence (U^n) such that $U^n = (u_1^n, ..., u_M^n)$ and u_i^n solves the variational inequality (V.I)

$$\begin{cases} a(u_i^n, v - u_i^n) \geq (f_i, v - u_i^n) \quad \forall v \in H_0^1(\Omega), \\ u_i^n \leq k + u_{i+1}^{n-1}, v \leq k + u_{i+1}^{n-1}, \\ u_{M+1}^{n-1} = u_1^{n-1}. \end{cases} \tag{9}$$

Theorem 1. *[5] The sequence* (U^n) *defined in* (9) *converge decreasingly to the solution* U *of of system (1). Moreover, there exists* $0 < \mu < 1$ *such that*

$$\left\|U^n - U\right\|_\infty \leq \mu^n \left\|U^0\right\|_\infty. \tag{10}$$

3. The discrete Problem

For the sake of simplicity we suppose that Ω is polyhedral.We then consider a regular and quasi-uniform triangulation τ_h of $\bar{\Omega}$, consisting of n-simplices K. Denote by $h = \max_{K\in\tau_h} h_K$, the meshsize of τ_h with h_K being the diameter of K. For each $K \in \tau_h$, denote by $P_1(K)$ the set of polynomials on K with degree no more than 1. The P_1- conforming finite element space is given by

$$\mathbb{V}_h = \left\{v : v \in H^1(\Omega) \cap C(\bar{\Omega}),\ v_{/K} \in P_1(K), \quad \forall K \in \tau_h\right\}.$$

Let M_i, $1 \leq i \leq N_h$ denote the the vertices of the triangulation τ_h, and let φ_i, $1 \leq i \leq m(h)$, denote the functions of V_h which satisfy

$$\varphi_i(M_j) = \delta_{ij},\ 1 \leq i, j \leq N_h$$

so that the functions φ_i form a basis of V_h. For every $v \in H^1(\Omega) \cap C(\bar{\Omega})$, the function

$$r_h v(x) = \sum_{i=1}^{N_h} v(M_i)\varphi_i(x)$$

represents the interpolate of v over τ_h.

Now, in order to establish existence and uniqueness of a solution to V.I (5), the stiffness matrix is required to be an M-Matrix.

Definition 1. A real matrix $d \times d$ matrix $C = (c_{ls})$ with $c_{ls} \leq 0$, $\forall l \neq s$, $1 \leq l, s \leq d$, is called an M-Matrix if C is nonsingular and $C^{-1} \geq 0$ (i.e., all entries of its inverse are nonnegative).

3.1. **Discrete Maximum Principle.** Denote by A^i the matrices with generic coefficient

$$a^i_{ls} = a_i(\varphi_l, \varphi_s), \quad 1 \leq l, s \leq N_h; \quad i = 1, ..., M. \tag{11}$$

Because the bilinear form $a_i(.,.)$ is coercive, we have

$$A^i \text{ is positive definite} \tag{12}$$

and

$$a^i_{ll} > 0 \quad \forall l = 1, ..., m(h). \tag{13}$$

Furthermore, if the matrix (a_{jk}) involved in the bilinear form (2) is symmetric $(a_{jk} = a_{kj})$, then mesh conditions for which the off-diagonal entries of A^i satisfy

$$a^i_{ls} \leq 0, \forall i \neq j, \quad 1 \leq l, s \leq m(h) \tag{14}$$

can be found in [8]. Therefore, combining (12), (13) and (14), we have the following lemma.

Lemma 1. *The matrices A^i, $i = 1, ..., M$ are M-Matrices.*

Proof. See [8], [9]. □

3.2. **A discrete Iterative Scheme.** Let $U^0_h = \left(u^0_{1h}, ..., u^0_{Mh}\right)$ such that $u^0_{i,h} \in \mathbb{V}_h$ solves the equation

$$a_i(u^0_{i,h}, v) = (f_i, v) \quad \forall v \in \mathbb{V}_h; \quad i = 1, ..., M. \tag{15}$$

Now, starting from $U^0_h = 0$, we define the discrete sequence (U^n_h) such that $U^n_h = (u^n_{1h}, ...u^n_{Mh})$ and $u^n_{ih} \in \mathbb{V}_h$ solves the variational inequality (V.I)

$$\begin{cases} a(u^n_{ih}, v - u^n_{ih}) \geq (f_i, v - u^n_{ih}) \quad \forall v \in \mathbb{V}_h, \\ u^n_{ih} \leq k + u^{n-1}_{i+1h}, v \leq k + u^{n-1}_{i+1h}, \\ u^{n-1}_{M+1h} = u^{n-1}_{1h}. \end{cases} \tag{16}$$

Theorem 2. *[5] Under conditions of lemma 1, the sequence (U^n_h) and $(U_{n,h})$ converges decreasingly to the unique solution solution U_h of Q.V.I (5).Moreover, there exists a constant $0 < \mu < 1$ such that*

$$\|U^n_h - U_h\|_\infty \leq \mu^n \left\|U^0_h\right\|_\infty, \tag{17}$$

$$\|U_{n,h} - U_h\|_\infty \leq \mu^n \left\|U^0_h\right\|_\infty. \tag{18}$$

3.3. **Discrete regularity.** Let $\omega \in H_0^1(\Omega)$ be the solution of the V.I

$$\begin{cases} a(\omega, v-\omega) \geq (g, v-\omega)\, \forall v \in H_0^1(\Omega), \\ v \leq r_h\psi,\ \omega \leq r_h\psi \end{cases} \tag{19}$$

and $\omega_h \in \mathbb{V}_h$,its discrete counterpart, the solution of the V.I

$$\begin{cases} a(\omega_h, v-\omega_h) \geq (g, v-\omega_h)\, \forall v \in \mathbb{V}_h, \\ v \leq r_h\psi,\ \omega_h \leq r_h\psi. \end{cases} \tag{20}$$

This concept of "discrete regularity", introduced in [10], can be regarded as the discrete counterpart of the Lewy-Stampaccia estimate $\|\mathcal{A}u\|_\infty \leq C$ ($\mathcal{A}$ being the operator associated with bilinear form $a(.,.)$), extended to the variational form through the $L^1 - L^\infty$ duality. The main role it plays, in the present paper, is in the regularization of the obstacles appearing in the discrete problems (16)

Lemma 2. *[10] We assume that there exists a constant C independent of h such that*

$$|a(\omega_h, \varphi_s)| \leq C\,\|\varphi_s\|_{L^1(\Omega)} \quad \forall s = 1, 2, ..., N_h. \tag{21}$$

Then, there exists a family of right hand sides $g^{(h)}$ such that

$$\left\|g^{(h)}\right\|_\infty \leq C$$

and

$$a(\omega_h, v) = (g^{n,(h)}, v) \quad \forall v \in \mathbb{V}_h.$$

Theorem 3. *Let conditions of lemma 2 hold. Then, there exists a sequence $\left(g^{n,(h)}\right)_{n\geq 1}$ and a constant $C > 0$ independent of h and n such that*

$$\left\|g^{n,(h)}\right\|_\infty \leq C,$$

$$a(u_{ih}^n, v) = (g^{(h)}, v) \quad \forall v \in \mathbb{V}_h,$$

where u_{ih}^n is defined in (16).

Proof. The proof will be carried out by induction. For $n = 1$, let u_{ih}^1 be the solution of the V.I

$$\begin{cases} a(u_{ih}^1, v-u_h^1) \geq (f_i, v-u_{ih}^1)\ \ \forall v \in \mathbb{V}_h, \\ v \leq k + u_{ih}^0,\ \ u_{ih}^1 \leq k + u_{ih}^0, \end{cases}$$

where

$$a(u_{ih}^0, v) = (f_i, v) \quad \forall v \in V_h,$$

So, clearly

$$\left|a(u_{ih}^0, \varphi_s)\right| \leq C\,\|\varphi_s\|_{L^1(\Omega)} \quad \forall s = 1, 2, ..., N_h \tag{22}$$

and, using the discrete Levy-Stampachia inequality [4], we have

$$-(f_i, \varphi_s) \wedge a(k + u_{ih}^0, \varphi_s) \leq a(u_{ih}^1, \varphi_i) \leq (f, \varphi_s).$$

But

$$a(k + u_{ih}^0, \varphi_s) = a(u_{ih}^0, \varphi_s) + (ka_0^i(x), \varphi_s)$$

and, using (22), there exists a constant C such that,

$$-(f_i,\varphi_i)\wedge(-C,\varphi_s)\leq a(u_{ih}^1,\varphi_s)\leq(f,\varphi_s)$$

which implies

$$\left|a(u_{ih}^1,\varphi_s)\right|\leq C\left\|\varphi_s\right\|_{L^1(\Omega)},\forall s=1,2,...,N_h.$$

Hence, making use of lemma 2, there exists a family of right-hands side $\left\{g_i^{1,(h)}\right\}\in L^\infty(\Omega)$ such that

$$\begin{cases} i) & \left\|g_i^{1,(h)}\right\|_\infty\leq C \\ & \text{and} \\ ii) & a(u_{ih}^1,v)=(g_i^{1,(h)},v)\quad\forall v\in\mathbb{V}_h. \end{cases}$$

Now, assume that there exists a constant C independent of n such that

$$a(u_{ih}^{n-1},\varphi_s)\leq C\left\|\varphi_s\right\|_{L^1(\Omega)},\quad\forall s=1,2,...,N_h. \tag{23}$$

So, using the discrete Levy-Stampachia inequality , we get

$$-(f,\varphi_s)\wedge a(k+u_{ih}^{n-1},\varphi_i)\leq a(u_{ih}^n,\varphi_s)\leq(f,\varphi_s)$$

or

$$-(f,\varphi_s)\wedge(a(k+u_{ih}^{n-1},\varphi_s)\leq a(u_{ih}^n,\varphi_s)\leq(f,\varphi_s)$$

and, as

$$a(k+u_h^{n-1},\varphi_s)=a(u_h^{n-1},\varphi_s)+(ka_0^i(x),\varphi_s)$$

using (23) as above, we have

$$-(f_i,\varphi_s)\wedge(-C,\varphi_s)\leq a(u_h^n,\varphi_s)\leq(f,\varphi_s)$$

which implies

$$|a(u_h^n,\varphi_s)|\leq C\left\|\varphi_s\right\|_{L^1(\Omega)}.$$

So, making use of lemma 2, there exists family of right-hands side $\left\{g_i^{n,(h)}\right\}\in L^\infty(\Omega)$ such that

$$\begin{cases} i) & \left\|g_i^{n,(h)}\right\|_\infty\leq C \\ & \text{and} \\ ii) & a(u_{ih}^n,v)=(g_i^{n,(h)},v)\quad\forall v\in\mathbb{V}_h \end{cases}$$

which completes the proof. □

Note that, as

$$a(u_{ih}^n,v)=(g_i^{n,(h)},v)\forall v\in\mathbb{V}_h$$

one can define

$$U^{n,(h)}=\left(u_1^{n,(h)},...,u_M^{n,(h)}\right),$$

the discrete analog of

$$U_h^n=(u_{1h}^n,...,u_{Mh}^n)$$

such that

$$\left\|u_i^{n,(h)}\right\|_{W^{2,p}(\Omega)}\leq C$$

and

$$a(u_i^{n,(h)}, v) = (g_i^{n,(h)}, v) \quad \forall v \in H_0^1(\Omega) \tag{24}$$

and, by standard maximum norm estimates

$$\left\| u_i^{n,(h)} - u_{ih}^n \right\|_\infty \leq Ch^2 \left| \log h \right|. \tag{25}$$

4. L^∞ – Error Analysis

From now on, C will denote an arbitrary constant independent of both h and n.

4.1. **Background.** We begin with recalling some useful properties enjoyed by elliptic variational inequalities. Indeed, let

Definition 2. $w \in H_0^1(\Omega)$ is said to be a subsolution for the VI (19) if

$$\begin{cases} a(w, v) \leq (g, v) \forall v \in H_0^1(\Omega), v \geq 0, \\ w \leq \psi. \end{cases} \tag{26}$$

Theorem 4. *[7] The solution ω of V.I (19) is the least upper bound of the set of subsolutions.*

Theorem 5. *[7] Let $\omega = \partial(\tilde{\psi})$ and $\tilde{\omega} = \partial(\tilde{\psi})$. Then, we have*

$$\left\| \omega - \tilde{\omega} \right\|_\infty \leq C \left\| \psi - \tilde{\psi} \right\|_\infty. \tag{27}$$

Remark 1. *Under conditions of lemma 1, the above properties of the solution of V.I (19) remain valid in the discrete case.*

Indeed, let $\omega_h = \partial_h(\psi) \in \mathbb{V}_h$ be the solution of the discrete variational inequality

$$\begin{cases} a(\omega_h, v - \omega_h) \geq (g, v - \omega_h) \forall v \in \mathbb{V}_h, \\ \omega_h \leq r_h\psi, v \leq r_h\psi. \end{cases} \tag{28}$$

Next, we shall give the discrete analog of Theorems 3, 4. Their respective proofs will be omitted as they are similar to their continuous counterparts.

Definition 3. $w_h \in \mathbb{V}_h$ is said to be a subsolution for the V.I (28) if

$$\begin{cases} a(w_h, \varphi_s) \leq (g, \varphi_s) \forall s = 1, ..., N_h, \\ w_h \leq r_h\psi. \end{cases} \tag{29}$$

Theorem 6. *Under conditions of lemma 1, the solution ω_h of V.I (28) is the least upper bound of the set of discrete subsolutions.*

Theorem 7. *Let $\omega_h = \partial_h(\psi)$ and $\tilde{\omega}_h = \partial_h(\tilde{\psi})$. Then, under conditions of lemma 1, we have*

$$\left\| \omega_h - \tilde{\omega}_h \right\|_\infty \leq C \left\| \psi - \tilde{\psi} \right\|_\infty. \tag{30}$$

Lemma 3. *[11] If $\psi \in W^{2,p}(\Omega)$ and $\omega \in W^{2,p}(\Omega)$, $2 \leq p < \infty$, then the following error estimate holds*

$$\left\| \omega - \omega_h \right\|_\infty \leq Ch^2 \left| \ln h \right|^2. \tag{31}$$

4.2. L^∞**- Error estimate for the Iterative Scheme.** In order to estimate the error between the continuous iterative scheme and its finite element counterpart, we introduce the following sequences of variational inequalities.

An auxiliary sequence of continuous variational inequalities: We introduce the sequence $\bar{U}^n = (\bar{u}_1^n, ..., \bar{u}_M^n)_{n\geq 1}$ such that $\bar{u}_i^n = \partial\left(k + \hat{u}_{i+1,h}^{n-1}\right) \in H_0^1(\Omega)$ solves the continuous V.I**:**

$$\begin{cases} a_i(\bar{u}_i^n, v - \bar{u}^n) \geq (f, v - \bar{u}_i^n) \quad \forall v \in H_0^1(\Omega), \\ \bar{u}_i^n \leq k + u_{i+1}^{n-1,(h)}, \quad v \leq k + u_{i+1}^{n-1,(h)}, \\ u_{M+1,h}^{n-1} = u_{i+1}^{n-1,(h)}, \end{cases} \tag{32}$$

where $u_{i+1}^{n-1,(h)}$ is defined in (24).

An auxiliary sequence of discrete variational inequalities We define the sequence $\bar{U}_h^n = \left(\bar{u}_{1,h}^n, ..., \bar{u}_{i,h}^n\right)_{n\geq 1}$ such that $\bar{u}_{i,h}^n = \partial_h\left(k + u_{i+1}^{n-1}\right) \in \mathbb{V}_h$ solves the discrete V.I

$$\begin{cases} a_i(\bar{u}_{i,h}^n, v - \bar{u}_h^n) \geq (f_i, v - \bar{u}_h^n) \quad \forall v \in \mathbb{V}_h, \\ \bar{u}_h^n \leq r_h\left(k + u_{i+1}^{n-1}\right), \ v \leq r_h\left(k + u_{i+1}^{n-1}\right), \\ u_{M+1}^{n-1} = u_1^{n-1}, \end{cases} \tag{33}$$

where u^0 and u^n are defined in (8) and (9), respectively.

Theorem 8. *We have*

$$\left\|U^n - \bar{U}_h^n\right\|_\infty \leq Ch^2 \left|\ln h\right|^2. \tag{34}$$

Proof. As $\bar{u}_{i,h}^n$ is the discrete counterparts of u_i^n and $\|u_i^n\|_{W^{2,p}(\Omega)} \leq C$ (independent of n) (see [5]), making use of (31), we get the desired error estimates. □

Theorem 9.

$$\|U^n - U_h^n\|_\infty \leq Ch^2 \left|\ln h\right|^2. \tag{35}$$

Proof. We proceed by induction. Indeed, consider V.I (32) for $n = 1$:

$$\begin{cases} a_i(\bar{u}_i^1, v - \bar{u}_i^1) \geq (f, v - \bar{u}_i^1) \quad \forall v \in H_0^1(\Omega), \\ \bar{u}_i^1 \leq k + u_{i+1}^{0,(h)}, \quad v \leq k + u_{i+1}^{0,(h)}, \\ u_{M+1}^{0,(h)} = u_1^{0,(h)}. \end{cases}$$

So

$$\left\|\bar{u}_i^1 - u_{i,h}^1\right\|_\infty \leq Ch^2 \left|\ln h\right|^2. \tag{36}$$

Indeed, let $\bar{u}_i^1 = \partial(k + u_{i+1}^{0,(h)}), \tilde{u}_{i,h}^1 = \partial_h(k + u_{i+1}^{0,(h)})$ and $u_{i,h}^1 = \partial_h(k + u_{i+1,h}^0)$. Then, as $\tilde{u}_{i,h}^1$ is the discrete analog of $\bar{u}_i^1$, making use of (34), we have

$$\left\|\bar{u}_i^1 - \tilde{u}_{i,h}^1\right\|_\infty \leq Ch^2 \left|\ln h\right|^2. \tag{37}$$

Moreover, using (30) and standard maximum error estimate, we get

$$\begin{aligned}\left\|u_{i,h}^{1}-\tilde{u}_{i,h}^{1}\right\|_{\infty} &\leq \left\|u_{i+1}^{0,(h)}-u_{i+1,h}^{0}\right\|_{\infty} \\ &\leq Ch^{2}\left|\ln h\right|.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\bar{u}_{i}^{1}-u_{i,h}^{1}\right\|_{\infty} &\leq \left\|\bar{u}_{i}^{1}-\tilde{u}_{i,h}^{1}\right\|_{\infty}+\left\|\tilde{u}_{i,h}^{1}-u_{i,h}^{1}\right\|_{\infty} \\ &\leq Ch^{2}\left|\ln h\right|^{2}.\end{aligned}$$

Now, as $\bar{u}_{i}^{1}$ is solution to a V.I, it is also a subsolution, i.e.,

$$\begin{aligned}&a(\bar{u}_{i}^{1},v)\leq(f_{i},v)\quad\forall v\in H_{0}^{1}(\Omega),v\geq 0,\\ &\bar{u}_{i}^{1}\leq k+u_{i+1}^{0,(h)}.\end{aligned}$$

But, as

$$\begin{aligned}\bar{u}_{i}^{1} &\leq k+\left\|u_{i+1}^{0,(h)}-u_{i+1,h}^{0}\right\|_{\infty}+u_{i+1}^{0}\leq \\ &\leq k+Ch^{2}\left|\ln h\right|^{2}+u_{i+1}^{0},\end{aligned}$$

we have

$$\begin{aligned}a(\bar{u}_{i}^{1},v)\leq(f,v)\forall v\in H_{0}^{1}(\Omega),v\geq 0,\\ \bar{u}_{i}^{1}\leq k+Ch^{2}\left|\ln h\right|+u_{i+1}^{0}.\end{aligned}$$

Hence, $\bar{u}_{i}^{1}$ is also a subsolution for the V.I with obstacle $k+Ch^{2}\left|\ln h\right|^{2}+u_{i+1}^{0}$. Let $\bar{\omega}_{i}^{1}=\partial(k+Ch^{2}\left|\ln h\right|^{2}+u_{i+1}^{0})$. Then, as $u_{i}^{1}=\partial(k+u_{i+1}^{0})$, making use of (27) and standard maximum error estimate

$$\left\|u_{i+1}^{0}-u_{i+1,h}^{0}\right\|_{\infty}\leq Ch^{2}\left|\ln h\right|, \tag{38}$$

we get

$$\begin{aligned}\left\|\bar{\omega}_{i}^{1}-u_{i}^{1}\right\|_{\infty} &\leq Ch^{2}\left|\ln h\right|^{2}+\left\|u_{i+1}^{0}-u_{i+1,h}^{0}\right\|_{\infty}\leq \\ &\leq Ch^{2}\left|\ln h\right|^{2}.\end{aligned}$$

Hence, making use of Theorem 4, we have

$$\bar{u}_{i}^{1}\leq\bar{\omega}_{i}^{1}\leq u_{i}^{1}+Ch^{2}\left|\ln h\right|^{2}.$$

Putting

$$\beta_{i}^{1}=\bar{u}_{i}^{1}-Ch^{2}\left|\ln h\right|^{2},\forall i=1,...,M,$$

we get

$$\beta_{i}^{1}\leq u_{i}^{1},\forall i=1,...,M. \tag{39}$$

Further more, using estimate (36), we get

$$\begin{aligned}\left\|\beta_{i}^{1}-u_{i,h}^{1}\right\|_{\infty} &\leq \left\|\bar{u}_{i}^{1}-u_{i,h}^{1}\right\|_{\infty}+Ch^{2}\left|\ln h\right|^{2}\leq \\ &\leq Ch^{2}\left|\ln h\right|^{2}.\end{aligned} \tag{40}$$

Now consider the discrete V.I (33) for $n = 1$:

$$\begin{cases} a_i(\bar{u}^1_{i,h}, v - \bar{u}^1_{i,h}) \geq (f_i, v - \bar{u}^1_{i,h}) \quad \forall v \in \mathbb{V}_h, \\ \bar{u}^1_{i,h} \leq r_h\left(k + u^0_{i+1}\right), \; v \leq r_h\left(k + u^0_{i+1}\right), \end{cases}$$

$\bar{u}^1_{i,h}$ being also a discrete subsolution, we have

$$\begin{aligned} &a(\bar{u}^1_{i,h}, \varphi_i) \leq (f, \varphi_i) \quad \forall \varphi_i, \\ &\bar{u}^1_{i,h} \leq r_h\left(k + u^0_{i+1}\right), \end{aligned}$$

and, from standard maximum error estimate

$$\left\| u^0 - u^0_h \right\|_\infty \leq Ch^2 \left| \ln h \right|.$$

So

$$\begin{aligned} \bar{u}^1_{i,h} &\leq k + \left\| u^0_{i+1} - u^0_{i+1,h} \right\|_\infty + r_h u^0_{i+1,h} \leq \\ &\leq k + Ch^2 \left| \ln h \right|^2 + r_h u^0_{i+1,h}. \end{aligned}$$

then

$$\begin{aligned} a_i(\bar{u}^1_{i,h}, \varphi_i) &\leq (f_i, \varphi_i) \quad \forall \varphi_i, \\ \bar{u}^1_{i,h} &\leq k + Ch^2 \left| \ln h \right|^2 + r_h u^0_{i+1,h} \end{aligned}$$

because r_h is Lipschitz. So, $\bar{u}^1_{i,h}$ is also a discrete subsolution for the V.I with obstacle $k + Ch^2 \left| \ln h \right|^2 + r_h u^0_{i+1,h}$. Let $\bar{\omega}^1_{i,h} = \partial_h(k + Ch^2 \left| \ln h \right|^2 + u^0_{i+1,h})$. As $u^1_{i,h} = \partial_h(k + u^0_{i+1,h})$, making use of (30) and (38), we get

$$\begin{aligned} \left\| \bar{\omega}^1_{i,h} - u^1_{i,h} \right\|_\infty &\leq \left\| u^0_{i+1,h} - u^0_{i+1,h} \right\|_\infty \leq \\ &\leq Ch^2 \left| \ln h \right|^2 \end{aligned}$$

and, applying Theorem 6, we get

$$\bar{u}^1_{i,h} \leq \bar{\omega}^1_{i,h} \leq u^1_{i,h} + Ch^2 \left| \ln h \right|^2.$$

Now, taking

$$\gamma^1_{i,h} = \bar{u}^1_{i,h} - Ch^2 \left| \ln h \right|^2, \quad \forall i = 1, ..., M,$$

we have

$$\gamma^1_{i,h} \leq u^1_{i,h}, \forall i = 1, ..., M. \tag{41}$$

Hence, as $u^1_{i,h}$ is the discrete analog of u^1_i, making use (30) and (34), we get

$$\begin{aligned} \left\| \gamma^1_{i,h} - u^1_i \right\|_\infty &\leq \left\| \bar{u}^1_{i,h} - u^1_i \right\|_\infty + Ch^2 \left| \ln h \right|^2 \leq \\ &\leq Ch^2 \left| \ln h \right|^2. \end{aligned} \tag{42}$$

Thus, combining (39), (40) and (41), (42), we obtain

$$\begin{aligned} u^1_i &\leq \gamma^1_{i,h} + Ch^2 \left| \ln h \right|^2 \\ &\leq u^1_{i,h} + Ch^2 \left| \ln h \right|^2 \\ &\leq \beta^1_i + Ch^2 \left| \ln h \right|^2 \\ &\leq u^1_i + Ch^2 \left| \ln h \right|^2. \end{aligned}$$

That is

$$\left\| u_i^1 - u_{i,h}^1 \right\|_\infty \le Ch^2 \left| \ln h \right|^2 .$$

Let us now assume that

$$\left\| u_i^{n-1} - u_{i,h}^{n-1} \right\|_\infty \le Ch^2 \left| \ln h \right|^2 . \tag{43}$$

Since $\tilde{u}_{i,h}^n = \partial_h(k + u_{i+1}^{n-1,(h)})$ is the discrete analog of $\bar{u}_i^n = \partial(k + u_{i+1}^{n-1,(h)})$, making use of (34), we get

$$\left\| \bar{u}_i^n - \tilde{u}_{i,h}^n \right\|_\infty \le Ch^2 \left| \ln h \right|^2 . \tag{44}$$

Let us now prove that

$$\left\| \bar{u}_i^n - u_{i,h}^n \right\|_\infty \le Ch^2 \left| \ln h \right|^2 . \tag{45}$$

Indeed, using (44) , (30), we get

$$\begin{aligned} \left\| \bar{u}_i^n - u_{i,h}^n \right\|_\infty &\le \left\| \bar{u}_i^n - \tilde{u}_{i,h}^n \right\|_\infty + \left\| \tilde{u}_{i,h}^n - u_{i,h}^n \right\|_\infty \\ &\le Ch^2 \left| \ln h \right|^2 + \left\| u_{i+1}^{n-1,(h)} - u_{i+1,h}^{n-1} \right\|_\infty \\ &\le Ch^2 \left| \ln h \right|^2 , \end{aligned}$$

On the other hand, the solution of V.I (32) is also a subsolution, that is

$$\begin{cases} a_i(\bar{u}_i^n, v) \le (f_i, v) \quad \forall v \in H^1(\Omega), \quad v \ge 0, \\ \bar{u}_i^n \le k + u_{i+1}^{n-1,(h)}. \end{cases}$$

So, using (43), we have

$$\begin{aligned} \bar{u}_i^n &\le k + \left\| u_{i+1}^{n-1} - u_{i+1,h}^{n-1} \right\|_\infty + u_{i+1,h}^{n-1} \\ &\le k + Ch^2 \left| \ln h \right|^2 + u_{i+1,h}^{n-1} \end{aligned}$$

and thus,

$$\begin{aligned} a_i(\bar{u}_i^n, v) &\le (f_i, v) \quad \forall v \in H^1(\Omega), \quad v \ge 0, \\ \bar{u}_i^n &\le k + \left\| u_{i+1}^{n-1} - u_{i+1,h}^{n-1} \right\|_\infty + u_{i+1,h}^{n-1}, \\ &\le k + Ch^2 \left| \ln h \right|^2 + u_{i+1,h}^{n-1}. \end{aligned}$$

So $\bar{u}_i^n$ is a subsolution for the V.I with obstacle $k + Ch^2 \left| \ln h \right|^2 + u_{i+1,h}^{n-1}$. Let $\bar{\omega}_i^n = \partial(k + Ch^2 \left| \ln h \right|^2 + u_{i+1,h}^{n-1})$. Then, as $u_i^n = \partial(k + u_{i+1}^{n-1})$, making use of (27), and (43), we get

$$\begin{aligned} \left\| \bar{\omega}_i^n - u_i^n \right\|_\infty &\le Ch^2 \left| \ln h \right|^2 + \left\| u_{i+1,h}^{n-1} - u_{i+1}^{n-1} \right\|_\infty \\ &\le Ch^2 \left| \ln h \right|^2 . \end{aligned}$$

Hence, applying Theorem 4, we have

$$\bar{u}_i^n \le \bar{\omega}_i^n \le u_i^n + Ch^2 \left| \ln h \right|^2 .$$

Now, putting

$$\beta_i^n = \bar{u}_i^n - Ch^2 \left| \ln h \right|^2 , \quad \forall i = 1, ..., M.$$

we obtain

$$\beta_i^n \le u_i^n, \forall i = 1, ..., M \tag{46}$$

and, using (45) ,

$$\begin{aligned}\left\|\beta_i^n - u_{i,h}^n\right\|_\infty &\le \left\|\bar{u}_i^n - Ch^2\left|\ln h\right|^2 - u_{i,h}^n\right\|_\infty \\ &\le \left\|\bar{u}_i^n - u_{i,h}^n\right\|_\infty + Ch^2\left|\ln h\right|^2 \\ &\le Ch^2\left|\ln h\right|^2.\end{aligned} \tag{47}$$

Now, consider the discrete V.I (33)

$$\begin{cases} a_i(\bar{u}_{i,h}^n, v - \bar{u}_{i,h}^n) \ge (f_i, v - \bar{u}_{i,h}^n) \quad \forall v \in \mathbb{V}_h, \\ \bar{u}_{i,h}^n \le r_h\left(k + u_{i+1}^{n-1}\right), \, v \le r_h\left(k + u_{i+1}^{n-1}\right), \end{cases} \tag{48}$$

$\bar{u}_{i,h}^n$ being also a subsolution, we have

$$\begin{cases} a_i(\bar{u}_{i,h}^n, \varphi_i) \le (f_i, \varphi_i) \quad \forall i = 1, ..., m(h), \\ \bar{u}_{i,h}^n \le r_h\left(k + u_{i+1}^{n-1}\right). \end{cases} \tag{49}$$

So, making use of (43), we have

$$\begin{aligned}\bar{u}_{i,h}^n &\le k + r_h u_{i+1}^{n-1} - r_h u_{i+1,h}^{n-1} + r_h u_{i+1,h}^{n-1} \\ &\le k + \left\|r_h u_{i+1}^{n-1} - r_h u_{i+1,h}^{n-1}\right\|_\infty + r_h u_{i+1,h}^{n-1} \\ &\le k + Ch^2\left|\ln h\right|^2 + r_h u_{i+1,h}^{n-1}\end{aligned}$$

and hence

$$\begin{aligned}a(\bar{u}_{i,h}^n, \varphi_i) \le (f_i, \varphi_i) \quad \forall \varphi_i, \\ \bar{u}_{i,h}^n \le k + Ch^2\left|\ln h\right|^2 + r_h \hat{u}_{i+1,h}^{n-1}.\end{aligned}$$

So, $\bar{u}_{i,h}^n$ is a subsolution for the V.I with obstacle $k + Ch^2\left|\ln h\right|^2 + r_h u_{i+1,h}^{n-1}$. Let $\bar{\omega}_{i,h}^n = \partial_h(k + Ch^2\left|\ln h\right|^2 + r_h u_{i+1,h}^{n-1})$. Then, as $u_{i,h}^n = \partial_h(k + r_h u_{i+1,h}^{n-1})$, making use of (30) and (43), we get

$$\left\|\bar{\omega}_{i,h}^n - u_{i,h}^n\right\|_\infty \le Ch^2\left|\ln h\right|^2 + \left\|u_{i+1,h}^{n-1} - u_{i+1,h}^{n-1}\right\|_\infty$$

and, due to Theorem 6, we have

$$\bar{u}_{i,h}^n \le \bar{\omega}_{i,h}^n \le u_{i,h}^n + Ch^2\left|\ln h\right|^2.$$

Now, taking

$$\gamma_{i,h}^n = \bar{u}_{i,h}^n - Ch^2\left|\ln h\right|^2, \quad \forall i = 1, ..., M.$$

we obtain

$$\gamma_{i,h}^n \le u_{i,h}^n. \tag{50}$$

Moreover, $\bar{u}_h^n$ being the discrete counterpart of u^n, using (34), we have

$$\left\|\bar{u}_{i,h}^n - u_i^n\right\|_\infty \le Ch^2\left|\ln h\right|^2, \quad \forall i = 1, ..., M$$

and therefore

$$\left\|\gamma_{i,h}^{n}-u_{i}^{n}\right\|_{\infty} \leq \left\|\bar{u}_{i,h}^{n}-u_{i}^{n}\right\|_{\infty}+Ch^{2}\left|\ln h\right|^{2} \leq Ch^{2}\left|\ln h\right|^{2}. \tag{51}$$

Finally, combining (46), (47) and (50), (51), we obtain

$$\begin{aligned} u_i^n &\leq \gamma_{i,h}^n + Ch^2\left|\ln h\right|^2 \\ &\leq u_{i,h}^n + Ch^2\left|\ln h\right|^2 \\ &\leq \beta_i^n + Ch^2\left|\ln h\right|^2 \\ &\leq u_i^n + Ch^2\left|\ln h\right|^2. \end{aligned}$$

That is

$$\left\|u_i^n - u_{i,h}^n\right\|_\infty \leq Ch^2\left|\ln h\right|^2 \quad \forall i = 1, ..., M.$$

□

4.3. L^∞**-Error estimate for the system of QVIs.** Now combining estimates (10), (17), and (35), we have:

Theorem 10.

$$\left\|U-U_h\right\|_\infty \leq Ch^2\left|\ln h\right|^2. \tag{52}$$

Proof. Indeed,

$$\begin{aligned} \left\|U-U_h\right\|_\infty &\leq \left\|U-U^n\right\|_\infty + \left\|U^n-U_h^n\right\|_\infty + \left\|U_h^n-U_h\right\|_\infty \\ &\leq \mu^n\left\|U^0\right\|_\infty + Ch^2\left|\ln h\right|^2 + \mu^n\left\|U_h^0\right\|_\infty. \end{aligned} \tag{53}$$

So, passing to the limit, as $n \to \infty$, the desired result follows. □

Remark 2. *For practical purposes, it is interesting to estimate the error between the exact solution and the actually computed approximations* U_h^n*, that is,*

$$\left\|U-U_h^n\right\|_\infty \leq \mu^n\left\|U^0\right\|_\infty + Ch^2\left|\ln h\right|^2. \tag{54}$$

Proof. Indeed,

$$\begin{aligned} \left\|U-U_h^n\right\|_\infty &\leq \left\|U-U^n\right\|_\infty + \left\|U^n-U_h^n\right\|_\infty \\ &\leq \mu^n\left\|U^0\right\|_\infty + Ch^2\left|\ln h\right|^2. \end{aligned}$$

□

5. Numerical example

Let $\Omega = (0,1)\times(0,1)$, $M=3$, $\mathcal{A}^i = -\triangle$, $f_1 = \sin^2 x$, $f_2 = \cos^2 x$, $f_3 = e^x$. We divide Ω into squares with edge $h = \frac{1}{10}$, then by diagonals with same direction divide every square into two triangles. Then the finite dimensional quasi-variational inequalities system is

$$\begin{cases} U_i \in K_i, \\ \left(A^i U_i - F_i, V - U_i\right) \geq 0, \quad \forall V \in K_i, \quad i = 1, ..., M, \end{cases} \tag{55}$$

where A^i are the stiffness matrices defined in (11), and the right-hand side $F_i = (f_i, \varphi_l), l = 1, ..., N_h,, K_i = \{V \in R^{N_h} \text{ such that } V \leq K + U_{i+1}\}, U_{M+1} = U_1$, $K = (k, ..., k))^T$ .The iterative scheme is,

$$\begin{cases} U_i^{n+1} \in K^{i,n+1}, \\ \left(A^i U^{i,n+1} - F^i, V - U^{i,n+1}\right) \geq 0, \quad \forall V \in K^{i,n+1}, \quad i = 1, ..., M, \end{cases} \tag{56}$$

where $K^{i,n+1} = \{V \in R^{N_h} \text{ such that } V \leq K + U^{i,n}\}$, $U^{M+1,n} = U^{1,n}$.

We take $k = 0.01$ and solve (56) (Jacobi type) with projected Gauss-Seidel as inner iteration. The stopping criteria for the inner iteration and outer iteration both are $\epsilon = 10^{-6}$, the initial value is $U^0 = (U_1^0, ..., U_M^0)$, such that $A^i U_i^0 = F^i, \quad i = 1, ..., M$.

The computation of the solution for h, $h/2$ and $h/4$ leads to a convergence order $p = 2.062$, which is in good agreement with the theory.

6. Conclusion

This paper addresses the finite element of the Dirichlet problem for an elliptic quasi-variational inequalities system. The optimal error estimate is derived, combining geometric convergence of an iterative scheme and its finite element error estimate, obtained by means of the concept of subsolutions and discrete regularity for variational inequalities. A numerical example is also given to support the theory.

In light of the findings of this work, we wonder whether these can be exploited to:
1. Extend the study to the noncoercive problem.
2. Derive a posteriori error estimate for this system of Q.V.I.
This will be the focus of our attention in future works.

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