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COUPLING OF LAGUERRE TRANSFORM AND FAST BEM FOR SOLVING DIRICHLET INITIAL-BOUNDARY VALUE PROBLEMS FOR THE WAVE EQUATION

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Резюме. Подано поглиблений аналіз двох підходів до розв'язування початково-крайової задачі Діріхле для однорідного хвильового рівняння, який базується на поєднанні перетворення Лагера за часовою змінною і методу граничних елементів (МГЕ) у необмеженій просторовій області. В результаті обидва підходи приводять до тієї ж самої нескінченної трикутної системи граничних інтегральних рівнянь. Аналіз проведено у вагових просторах Соболєва, елементами яких є функції часової змінної, які набувають значень у відповідних просторах Соболєва.

Для зменшення потреби в обчислювальних ресурсах реалізовано швидкий МГЕ, використовуючи адаптивну перехресну апроксимацію отриманих матриць. Крім того, метод поширено на розв'язування задачі Діріхле в області з включенням. Також подано чисельні результати для модельних задач, які ілюструють точність і очікуваний порядок збіжності запропонованого методу.

Abstract. We present an improved analysis of two approaches to solving of the Dirichlet initial-boundary value problem for a homogeneous wave equation, which are based on the combination of the Laguerre transform for the time variable with the Galerkin-BEM in an unbounded spatial domain. Both approaches lead to the same infinite triangular system of boundary integral equations as a result. The analysis is done in weighted Sobolev spaces of functions of the time variable taking values in suitable Sobolev spaces.

For reducing both storage and computational costs we implement the fast BEM using adaptive cross approximation of obtained matrices. Furthermore, we extend this method for solving the Dirichlet problem in the domain with an inclusion. We also present numerical results for some model problems which illustrate the accuracy and estimated convergence order of the proposed method.

1. Introduction

In recent years, many studies have been dedicated to the development of effective methods for the numerical solution of time domain boundary integral equations (TDBIEs), which arise from initial-boundary value problems (IBVPs) for the wave equation. Comprehensive lists of related works are presented in [11, 35]. A common feature of these studies is the usage of deep analytical concepts to take into account the dependence of the solutions on the time variable. However, as noted in [10], the computational complexity of proposed

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approaches is still high for problems in 3D domains and the development of effective numerical methods remains actual.

In this paper we present new results of solving both IBVPs and TDBIEs by approach, which is based on the Laguerre transform (LT) [18, 25] in the time variable. The advantage of this transform is that an inverse LT is easy to calculate. Moreover, for solving both boundary value problems (BVPs) and boundary integral equations (BIEs) in the Laguerre domain, efficient recursive algorithms can be constructed using techniques well developed for elliptic problems and their BIEs.

We distinguish two approaches with respect to the order in which the LT is applied in solving IBVPs. In the first case, the transform is applied directly to the IBVP, and as a result, a BVP for infinite triangular system of elliptic equations is obtained. Such approach was used (without much theoretical justification) for solving different evolutional IBVPs in papers [4,5,13,28,29,33,37], in which for the problems in the Laguerre domain a suitable representation of the solution was also constructed and corresponding BIEs were derived. Variational formulations for such problems and associated BIEs were proposed and justified for the first time in [30].

Theoretical aspects of another approach, when the LT is directly applied to retarded potentials, were investigated in [24, 25]. The results for Dirichlet and Neumann IBVPs obtained therein have enabled to substantiate the equivalence between each of these problems and infinite triangular systems of corresponding BIEs in the Laguerre domain and also to define the scope of the problems that can be solved with help of the LT.

Both aforementioned approaches lead to the same infinite triangular system of BIEs. This fact creates a basis for the justification of the first approach, as well as for the effective implementation of the BEM for numerical solution of the system of BIEs. These two aspects determine the main research goal of this article.

We begin in Section 1 with a brief description of the second approach, where the LT is applied to the TDBIE, which arose from the Dirichlet IBVP by using a retarded single layer potential. We introduce the needed functional spaces, give a definition of the LT and obtain an infinite sequence of BIEs.

In Section 2 we transform the IBVP to the BVP for an infinite system of elliptic equations and explain how this approach leads to a sequence of BIEs. After that we derive the representation of the solution of the IBVP in the form of the Fourier-Laguerre series, which coefficients represent the solution of the BVP in the Laguerre domain. Then in Section 3 we consider the IBVP in the half-space with some inclusion and obtain the representation of its solution using a Green's function for such domain. At the end in Section 4 we demonstrate the implementation of the Galerkin-BEM and its fast modification, and present the results of the numerical experiments.

2. Reduction of the IBVP to the infinite system of BIEs

Let Ω^- be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary Γ, $\Omega := \mathbb{R}^3 \backslash \overline{\Omega^-}$, $\mathbb{R}_+ := (0, \infty)$, $Q := \Omega \times \mathbb{R}_+$ and $\Sigma := \Gamma \times \mathbb{R}_+$. We consider the initial-boundary value problem for the homogeneous wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = 0, \quad (x,t) \in Q, \tag{1}$$

where $\Delta := \sum_{i=1}^{3} \partial^2/\partial x_i^2$ is the Laplace operator. We find a function $u(x,t)$, $(x,t) \in \overline{Q}$, which satisfies (in some sense) the equation(1), homogeneous initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in \Omega, \tag{2}$$

and the Dirichlet boundary condition

$$u(x,t) = g(x,t), \quad (x,t) \in \Sigma, \tag{3}$$

where function g is given on Σ. We also call (1)-(3) a Dirichlet problem.

To solve the IBVP (1)-(3) we use a retarded single layer potential

$$(\mathcal{S}\mu)(x,t) := \frac{1}{4\pi} \int_\Gamma \frac{\mu(y, t - |x-y|)}{|x-y|} d\Gamma_y, \quad (x,t) \in \overline{Q}, \tag{4}$$

where $\mu : \Gamma \times \mathbb{R} \to \mathbb{R}$ is an unknown density. It is known (see, e.g., [34]) that if an arbitrary function $\mu(y,\tau)$ is smooth enough and $\mu(y,\tau) = 0$ for $y \in \Gamma$ and $\tau \leq 0$, then function

$$u(x,t) = (\mathcal{S}\mu)(x,t), \quad (x,t) \in \overline{Q}, \tag{5}$$

satisfies (in the classical sense) the wave equation and initial conditions. The function u satisfies also the boundary condition (3), if μ is a solution of such TDBIE

$$(\mathcal{V}\mu)(x,t) := \frac{1}{4\pi} \int_\Gamma \frac{\mu(y, t - |x-y|)}{|x-y|} d\Gamma_y = g(x,t), \quad (x,t) \in \Sigma. \tag{6}$$

Let X be a Hilbert space with an inner product $(\cdot,\cdot)_X$ and an induced norm $||\cdot||_X$. In order to construct a generalized solution of the IBVP (1)-(3) we consider spaces of functions of the time variable which have values in some Hilbert space X. For such functions the weighted Lebesgue space $L^2_\sigma(\mathbb{R}_+; X)$ [9] with weight $\rho_\sigma(t) = e^{-\sigma t}$ ($t \in \mathbb{R}_+$ and parameter $\sigma > 0$) is the simplest Hilbert space. Elements $v \in L^2_\sigma(\mathbb{R}_+; X)$ are measurable functions $v : \mathbb{R}_+ \to X$ such that $\int_{\mathbb{R}_+} ||v(t)||^2_X \, e^{-\sigma t} dt < \infty$. This space is equipped with the inner product

$$(v,w)_{L^2_\sigma(\mathbb{R}_+;X)} := \int_{\mathbb{R}_+} \big(v(t), w(t)\big)_X \, e^{-\sigma t} dt, \;\; v, w \in L^2_\sigma(\mathbb{R}_+; X), \tag{7}$$

and the norm

$$\|v\|_{L^2_\sigma(\mathbb{R}_+;X)} := \sqrt{(v,v)_{L^2_\sigma(\mathbb{R}_+;X)}}, \;\; v \in L^2_\sigma(\mathbb{R}_+; X). \tag{8}$$

We also consider the weighted Sobolev spaces

$$H^m_\sigma(\mathbb{R}_+;X) := \{\ v \in L^2_\sigma(\mathbb{R}_+;X) \mid v^{(k)} \in L^2_\sigma(\mathbb{R}_+;X),\ v^{(k)}(0) = 0,\ k = \overline{0,m}\} \tag{9}$$

where $m \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers), with norm

$$\|v\|_{H^m_\sigma(\mathbb{R}_+;X)} := \left(\sum_{k=0}^{m} \left\| v^{(k)} \right\|^2_{L^2_\sigma(\mathbb{R}_+;X)} \right)^{1/2}. \tag{10}$$

Here derivatives $v^{(k)}$, $k \in \mathbb{N}$, are understood in terms of the space $\mathcal{D}'(\mathbb{R}_+;X)$, elements of which are distributions with values in the space X. We assume that elements of the space $H^m_\sigma(\mathbb{R}_+;X)$ are extended with zero for non-positive arguments.

It is well known [18], that Laguerre polynomials $\{L_k(\sigma\cdot)\}_{k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}}$ form an orthogonal basis in the space $L^2_\sigma(\mathbb{R}_+) := L^2_\sigma(\mathbb{R}_+;\mathbb{R})$, that is, for every function $f \in L^2_\sigma(\mathbb{R}_+)$ there exists its expansion in the Fourier-Laguerre series

$$f(t) = \sum_{k=0}^{\infty} f_k\, L_k(\sigma t),\ t \in \mathbb{R}_+, \tag{11}$$

where Fourier-Laguerre coefficients $f_0, f_1, ..., f_k, ...$ have the representation formula

$$f_k := \sigma \int\limits_{\mathbb{R}_+} f(t)\, L_k(\sigma t)\, e^{-\sigma t} dt, \quad k \in \mathbb{N}_0. \tag{12}$$

We write a sequence of any elements of the set X as a vector-column $\mathbf{v} := (v_0, v_1, ...)^\top$ and denote by X^∞ a set of all possible sequences of elements of the set X. In particular, we consider a space of numerical sequences $l^2 := \big\{\mathbf{v} \in \mathbb{R}^\infty \mid \sum\limits_{j=0}^{\infty} |v_j|^2 < +\infty\big\}$ with the inner product $(\mathbf{v},\mathbf{w}) = \sum\limits_{j=0}^{\infty} v_j w_j$ and the norm $||\mathbf{v}||_{l^2} := \left(\sum\limits_{j=0}^{\infty} |v_j|^2 \right)^{1/2}$ for $\mathbf{v},\mathbf{w} \in l^2$.

We recall [18] that the Laguerre transform (LT) is a mapping $\mathcal{L} : L^2_\sigma(\mathbb{R}_+) \to l^2$, which maps an arbitrary function f to a sequence $\mathbf{f} = (f_0, f_1, ..., f_k, ...)^\top$ according to the rule (12). We will also use the notation $\mathcal{L}_k f \equiv (\mathcal{L}f)(k) := f_k\ \ \forall k \in \mathbb{N}_0$. Note that the Parseval equality holds

$$||f||^2_{L^2_\sigma(\mathbb{R}_+)} = \frac{1}{\sigma} \sum_{k=0}^{\infty} |f_k|^2. \tag{13}$$

The LT $\mathcal{L}$ is a bijective mapping and its inverse $\mathcal{L}^{-1} : l^2 \to L^2_\sigma(\mathbb{R}_+)$ maps an arbitrary sequence $\mathbf{h} = (h_0,\, h_1, ...,\, h_k, ...)^\top$ to a function

$$(\mathcal{L}^{-1}\mathbf{h})(t) := \sum_{k=0}^{\infty} h_k\, L_k(\sigma t),\ t \in \mathbb{R}_+. \tag{14}$$

For the arbitrary function $f \in L^2_\sigma(\mathbb{R}_+)$ we have an equality

$$\mathcal{L}^{-1}\mathcal{L}f = f. \tag{15}$$

In [24] the LT was extended on functions of time variable with values in the Hilbert space X. LT was considered as a mapping $\mathcal{L} : L^2_\sigma(\mathbb{R}_+; X) \to X^\infty$ which operates according to the rule (12).

Let

$$l^2(X) := \{\mathbf{v} \in X^\infty \mid \sum_{j=0}^{\infty} \|v_j\|^2_X < +\infty\}$$

be a Hilbert space with the inner product $(\mathbf{v}, \mathbf{w}) = \sum\limits_{j=0}^{\infty} (v_j, w_j)_X$ and the norm $||\mathbf{v}||_{l^2(X)} := \left(\sum\limits_{j=0}^{\infty} \|v_j\|^2_X \right)^{1/2}$, $\mathbf{v}, \mathbf{w} \in l^2(X)$.

Proposition 1 ([24], Theorem 2)**.** *The mapping $\mathcal{L} : L^2_\sigma(\mathbb{R}_+; X) \to X^\infty$ that maps an arbitrary function f to a sequence $\mathbf{f} := (f_0,\, f_1, \ldots,\, f_k, \ldots)^\top$ according to the formula (12), is injective and its image is the space $l^2(X)$, and*

$$\|f\|^2_{L^2_\sigma(\mathbb{R}_+;X)} = \frac{1}{\sigma} \sum_{k=0}^{\infty} ||f_k||^2_X. \tag{16}$$

In addition, for the arbitrary function $f \in L^2_\sigma(\mathbb{R}_+; X)$ we have an equality

$$\mathcal{L}^{-1}\mathcal{L}f = f, \tag{17}$$

where the mapping $\mathcal{L}^{-1} : l^2(X) \to L^2_\sigma(\mathbb{R}_+; X)$ is the inverse to $\mathcal{L}$ and maps the arbitrary sequence $\mathbf{h} := (h_0,\, h_1, \ldots,\, h_k, \ldots)^\top$ to the function h according to the formula (14).

Definition 4 ([24])**.** Let $\sigma > 0$ and X be a Hilbert space. Mappings

$$\mathcal{L} : L^2_\sigma(\mathbb{R}_+; X) \to l^2(X) \quad \text{and} \quad \mathcal{L}^{-1} : l^2(X) \to L^2_\sigma(\mathbb{R}_+; X),$$

mentioned in theorem 1, are called, respectively, *direct and inverse Laguerre transforms*, and the formula (16) is an analogue of the Parseval equality.

Definition 5 ([23])**.** Let X, Y, Z be arbitrary sets and $q : X \times Y \to Z$ be some mapping. By a *q-convolution* of sequences $\mathbf{u} \in X^\infty$ and $\mathbf{v} \in Y^\infty$ we understand the sequence $\mathbf{w} := (w_0, w_1, \ldots, w_j, \ldots)^\top \in Z^\infty$, whose elements are obtained by the rule

$$w_j := \sum_{i=0}^{j} q\left(u_{j-i}, v_i\right) \equiv \sum_{i=0}^{j} q\left(u_i, v_{j-i}\right), \;\; j \in \mathbb{N}_0; \tag{18}$$

the q-convolution of $\mathbf{u}$ and $\mathbf{v}$ is shortly written in form $\mathbf{w} = \mathbf{u} \underset{q}{\circ} \mathbf{v}$.

If $X = \mathcal{L}(Y, Z)$ is a space of linear operators acting from the space Y into the space Z and $q(A, v) = Av$, $A \in \mathcal{L}(Y, Z)$, $v \in Y$, then components of the q-convolution of arbitrary sequences $\mathbf{A} \in \big(\mathcal{L}(Y,Z)\big)^\infty$ and $\mathbf{v} \in Y^\infty$ are represented by the formula

$$w_j = \sum_{i=0}^{j} A_{j-i} v_i, \quad j \in \mathbb{N}_0. \tag{19}$$

In this case we write $\mathbf{w} = \mathbf{A} \underset{Z}{\circ} \mathbf{v}$.

Note that for any function $f \in L^2_\sigma(\ R_+; X)$ the Fourier-Laguerre series of the function $f(t-a)$, $a > 0$, can be expressed in terms of the sequence $\mathbf{f} := \mathcal{L}f$ [24, Lemma 1]:

$$f\big(\,\cdot - a\big) = e^{-\sigma a} \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \zeta_{j-i}(\sigma a) f_i \right) L_j(\sigma \cdot) \ \text{ in } L^2_\sigma(\mathbb{R}_+; X), \tag{20}$$

where

$$\zeta_0(s) := 1, \ \ \zeta_k(s) := L_k(s) - L_{k-1}(s), \ \ s \in \overline{\mathbb{R}_+} = [0,\infty), \qquad k \in \mathbb{N}. \tag{21}$$

Let $H^1(\Omega)$ and $H^{1/2}(\Gamma)$ denote the usually defined (see, e.g., [17]) Sobolev spaces and $H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))'$. Consider now the retarded single layer potential (4) and TDBIE (6). Assuming the density $\mu \in L^2_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$ is sufficiently smooth, we can write the expansion [24]:

$$(\mathcal{S}\mu)(x,t) = \sum_{j=0}^{\infty} u_j(x)\, L_j(\sigma t), \ \ (x,t) \in Q, \tag{22}$$

where coefficients $u_j := \mathcal{L}_j\, \mathcal{S}\mu$, $j \in \mathbb{N}_0$, are components of the q-convolution

$$\mathbf{u}(x) := \big(\mathbf{S} \underset{H^1(\Omega)}{\circ} \boldsymbol{\mu}\big)(x), \ \ x \in \Omega. \tag{23}$$

Here $\boldsymbol{\mu} := \mathcal{L}\mu$ and the sequence $\mathbf{S}$ consists of operators $S_k : H^{-1/2}(\Gamma) \to H^1(\Omega)$, $k \in \mathbb{N}_0$, acting on any function $\xi \in L^2(\Gamma)$ according to the rule

$$(S_k\xi)(x) := \int_\Gamma \xi(y) e_k(x-y)\, d\Gamma_y, \ \ x \in \Omega, \tag{24}$$

where

$$e_0(z) := \frac{e^{-\sigma|z|}}{4\pi|z|}, \ e_k(z) := \frac{e^{-\sigma|z|}}{4\pi|z|}\big(L_k(\sigma|z|) - L_{k-1}(\sigma|z|)\big), \ \ z \in \mathbb{R}^3 \setminus \{0\}, \ \ k \in \mathbb{N}. \tag{25}$$

One can extend the expression (24) to the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ duality product $(S_k\xi)(x) = \big\langle \xi(\cdot), e_k(x - \cdot)\big\rangle_\Gamma$, $x \in \Omega$, for elements $\xi \in H^{-1/2}(\Gamma)$ [24].

Similarly, applying the LT to the equation (6), we obtain an infinite triangular system of BIEs

$$\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu} = \mathbf{g} \quad \text{on } \Gamma, \tag{26}$$

where $\mathbf{g} := \mathcal{L}g$ and $\mathbf{V}$ is a sequence of boundary operators $V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $k \in \mathbb{N}_0$, which may be expressed as a composition $V_k := \gamma_0 \circ S_k$ of operator S_k with trace operator γ_0. In case of $\xi \in L^2(\Gamma)$ we have

$$(V_k\xi)(x) = \int_\Gamma \xi(y)e_k(x-y)\,d\Gamma_y, \quad x \in \Gamma. \tag{27}$$

Proposition 2 ([24], Theorem 1)**.** *Let $g \in H^{m+4}_{\sigma_0}(\mathbb{R}_+; H^{1/2}(\Gamma))$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then there exists a unique generalized solution of the problem (1)-(3), it belongs to the space $H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$ and for any $\sigma \geq \sigma_0$ such an inequality holds*

$$||u||_{H^{m+1}_\sigma(\mathbb{R}_+;H^1(\Omega))} \leq C||g||_{H^{m+4}_\sigma(\mathbb{R}_+;H^{1/2}(\Gamma))}, \tag{28}$$

where $C > 0$ is a constant that is not dependent on g.

In addition, the generalized solution of the problem (1)-(3) can be represented as a sum of series (22), that is convergent in the space $L^2_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$, which coefficients $\mathbf{u}$ are defined by formula (23), where the sequence $\boldsymbol{\mu} \in l^2(H^{-1/2}(\Gamma))$ is a solution of the system of the BIEs (26) with $\mathbf{g} := \mathcal{L}g$.

Note that the assumption about the function g in the proposition 2 guarantees the applicability of the LT at all stages of constructing of the numerical solution to the problem (1)-(3) without any additional assumption about relation between parameters m and σ_0. On theoretical aspects of generalized solutions to such problems in other functional spaces, see, for example, in [21].

3. System of the convolutional type and its solution

We can also obtain both the representation (22) of the generalized solution of the problem (1)-(3) and the system of the BIEs (26) in another way. For this we use such property of the LT for the derivatives of the function $f \in H^2_\sigma(\mathbb{R}_+; X)$:

$$\mathcal{L}_k\left(\frac{\partial^2 f(t)}{\partial t^2}\right) = \sigma^2 \sum_{l=0}^{k}(k-l+1)\mathcal{L}_l\big(f(t)\big), \quad k \in \mathbb{N}_0. \tag{29}$$

By applying the LT to the wave equation (1) directly and using (29), in Ω we obtain the following infinite triangular system of elliptic equations

$$\begin{cases} Pu_0 = 0, \\ c_1u_0 + Pu_1 = 0, \\ c_2u_0 + c_1u_1 + Pu_2 = 0, \\ \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ c_ku_0 + c_{k-1}u_1 + ... + Pu_k = 0, \\ \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \end{cases} \tag{30}$$

where $u_k := \mathcal{L}_k u$, $k \in \mathbb{N}_0$, are the unknown functions and $P := c_0 I - \Delta$, $c_k := (k+1)\sigma^2$, I is the identity operator. Henceforth we denote $\mathbf{u} := (u_0, u_1, ...)^\top$ and $\mathbf{G}$ the infinite triangular matrix in the left hand side of (30). This allows us to rewrite the system in form

$$\mathbf{G}\mathbf{u}{=}\mathbf{0} \text{ in } \Omega. \tag{31}$$

By the LT we obtain from the condition (3) a sequence of boundary conditions regarding the unknown functions

$$\gamma_0 \mathbf{u} = \mathbf{g} := \mathcal{L}g \text{ on } \Gamma. \tag{32}$$

Theorem 1. *Let the given function g satisfies the condition of the proposition 2, that is, $g \in H^{m+4}_{\sigma_0}(\mathbb{R}_+; H^{1/2}(\Gamma))$ for some $\sigma_0 > 0$ and $m \in \mathbb{N}_0$. Then the unique generalized solution $u \in H^{m+1}_{\sigma_0}(\mathbb{R}_+; H^1(\Omega))$ of the problem (1)-(3) can be represented by the solution $\mathbf{u} := (u_0, u_1, ...)^\top$ of the boundary value problem (31), (32) as the sum of a series*

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x)\, L_j(\sigma t), \quad (x,t) \in Q. \tag{33}$$

Proof. Let us consider a top part $\mathbf{G}^k \mathbf{u}^k{=}\mathbf{0}$ of the system (31) for any fixed $k \in \mathbb{N}_0$, which consists from the $k+1$ equations. According to the [30, Lemma 2] its any solution $\mathbf{u}^k := (u_0, u_1, ..., u_k)^\top$ can be represented in Ω by the formula

$$u_j(x) = \sum_{i=0}^{j} \left\langle \mu_i(\cdot), e_{j-i}(x-\cdot) \right\rangle_\Gamma, \; x \in \Omega, \; j \in \mathbb{N}_0, \tag{34}$$

where μ_j, $j \in \mathbb{N}_0$, are some elements of the space $H^{-1/2}(\Gamma)$ and functions e_j, $j \in \mathbb{N}_0$, may be expressed through a fundamental solution $\mathbf{E} := (E_0, E_1, ...)^\top$ of the operator $\mathbf{G}$ in form

$$e_0 := E_0, \;\; e_j := E_j - E_{j-1}, \; j \in \mathbb{N}. \tag{35}$$

In addition, if the sequence $\boldsymbol{\mu}^k := \big(\mu_0, \mu_1, ..., \mu_k\big)^\top$ is obtained as a solution of the system of BIEs

$$\sum_{i=0}^{j} \left\langle \mu_i(\cdot), e_{j-i}(x-\cdot) \right\rangle_\Gamma = g_j, \; x \in \Gamma, \; j \in \overline{0,k}, \tag{36}$$

then the sequence $\mathbf{u}^k$ will be the solution of suitable Dirichlet problem for the system $\mathbf{G}^k \mathbf{u}^k{=}\mathbf{0}$.

Notice that (35) may be reduced to form (25) [31, Theorem 1]. Therefore, the formula (34) coincides with the representation of the Fourier-Laguerre coefficients of the retarded potential (4) and BIEs in the system (36) are the same as in the infinite system (26). So sequence $\boldsymbol{\mu} := \big(\mu_0, \mu_1, ...\big)^\top$ coincides with LT of the solution μ of the TDBIE (6) and, as a consequence, the solution $\mathbf{u}$ of the problem (31), (32) coincides with LT of the solution u of the problem (1)-(3). As a conclusion from the Proposition 2 we have that $\boldsymbol{\mu} \in l^2(H^{-1/2}(\Gamma))$ and $\mathbf{u} \in l^2(H^1(\Omega))$.

Using the notation (21), in the case $\mu_i \in L^2_\sigma(\Gamma)$ we can rewrite the formula (34)

$$\begin{aligned} u_j(x) &= \sum_{i=0}^{j} \int_\Gamma \mu_i(y) e_{j-i}(x-y) d\Gamma_y = \\ &= \int_\Gamma \frac{e^{-\sigma|x-y|}}{4\pi|x-y|} \sum_{i=0}^{j} \mu_i(y) \zeta_{j-i}(\sigma|x-y|) d\Gamma_y. \end{aligned} \tag{37}$$

By substituting the expression (37) into the partial sum

$$\widetilde{u}^k(x,t) := \sum_{j=0}^{k} u_j(x) L_j(\sigma t), \quad (x,t) \in Q, \tag{38}$$

and taking the external sum into the integral over Γ we obtain

$$\widetilde{u}^k(x,t) = \int_\Gamma \frac{e^{-\sigma|x-y|}}{4\pi|x-y|} \sum_{j=0}^{k} \sum_{i=0}^{j} \mu_i(y) \zeta_{j-i}(\sigma|x-y|) L_j(\sigma t) d\Gamma_y, \quad (x,t) \in Q. \tag{39}$$

Taking into account, that $\boldsymbol{\mu} \in l^2(H^{-1/2}(\Gamma))$ and formula (20) holds for this sequence, putting $k \to \infty$ we finally get

$$u(x,t) = \int_\Gamma \frac{1}{4\pi|x-y|} \mu(y, t - |x-y|) d\Gamma_y, \quad (x,t) \in Q, \tag{40}$$

where $\mu = \mathcal{L}^{-1}\boldsymbol{\mu}$. Since μ is the solution of the TDBIE (6), the retarded potential (40) coincides with potential (4). Therefore, (40) is the solution of the problem (1)-(3). $\square$

Taking into account that the system (26) is triangular we rewrite it as a sequence of BIEs

$$\begin{cases} (V_0\mu_0)(x) = g_0(x), \\ (V_0\mu_1)(x) = \tilde{g}_1(x), \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ (V_0\mu_k)(x) = \tilde{g}_k(x), \quad k \in \mathbb{N}, \quad x \in \Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{41}$$

with recurrent expressions in right-hand sides

$$\tilde{g}_k(x) := g_k(x) - \sum_{i=0}^{k-1} (V_{k-i}\mu_i)(x), \quad k \in \mathbb{N}. \tag{42}$$

Since the boundary operator V_0 is $H^{-1/2}(\Gamma)$-elliptical [6,17], for arbitrary fixed $k \in \mathbb{N}_0$ the $k-$th equation in (41) with $g_k \in H^{1/2}(\Gamma)$ has a unique solution $\mu_k \in H^{-1/2}(\Gamma)$. We can choose (by some criteria) the value of parameter N and find from (41) the first components for the sequence $\boldsymbol{\mu}^N := \big(\mu_0, \mu_1, ..., \mu_N, 0, 0, ...\big)^\top$.

Using it for calculation a sequence $\mathbf{u}^N := \big(u_0, u_1, ..., u_N, 0, 0, ...\big)^\top$ by the formula

$$\mathbf{u}^N(x) = \big(\mathbf{S} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu}^N\big)(x), \;\; x \in \Omega, \tag{43}$$

we obtain an approximate solution $\widetilde{u}^N(x,t)$ of the problem (1)-(3) as a partial sum (38) of the expansion (22) of the exact solution $u(x,t)$.

4. Problems in the domain with an inclusion

Reducing the IBVP (1)-(3) to the BVP (31), (32) allows us to solve it by numerical approaches, which have been successfully used for solution of the elliptic problems. In particular, it concerns the use of surface potentials, which are based on Green's function [8] for specific domain Ω_0 instead of the fundamental solution (25) for operator $\mathbf{G}$ in $\mathbb{R}^3$. Suppose Γ_0 is a Lipschitz boundary of Ω_0.

Definition 6 ([31]). Let $\mathbf{N}(x,y) := (N_0(x,y), N_1(x,y), ...)^\top$, $(x,y) \in \overline{\Omega}_0 \times \Omega_0$ be a solution of the equation

$$\mathbf{G}\mathbf{u} = \bar{\boldsymbol{\delta}}_y \;\text{in}\; \big(\mathcal{D}'(\Omega_0)\big)^\infty, \tag{44}$$

where $\bar{\boldsymbol{\delta}}_y := (\,\delta(\cdot - y),\; 0,\; 0,\; ...\,)^\top$. We say that $\mathbf{N}$ is *Green's function for the Dirichlet problem for the system* (31) *in the domain* Ω_0 if all its components vanish for $(x,y) \in \Gamma_0 \times \Omega_0$.

Building the Green's function for the domain with arbitrary geometry isn't a simple task in general. But for domains with a certain type of symmetry it can be built analytically by the reflection method [31]. Without loss of generality we present here the Green's function for the Dirichlet problem in case of the half-space $\Omega_0 = \mathbb{R}^2 \times \mathbb{R}_+$:

$$N_k(x,y) = e_k(x-y) - e_k(x-y^*), \; k \in \mathbb{N}_0, \tag{45}$$

where y^* is a point symmetric to the point y in regards to the plane Γ_0 and functions e_k are defined by (25).

Let us denote the unit exterior normal vector to the surface Γ_0 as $\boldsymbol{\nu}$. Consider a sequence $\mathbf{D}$ which consists of operators $D_k : H^{1/2}(\Gamma_0) \to H^1(\Omega))$, $k \in \mathbb{N}_0$, that act on an arbitrary function $\xi \in H^{1/2}(\Gamma_0)$ according to the rule

$$(D_k\xi)(x) := \int_{\Gamma_0} \xi(y)\, \partial_\nu N_k(x,y)\, d\Gamma_y, \;\; x \in \Omega_0, \tag{46}$$

where ∂_ν is the notation of the normal derivative. If $\boldsymbol{\lambda} \in l^2(H^{1/2}(\Gamma_0))$ is an arbitrary sequence then a sequence

$$\mathbf{u}(x) := -\big(\mathbf{D} \underset{H^1(\Omega)}{\circ} \boldsymbol{\lambda}\big)(x), \;\; x \in \Omega_0, \tag{47}$$

satisfies the system (31) [31].

Let bounded domain Ω^- with a Lipschitz boundary Γ is an inclusion in the domain Ω_0 ($\Gamma_0 \cap \Gamma = \oslash$) and $\Omega := \Omega_0 \setminus \overline{\Omega^-}$. For an arbitrary function

$\mu \in L^2_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$ let us consider q-convolution

$$\mathbf{u}(x) := \big(\widetilde{\mathbf{S}} \underset{H^1(\Omega)}{\circ} \boldsymbol{\mu}\big)(x), \quad x \in \Omega, \tag{48}$$

of sequences $\boldsymbol{\mu} := \mathcal{L}\mu$ and $\widetilde{\mathbf{S}} := (\widetilde{S}_0, \widetilde{S}_1, ...)^\top$, where operators $\widetilde{S}_k : H^{-1/2}(\Gamma) \to H^1(\Omega))$, $k \in \mathbb{N}_0$, act on an arbitrary $\xi \in L^2(\Gamma)$ according to the rule

$$(\widetilde{S}_k\xi)(x) := \int_\Gamma \xi(y)\ N_k(x,y)\, d\Gamma_y, \quad x \in \Omega. \tag{49}$$

For $\xi \in H^{-1/2}(\Gamma)$ one can extend the expression (49) to the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ duality product $(\widetilde{S}_k\xi)(x) = \big\langle \xi(\cdot), N_k(x - \cdot)\big\rangle_\Gamma$ with $x \in \Omega$. It is easy to see that for arbitrary functions $\mu \in L^2_\sigma(\mathbb{R}_+; H^{-1/2}(\Gamma))$ and $\lambda \in L^2_\sigma(\mathbb{R}_+; H^{1/2}(\Gamma_0))$ a combination of the sequences

$$\mathbf{u}(x) := \big(\widetilde{\mathbf{S}} \underset{H^1(\Omega)}{\circ} \boldsymbol{\mu}\big)(x) - \big(\mathbf{D} \underset{H^1(\Omega)}{\circ} \boldsymbol{\lambda}\big)(x), \ x \in \Omega, \tag{50}$$

satisfies the system (31) in Ω and the boundary condition $\gamma_0\mathbf{u} = \boldsymbol{\lambda}$ on Γ_0.

Suppose u satisfies the wave equation (1) and initial conditions (2) in Ω and traces $\gamma_{0,0}u = \lambda$ and $\gamma_{0,1}u = g$ are given on the cylinders $\Sigma_0 := \Gamma_0 \times \mathbb{R}_+$ and $\Sigma = \Gamma \times \mathbb{R}_+$ respectively. Then unknown sequence $\boldsymbol{\mu}$ for the representation (50) can be obtained from the system of BIEs

$$\widetilde{\mathbf{V}} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu} = \mathbf{g} + \gamma_{0,1}\big(\mathbf{D} \underset{H^1(\Omega)}{\circ} \boldsymbol{\lambda}\big) \quad \text{on } \Gamma, \tag{51}$$

where $\mathbf{g} := \mathcal{L}g$ and the components of the sequence $\widetilde{\mathbf{V}}$ are boundary operators $\widetilde{V}_k := \gamma_{0,1} \circ \widetilde{S}_k$, $\widetilde{V}_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $k \in \mathbb{N}_0$. Note that the resulting system can be reduced to the sequence of BIEs similar to (41) and has only one solution.

5. Fast BEM and results of numerical experiments

Both (26) and (51) systems are triangular so one can solve their equations sequentially. For this we use Galerkin-BEM and it fast modification [16, 36].

Let $\Gamma^M = \bigcup_{l=1}^M \overline{\tau}_l$ be some approximation of the boundary Γ by triangular boundary elements $\{\tau_l\}_{l=1}^M$ and $\{\varphi_l^0\}_{l=1}^M$ be a set of linearly-independent on Γ^M piece-wise constant functions

$$\varphi_l^0(x) = \begin{cases} 1, & x \in \tau_l, \\ 0, & x \notin \tau_l. \end{cases} \tag{52}$$

Treating a value $h := \max\limits_{l=\overline{1,M}} \big(\int_{\tau_l} \mathrm{d}s\big)^{1/2}$ as a parameter of the spatial approximation, we will consider a finite-dimensional space $S_h^0(\Gamma) := \operatorname{span}\{\varphi_l^0\}_{l=1}^M$ and represent a numerical solution of the system (41) by a sequence $\boldsymbol{\mu}^{N,h} :=$

$\left(\mu_0^h, \mu_1^h, ..., \mu_N^h, 0, 0, ...\right)^\top$ which components are linear combinations of piecewise constant functions

$$\mu_k^h = \sum_{l=1}^{M} \mu_{k,l}^h \varphi_l^0 \ \in S_h^0(\Gamma), \ \ k \in \mathbb{N}_0. \tag{53}$$

Here $\left\{\mu_{k,l}^h\right\}_{l=1}^M =: \boldsymbol{\mu}_k^h \in \mathbb{R}^M$ is a vector of unknown coefficients which can be found from the following system of linear algebraic equations

$$\mathbf{V}_0^h \boldsymbol{\mu}_k^h = \mathbf{g}_k^h - \sum_{j=0}^{k-1} \mathbf{V}_{k-j}^h \boldsymbol{\mu}_j^h, \ \ k \in \mathbb{N}_0, \tag{54}$$

where $g_k^h[i] = \int_{\tau_i} g_k(x) ds_x, \ \ i = \overline{1, M}$, and elements of the matrix $\mathbf{V}_p^h$ have following form

$$V_p^h[i,l] = \int_{\tau_i} \int_{\tau_l} e_p(x-y) ds_y ds_x, \ \ i,l = \overline{1,M}, \ p \in \mathbb{N}_0. \tag{55}$$

Notice, that for any $k \geq 1$ the components $\mu_0, \mu_1, ..., \mu_{k-1}$, obtained from BIE (41) on previous steps, are included into the expression in the right-hand side of the current equation. The evaluation of the surfaces integrals (55) has been discussed in [32].

We interpret sequences

$$\boldsymbol{\mu}^{N,h} := \left(\mu_0^h, \mu_1^h, ..., \mu_N^h, 0, 0, ...\right)^\top$$

and

$$\mathbf{u}^{N,h} := \left(u_0^h, u_1^h, ..., u_N^h, 0, 0, ...\right)^\top$$

with some fixed value of the parameter N as numerical solutions of the systems of BIEs (26) and the BVP (31)-(32), respectively. As well, a partial sum $\widetilde{u}^{N,h}(x,t) := \sum\limits_{j=0}^{N} u_j^h(x) L_j(\sigma t)$ we use as a numerical solution of the problem (1)-(3).

Let us assess the accuracy of the proposed method. Taking into account an obvious inequality $\|u - \widetilde{u}^{N,h}\|_{H^1_\sigma(\mathbb{R}_+;H^1(\Omega))} \leq \|u - \widetilde{u}^N\|_{H^1_\sigma(\mathbb{R}_+;H^1(\Omega))} + \|\widetilde{u}^N - \widetilde{u}^{N,h}\|_{H^1_\sigma(\mathbb{R}_+;H^1(\Omega))}$, in this paper we restrict ourselves to examining the posteriori error of the numerical solution, which corresponds to the second term in the right hand part of this inequality. An asymptotic error of the numerical solution in this case has been investigated in [22].

In the following we demonstrate numerical solutions of some model problems for the wave equation in the domain $\Omega = \mathbb{R}^3 \backslash \overline{\Omega^-}$, where $\Omega^- = (-1,1)\times(-1,1)\times(-1,1)$. For generating boundary values we use a spherical impulse represented by the formula

$$w(x,t) := |x|^{-1} w^*(t - |x| + 1) \vartheta(t - |x| + 1), \ (x,t) \in \mathbb{R}^3 \setminus \{0\} \times \mathbb{R}_0, \tag{56}$$

with a cubic B-spline w^* and the Heaviside step function $\vartheta(t)$. Notice that the function w satisfies (1) and (2).

Example 1. We consider the problem (1)-(3) in $\Omega \times \mathbb{R}_+$ with the given trace data $g = w$ on Σ and analyze accuracy and convergence of numerical solutions u_k^h and $\widetilde{u}^{N,h}$ on the sequence of discretization Γ^M with increasing M and with $N = 20$.

TABL. 1. Convergence analysis of u_0^h, u_{10}^h and $\widetilde{u}^{N,h}$ for Example 1 with $\sigma = 4$, $N = 20$ and increasing M

M	$u_0^h(x)$			u_{10}^h			$\widetilde{u}^{N,h}$		
	δ_0^h	eoc_0	ϵ_0^h	δ_{10}^h	eoc_{10}	ϵ_{10}^h	$\widetilde{\delta}^{N,h}$	$\widetilde{eoc}^{N,h}$	$\widetilde{\epsilon}^{N,h}$
108	$1.92{\cdot}10^{-4}$		3.24	$2.92{\cdot}10^{-3}$		22.21	$2.40{\cdot}10^{-2}$		4.66
300	$7.01{\cdot}10^{-5}$	2.03	1.18	$8.46{\cdot}10^{-4}$	2.43	6.43	$8.11{\cdot}10^{-3}$	2.13	1.57
768	$3.22{\cdot}10^{-5}$	2.42	0.54	$2.97{\cdot}10^{-4}$	2.23	2.26	$3.09{\cdot}10^{-3}$	2.05	0.60
1452	$1.83{\cdot}10^{-5}$	2.24	0.31	$1.49{\cdot}10^{-4}$	2.16	1.14	$1.62{\cdot}10^{-3}$	2.03	0.31
1728	$1.55{\cdot}10^{-5}$	2.16	0.26	$1.24{\cdot}10^{-4}$	2.14	0.94	$1.36{\cdot}10^{-3}$	2.02	0.26
2700	$1.02{\cdot}10^{-5}$	2.14	0.17	$7.72{\cdot}10^{-5}$	2.12	0.59	$8.63{\cdot}10^{-4}$	2.03	0.17
4800	$5.93{\cdot}10^{-6}$	2.11	0.10	$4.22{\cdot}10^{-5}$	2.10	0.32	$4.83{\cdot}10^{-4}$	2.02	0.09

At first we consider the impact of the parameter h on the approximation error of numerical solutions u_k^h, $k \in \overline{0,N}$, and $\widetilde{u}^{N,h}$ with some fixed value of the parameter N. For this we compute values $\delta_k^h := ||u_k^h - u_k||_{L^2(\Omega_{a,b})}$ and $\epsilon_k^h := \delta_k^h/||u_k||_{L^2(\Omega_{a,b})} * 100\ \%$, and also values $\widetilde{\delta}^{N,h} := ||\widetilde{u}^{N,h} - \widetilde{u}^N||_{L^2_\sigma(\mathbb{R}_+;L^2(\Omega_{a,b}))}$ and $\widetilde{\epsilon}^{N,h} := \widetilde{\delta}^{N,h}/||\widetilde{u}^N||_{L^2_\sigma(\mathbb{R}_+;L^2(\Omega_{a,b}))} * 100\ \%$, where $(a,b) =: \Omega_{(a,b)}$ is a spatial interval from which observation points are taken. Notice that we provide estimates in the norm of such Lebesgue space with aim to simplify calculations in the unbounded exterior domain Ω. Using a sequence of finite-dimensional spaces $S_h^0(\Gamma)$ with decreasing h for both kinds of numerical solutions we evaluate estimated orders of convergence [36] $eoc_k := \ln(\delta_k^{h_{j-1}}/\delta_k^{h_j})/\ln(h_{j-1}/h_j)$, $k \in \overline{0,N}$, and $\widetilde{eoc}^{N,h} := \ln(\widetilde{\delta}^{N,h_{j-1}}/\widetilde{\delta}^{N,h_j})/\ln(h_{j-1}/h_j)$, where h_{j-1} and h_j are consequent values of the parameter h.

Computed in $\Omega_{(a,b)}$ with $a = (1.2, 0, 0)$ and $b = (10, 0, 0)$, some results of the series of numerical experiments are given in Table 1. They highlight that $eoc \approx 2$ for both numerical solutions u_k^h and $\widetilde{u}^{N,h}$.

Now we assume that the cube Ω^- is included in the half space $\Omega_0 = \mathbb{R}^2 \times (-2, \infty)$ and $\Omega = \Omega_0 \setminus \overline{\Omega^-}$. For generating boundary functions in this case we use a function $\widehat{w}(x,t) := w(x,t) - w(x^*,t)$, where x^* is a point symmetric to the point x with respect to the plane $\Gamma_0 = \{(x_1, x_2, x_3) \mid x_3 = -2\}$. It is obvious that function $\widehat{w}$ satisfies (1) and (2) and $\widehat{w}(x,t) \equiv 0$ on Γ_0.

Example 2. We consider the problem (1)-(3) in $\Omega \times \mathbb{R}_+$ with traces $\gamma_{0,0}u = \lambda \equiv 0$ and $\gamma_{0,1}u = g = \widehat{w}$ given on the cylinders $\Sigma_0 := \Gamma_0 \times \mathbb{R}_+$ and $\Sigma = \Gamma \times \mathbb{R}_+$ respectively, and analyze accuracy and convergence of numerical solutions u_k^h and $\widetilde{u}^{N,h}$ on the sequence of discretization Γ^M with increasing M and with $N = 20$.

We solve this problem by modified BEM using the representation (50) based on Green's functions for the Dirichlet problem for the system (31) in the domain Ω_0. In this approach after discretization of BIEs we obtain matrices $\widetilde{\mathbf{V}}_k^h$ similar to the $\mathbf{V}_k^h$, $k \in \mathbb{N}_0$. Results of the numerical experiment are plotted in Figure 1.

As we can see from the Table 2 numerical solutions, obtained in this approach, have the same accuracy and the convergence order as in the previous example. Notice that some complication of the method due to the use of Green's functions does not lead to significant increase of computational resources for solving the problem in the domain with inclusion. The fact that we have avoided solving BIEs on the unbounded surface Γ_0 is an advantage of the modified BEM in solving such problems.

TABL. 2. Convergence analysis of u_0^h, u_{10}^h and $\widetilde{u}^{N,h}$ for Example 2 with $\sigma = 4$, $N = 20$ and increasing M

M	$u_0^h(x)$			u_{10}^h			$\widetilde{u}^{N,h}$		
	δ_0^h	eoc_0	ϵ_0^h	δ_{10}^h	eoc_{10}	ϵ_{10}^h	$\widetilde{\delta}^{N,h}$	$\widetilde{eoc}^{N,h}$	$\widetilde{\epsilon}^{N,h}$
108	$8.58{\cdot}10^{-5}$		3.24	$1.36{\cdot}10^{-3}$		7.59	$1.78{\cdot}10^{-2}$		3.35
300	$3.14{\cdot}10^{-5}$	2.03	1.19	$3.33{\cdot}10^{-4}$	2.76	1.85	$4.96{\cdot}10^{-3}$	2.50	0.94
768	$1.44{\cdot}10^{-5}$	2.42	0.55	$9.97{\cdot}10^{-5}$	2.56	0.56	$1.77{\cdot}10^{-3}$	2.20	0.33
1452	$8.14{\cdot}10^{-6}$	2.23	0.31	$4.64{\cdot}10^{-5}$	2.40	0.26	$9.06{\cdot}10^{-4}$	2.10	0.17
1728	$6.93{\cdot}10^{-6}$	2.16	0.26	$3.79{\cdot}10^{-5}$	2.31	0.21	$7.57{\cdot}10^{-4}$	2.05	0.14
2700	$4.56{\cdot}10^{-6}$	2.13	0.17	$2.27{\cdot}10^{-5}$	2.29	0.13	$4.79{\cdot}10^{-4}$	2.06	0.09

We now wish to notice that matrices $\mathbf{V}_k^h$ and $\widetilde{\mathbf{V}}_k^h$, $k \in \overline{0,N}$, which arise after discretization of boundary operators in equations (26) and (51), are fully populated and can reach large sizes. So for their calculation we apply the Fast BEM which based on adaptive cross approximation (ACA) of these matrices [3, 12]. Because this approach is universal in relation to the function in the kernel of boundary operators, an efficient algorithm can be constructed for calculating all the above matrices.

It can be checked that functions in the sequence $\mathbf{e}(x-y) = \big(e_0(x-y), e_1(x-y), ...e_k(x-y), ...\big)^\top$ are asymptotically smooth [3, Definition 3.2.]. This ensures that for each of the matrices $\mathbf{V}_k^h$ ACA algorithm admits admissible partitions into blocks that can be approximated by the product of matrices of smaller rank. For example, if some block $\mathbf{A} \in \mathbb{R}^{m\times n}$ in $\mathbf{V}_k^h$ is admissible it can be approximated with arbitrary small error ε in Frobenius norm by the matrix $\mathbf{S}_r := \mathbf{Q}\mathbf{T}^\top$, where $\mathbf{Q} \in \mathbb{R}^{m\times r}$ and $\mathbf{T} \in \mathbb{R}^{n\times r}$ are matrices of rank $r \leq \min(m,n)$. To do this we have to calculate and store in RAM only a subset of elements of the block $\mathbf{A}$ [3, Chapter 3].

In order to demonstrate efficiency of the ACA we apply Fast BEM to the problem which we have considered in the Example 1. As we can see from the Figure 2, memory consumption for storing data of the approximated matrix $\mathbf{V}_0^h$ depends on the parameter M almost linearly. By contrast, we need to store

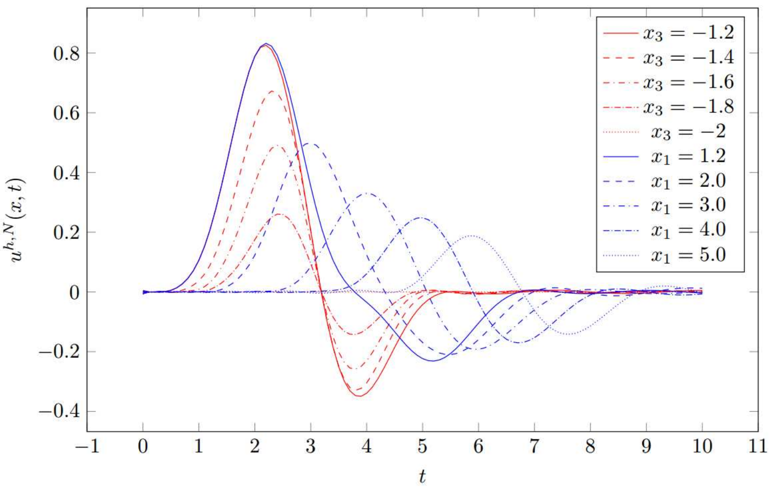

FIG. 1. Numerical solution of the problem in Example 2 in two sets of the observation points $\{(x_1, 0, 0)\}$ and $\{(0, 0, x_3)\}$

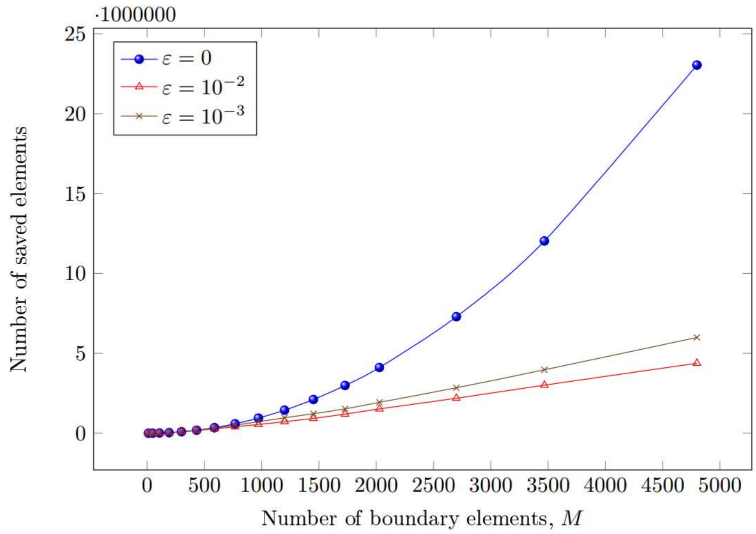

FIG. 2. Memory consumption for storing data of the matrix $\mathbf{V}_0^h$ for the Fast BEM ($\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$) and for the ordinary BEM ($\varepsilon = 0$)

M^2 elements of $\mathbf{V}_0^h$ using ordinary BEM. The same dependency concerns the time needed for calculating data of $\mathbf{V}_0^h$ by the fast and the ordinary BEM.

Note that according to the ACA algorithm admissible blocks are allocated outside of the main diagonal of the matrix. So their approximation doesn't require high accuracy. On Figure 3 we demonstrate the error of the numerical

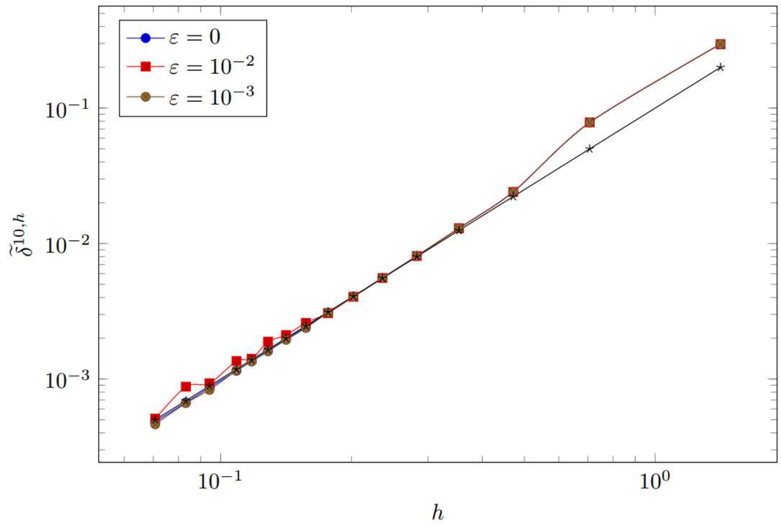

FIG. 3. Error $\widetilde{\delta}^{10,h}$ of numerical solutions for Example 1, which was obtained by the Fast BEM ($\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$) and by the ordinary BEM ($\varepsilon = 0$)

solutions for Example 1, which were obtained by the Fast BEM with approximation of admissible blocks in matrices $\mathbf{V}_k^h$ with some fixed values of the error ε. As we can see, the numerical solution in case of $\varepsilon = 10^{-3}$ has almost the same error $\widetilde{\delta}^{N,h}$ as in case of the application the ordinary BEM, when all elements of matrices $\mathbf{V}_k^h$ were calculated (on the figure we denote this solution by $\varepsilon = 0$).

6. Conclusions

We have described two approaches based on the Laguerre transform in the time domain, that require the solution of a sequence of boundary integral equations to obtain an approximate solution of the Dirichlet problem for the wave equation. After an additional justification for such transform, we have shown the application of the boundary elements method for solving integral equations in the Laguerre domain and derived a representation of the approximate solution of the wave equation.

In solving evolutional problems the coupling of the LT and the BEM makes it possible to use other techniques, that have been developed for elliptical problems. In particular, we have modified this method for solving Dirichlet problem in the domain with an inclusion, using Green's functions for the representation of the solution. Also we have implemented the Fast BEM using adaptive cross approximation for reducing both the storage and computational costs.

Finally, we can point out that in this article we have confined ourselves to considering a problem with a Dirichlet boundary condition in order to simplify the presentation. For other boundary conditions the approaches considered above will lead to other boundary integral equations that will need to be solved by another implementation of the BEM. We also remark that the Laguerre transform can be combined with other suitable methods. For example, for solving more general second-order hyperbolic equations, which coefficients are variable in the space domain, the Laguerre transform can be similarly combined with the finite elements method.

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