UDC 517.988

LAGRANGE INTERPOLATION FORMULA IN LINEAR SPACES

O. F. Kashpur, V. V. Khlobystov

Резюме. В лінійному нескінченновимірному просторі зі скалярним добутком і в скінченновимірному евклідовому просторі досліджена точність формули Лагранжа на поліномах відповідного степеня.

Abstract. In a linear infinite-dimensional space with scalar product and in a finite-dimensional Euclidean space the accuracy of the Lagrange formula on polynomials of the corresponding degree is investigated.

The problem of polynomial approximation of nonlinear operators is an actual in both the theoretical and in the applied senses. One of the methods of its solution is interpolation. A partial case of this problem is the polynomial interpolation of many-variable functions. It was shown in [1] that for the construction of the unique interpolation polynomial in the Euclidean space E_k it is necessary that the relation (between the n-th degree of the polynomial and the number of nodes m) $m = (n+k)!/n!k!$ be executed. Moreover constructing an n-th degree interpolant in E_k induces some difficulties. In practice, there are cases where the number of interpolation nodes is given less than what is needed to construct of the unique interpolant of the corresponding degree. In [2], it is shown that the number of nodes can be chosen less than dimension of the space of polynomials used for seeking the solution, with the problem will be invariantly solvable and will be have the unique solution with minimum norm generated by a scalar product by the Gaussian measure [3, 7]. We call an interpolation task invariantly solvable if it has a solution at arbitrary values of the function in the nodes.

In [4] interpolation operator polynomials in Hilbert spaces are given. In the article one of these interpolants is considered. It is shown that it is an interpolation Lagrange formula with fundamental functional polynomials in a linear space with a scalar product. This interpolation Lagrange formula (the number of nodes m and the degree of polynomial n are not interconnected) is studied both for the case of an infinite-dimensional linear space and for the case of the finite-dimensional Euclidean space E_k, the conditions for the accuracy of the Lagrange formula on polynomials of the corresponding degree are determined.

Key words. Hilbert space, Euclidean space, operator, interpolation polynom, invariance of solution.

It was shown in [4] that the interpolation operator polynomial of n-th degree for the operator f has the form

$$P_n(x) = \left\langle \overline{f}, \Gamma_m^+ \sum_{p=0}^{n} (x_i, x)^p |_{i=1}^{m} \right\rangle, \quad (1)$$

where x_i is an interpolation node, $P_n(x_i) = f(x_i) = f_i$, $i = \overline{1,m}$, $\overline{f} = (f_1, f_2, ..., f_m)$, x_i, $x \in H, H$ is the Hilbert space, $f : H \to Y$, Y is a linear space, $f_i \in Y$, Γ_m^+ is the Moore-Penrose pseudo-inverse matrix to the matrix

$$\Gamma_m = \| \sum_{p=0}^{n} (x_i, x_j)^p \|, \; < \cdot, \cdot >= \sum_{i=1}^{m} f_i \alpha_i, \; \alpha_i \in R_1.$$

In [4], in the event of fulfillment of the necessary and sufficient condition for solvability of operator interpolation task, such as

$$A_0 \overline{f} = \overline{0}, \; A_0 = E - \Gamma_m^+ \Gamma = E - \Gamma \Gamma_m^+, \quad (2)$$

A_0 is an idempotent symmetric matrix. Based on (2), we get: if the matrix Γ_m is nonsingular ($\Gamma_m^+ = \Gamma_m^{-1}$), then the problem will be invariantly solvable, that is, the solution will exist for any values of the operator in the nodes.

We denote $\Gamma_m^k = \|(x_i, x_j)^k\|$. In [4] it is shown that in the case of fulfillmenting of the condition

$$rg(\Gamma_m^0 + \Gamma_m^1) + n - 1 \geq m \quad (3)$$

the operator interpolation problem is invariantly solvable.

Consequently, let us consider the case when the problem is invariantly solvable: $\Gamma_m^+ = \Gamma_m^{-1}$, and the formula (1) turn in to the form:

$$P_n(x) = \left\langle \overline{f}, \Gamma_m^{-1} \sum_{p=0}^{n} (x_i, x)^p |_{i=1}^{m} \right\rangle. \quad (4)$$

In the following, the formula (4) will be rewritten in a different form and we reduce it to the Lagrange formula in a linear space with a scalar product. Let X, Y be linear spaces, X with a scalar product $(\cdot, \cdot)$, $f : X \to Y$, $P_n(x)$ be an interpolation operator polynomial of n-th degree for f with nodes $x_1, x_2, ..., x_m$, $P_n(x_i) = f(x_i) = f_i$, $x, x_i \in X$, $i = \overline{1,m}$, and the nodes x_i are chosen in such a way that the matrix $\|P_{ni}(x_j)\|$ will be nonsingular, where

$$P_{ni}(x) = \sum_{k=0}^{n} L_{ki} x^k, L_{ki} x^k = (x_i, x)^k, \; L_{0i} = 1, \; P_{ni} : X \to R_1, \; i = \overline{1,m}.$$

The invertibility of the matrix for a finite-dimensional Euclidean space is considered in [2] by the choice of independent vectors related with nodes. In the following, we denote: $\overline{P}_n(x) = (P_{n1}(x), P_{n2}(x), ..., P_{nm}(x))$, and by $P_{ni}^{-1}(x_j)$ the elements of the matrix $\|P_{ni}(x_j)\|^{-1}$. According to [4] we get

$$P_n(x) = \langle \overline{f}, \|P_{ni}(x_j)\|^{-1} \overline{P}_n(x) \rangle =$$

$$= \langle \overline{f}, \|P_{ni}^{-1}(x_j)\| \overline{P}_n(x) \rangle =$$

$$= \sum_{i=1}^{m} f_i \sum_{j=1}^{m} P_{ni}^{-1}(x_j) P_{nj}(x) = \sum_{i=1}^{m} f_i l_i(x), \qquad (5)$$

where

$$l_i(x) = \sum_{j=1}^{m} P_{ni}^{-1}(x_j) P_{nj}(x),$$

$$l_i(x_k) = \sum_{j=1}^{m} P_{ni}^{-1}(x_j) P_{nj}(x_k) = \delta_{ik}, \qquad (6)$$

δ_{ik} is the Kronecker symbol. Since (5), (6), we obtain

$$P_n(x_k) = \sum_{i=1}^{m} f_i l_i(x_k) = f_k = f(x_k), k = \overline{1, m}.$$

Thus, the formula (5) is the Lagrange formula for an interpolation polynomial in a linear space with a scalar product

$$P_n(x) = \sum_{i=1}^{m} f_i l_i(x),\ l_i(x_k) = \delta_{ik},\ i, k = \overline{1, m}, \qquad (7)$$

where $l_i(x)$ are fundamental functional Lagrange polynomials of n-th degree, $l_i : X \to R_1$.

Note that the interpolant (7) with the nodes $x_i, i = \overline{1, m}$ is not a unique polynomial in X. Indeed, if $p_n : X \to Y$ is an arbitrary operator polynomial of n-th degree [5], then formula

$$P_n(x) = p_n(x) + \sum_{i=1}^{m} (f_i - p_n(x_i)) l_i(x) \qquad (8)$$

defines the set of interpolation operator polynomials of n-th degree for the operator f,

$$P_n(x_k) = p_n(x_k) + \sum_{i=1}^{m} (f_i - p_n(x_i)) l_i(x_k) =$$

$$= p_n(x_k) + \sum_{i=1}^{m} (f_i - p_n(x_i)) \delta_{ik} = f_k = f(x_k), k = \overline{1, m}.$$

In [4] it is proved that the interpolant (7) belonging to the set (8) has a minimal norm generated by a scalar product by the Gaussian measure [3, 7].

It is known that in infinite-dimensional spaces, the finite set of nodes does not guarantee the uniqueness of the interpolant and its invariance with respect to polynomials of the corresponding degree. It was shown in [6-8] that the continuum information used to construct an interpolation polynomial does not provide the uniqueness of the interpolation formula. The so-called "Kergin insterpolation" for many-variable functions and in the Banach space was considerated in the paper [8]. We note, firstly, that the interpolation formulas (see [8])are convergence with the formulas from [6, 7] obtained in the 1960s up to equivalent integral transformations, and secondly, the classical Newton interpolation formulas for many-variable functions can not be derived from this formulas [9].

It has been known that the expression

$$p_n(x) - \sum_{i=1}^{m} p_n(x_i) l_i(x) \tag{9}$$

does not turn into a zero element of the infinite-dimensional linear space Y [5], that is, the Lagrange formula is not exact on the operator polynomial of the corresponding degree, and when constructing polynomial (5) the numbers m and n are not related.

Example 1. Let's put in (9) $n = 1$, where $p_1 : C[0,1] \to C[0,1], p_1(x) = \int_0^1 K(t,s)x(s)ds, K(t,s)$ is a continuous function on $[0,1] \times [0,1]$. Taking into account the form $l_i(x)$, we obtain that $p_1(x) - \sum_{i=1}^{m} p_1(x_i) l_i(x) \neq 0$. Consequently, in an infinite-dimensional linear space, the Lagrange formula is not exact on polynomials of the corresponding degree.

Let us consider the partial case where X is a finite-dimensional Euclidean space on an example of the space $E_2, f : E_2 \to R_1, u \in E_2, u = (x,y)$, $u_i = (x_i, y_i), i = \overline{1,m}$, where u_i is selected so that the matrix $\| \sum_{p=0}^{n} (x_i x_j + y_i y_j)^p \|$ has to be nonsingular (see [2]). From (5) we get

$$\begin{aligned} P_n(x,y) &= \left(\overline{f}, \left\| \sum_{p=0}^{n} (x_i x_j + y_i y_j)^p \right\|^{-1} \sum_{p=0}^{n} (x x_i + y y_i)^p |_{i=1}^{m} \right) = \\ &= \sum_{i=1}^{m} f_i l_i(x,y). \end{aligned} \tag{10}$$

Then

$$l_i(x,y)|_{i=1}^{m} = \left\| \sum_{p=0}^{n} (x_i x_j + y_i y_j)^p \right\|^{-1} \sum_{p=0}^{n} (x x_i + y y_i)^p |_{i=1}^{m},$$

$$l_i(x_k, y_k) = \delta_{ik},\ i,k = \overline{1,m}.$$

Taking into account (10), we obtain

$$P_n(x_k, y_k) = \sum_{i=1}^{m} f_i l_i(x_k, y_k) = f_k = f(x_k, y_k),\ k = \overline{1,m}$$

and the formula (6) is the interpolation Lagrange formula for $f : E_2 \to R_1$, where $l_i(x,y)$ are the fundamental Lagrange n-th degree polynomials of two variables. Also on the basis of [4] $P_n(x,y)$ is the minimum norm interpolant [3, 7] on the set of n-th degree interpolants of two variables.

In the following, we assume that the number m is given (fixed), and the n-th degree of the interpolation polynomial is chosen from the inequality $m \leq \min p = \overline{p}$, where p is the dimension of the space of n-th degree polynomials in $E_2, p = (n+1)(n+2)/2$ [10].

Example 2. Let $m = 2$, $u_i = (x_i, y_i)$, $i = 1, 2$, $u_1 = (0, 1)$, $u_2 = (1, 0)$. Then

$$m = 2 \leq \min(n + 1)(n + 2)/2 = \overline{p} = 3,\ n = 1.$$

Let us verify the condition (3) of the invariant solvability of the problem:

$$rg(\Gamma^0_m + \Gamma^1_m) + n - 1 = 2 + 1 - 1 = 2 \geq m,\ m = 2.$$

Thus, with such a choice of nodes, the problem is invariantly solvable, that is, the matrix Γ_2 has an invertible.

Let us construct the interpolation polynomial. We get

$$\left\| \sum_{p=0}^{1} (u_i, u_j)^p \right\|^{-1} = \frac{1}{3} \left\| \begin{matrix} 2 & -1 \\ -1 & 2 \end{matrix} \right\|,$$

$$\left\| \sum_{p=0}^{1} (u_i, u_j)^p \right\|^{-1} \sum_{p=0}^{1} (u_i, u_j)^p |_{i=1}^{2} = \frac{1}{3} \left\| \begin{matrix} 1 - x + 2y \\ 1 + 2x - y \end{matrix} \right\|,$$

$$l_1(x, y) = \frac{1}{3}(1 - x + 2y),$$

$$l_2(x, y) = \frac{1}{3}(1 + 2x - y),$$

$$l_i(u_j) = \delta_{ij},\ i, j = 1, 2,$$

$$P_1(x, y) = \sum_{i=1}^{2} f_i l_i(x, y).$$

Let $f(x, y) = 1 + 2x + 3y$. Then $f_1 = f(0, 1) = 4,\ f_2 = f(1, 0) = 3,$

$$P_1(x, y) = 4 \cdot \frac{1}{3}(1 - x + 2y) + 3 \cdot \frac{1}{3}(1 + 2x - y) =$$

$$= \frac{1}{3}(7 + 2x + 5y) \neq 1 + 2x + 3y,$$

that is, in the case of $m = 2, \overline{p} = 3, n = 1$, the interpolant $P_1(x, y)$ is not exact on the polynomial of the 1-st degree.

Example 3. Let $m = 3, u_i = (x_i, y_i), i = 1, 2, 3, u_1 = (0, 1), u_2 = (1, 0), u_3 = (0, -1)$. Then

$$m = 3 \leq \min(n + 1)(n + 2)/2 = \overline{p} = 3,\ n = 1.$$

Check the condition (3):

$$rg(\Gamma^0_m + \Gamma^1_m) + n - 1 = 3 + 1 - 1 = 3 \geq m,\ m = 3.$$

The condition is fulfilled, hence there exists Γ_3^{-1}. Let us construct the interpolation polynomial. We obtain

$$\left\| \sum_{p=0}^{1} (u_i, u_j)^p \right\|^{-1} = \frac{1}{4} \left\| \begin{matrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{matrix} \right\|,$$

$$\left\|\sum_{p=0}^{1}(u_i,u_j)^p\right\|^{-1}\sum_{p=0}^{1}(u_i,u)^p|_{i=1}^{3}=\frac{1}{2}\left\|\begin{matrix}1-x+y\\2x\\1-x-y\end{matrix}\right\|,$$

$$P_3(u)=\sum_{i=1}^{3}f_il_i(u),$$

$$l_1(x,y)=1/2(1-x+y),\ l_2(x,y)=x,\ l_3(x,y)=1/2(1-x-y),$$

$$l_i(u_j)=\delta_{ij},i,j=1,2,3.$$

Let $f(u)=1+2x+3y$, then

$$f_1=f(0,1)=4,\ f_2=f(1,0)=3,\ f_3=f(0,-1)=-2.$$

We get

$$P_1(x,y)=4\cdot\frac{1}{2}(1-x+y)+3x-2\cdot\frac{1}{2}(1-x-y)=1+2x+3y,$$

that is, in the case of $m=3,\overline{p}=3,n=1$, the Lagrange interpolant (10) is exact on the first degree polynomial of two variables.

Thus, for the finite-dimensional Euclidean space E_2, the conclusion is as follows: in the case of $m<\overline{p}$ we have the unique Lagrange interpolant with minimum norm, herewith it is not exact on polynomials of the corresponding degree (Example 2). In the paper [2] this interpolant is called underdetermined. If $m=\overline{p}$, then the Lagrange interpolation polynomial is unique and is exact on the polynomial of the corresponding degree [1] (example 3).

Similar considerations and transformations can be made for the Euclidean space E_k, $u\in E_k$, $u=(x_1,x_2,...,x_k)$, where the number of nodes m is given (fixed), and the n-th degree of the interpolant is determined from the condition

$$m\le\min p=\overline{p},p=(n+k)!/n!k!,\ k\ge 2, \tag{11}$$

where p is the dimension of the space of n-th degree polynomials in E_k [1]. We select the nodes $u_1,u_2,...,u_m$ in such a way that there exists the inverse matrix in (5), and the degree of the interpolation polynomial is determined from inequality (11).

Let us formulate the following conclusion for the space E_k. We get

Theorem 1. Let $f:E_k\to R_1$, $k\ge 2$, m be given. Then, if $m=\overline{p}$, then the Lagrange interpolant $P_n(u)$, $u\in E_k$ will be exact on all polynomials of degree not higher than n, and if $m<\overline{p}$, then the minimum norm interpolant $P_n(u)$, does not have such a property.

We fix the degree of the interpolation polynomial and the number of nodes, for example, $n=2,m=4$. For this case, we construct interpolants in spaces R_1, E_k, $k=2,3,\ldots$. Let p_k be the dimension of polynomials of the second degree in E_k.

In the space E_2, when $n=2,m=4$, we obtain that $p_2=6$. So, for unambiguous definition of $P_2(x,y)$, there are not enough two interpolation nodes. If we consider the construction of the interpolation polynomial of the second degree in E_3, in the case of $m=4$, we obtain that $p_3=10$ and for the unambiguous construction of the interpolant there are not enough 6 nodes. If we continue this process, then it is clear that as the dimension of the space E_k grows, the dimension of the polynomial space of the two variables p_k increases, and therefore, when constructing the interpolation polynomial of the 2-nd degree for 4 nodes, we are in a situation of "underdeterminacy". As you can see, the larger the dimension of the space E_k, the more indeterminacy (uncertainty) and less accurate of the constructed interpolation polynomial. We arrive at the following conclusion: in the case of decreasing of the Euclidean space dimension, the "underdeterminacy" of the Lagrange interpolant is decreases, and in the case $f : R_1 \to R_1$ we have $m = \overline{p} = n + 1$, that is, we obtain the classical n-th degree Lagrange polynomial with $n+1$ nodes for the function of one variable. In the space R_1 for $m = 4$ we get that $\overline{p} = 3$, that is, we can construct the interpolation polynomial of the third degree, herewith the resulting interpolant is unique.

As regards the linear space X with a scalar product, the following statement holds. If the interpolation nodes are chosen so that the corresponding matrix is nonsingular, then there is always the unique Lagrange interpolation polynomial with minimum norm [3, 7], but this interpolant is not exact on the operator polynomials of the corresponding degree (Example 1). We note that, the numbers m (number of nodes) and n (interpolation degree) are not related to each other when the interpolation operator Lagrange polynomial is constructed[4].

Remark. We consider the polynomial (8) in the following form

$$P_n(x) = p_n(x, f) + \sum_{i=1}^{m} (f_i - p_n(x_i, f)) l_i(x), \; x \in X, \tag{12}$$

where $p_n(x, f)$ is a c-polynomial, that is $p_n(x, f) = f$, if $f = p_n(x)$ is an arbitrary polynomial operator of degree not higher than n [4]. Then the formula (12) defines an exact interpolant on polynomials of the corresponding degree. Several examples of constructing a c-polynomial are considered in [4].

Bibliography

1. Babenko K.I. Foundations of numerical analysis / K.I. Babenko. – Moscow: Izhevsk: RC "Regular and chaotic dynamics", 2002. (in Russian).
2. Kashpur O.F. To some questions of a polynomial interpolation in euclidean spaces / O.F. Kashpur, V.V. Khlobystov // Dopov. Nac. Akad. Nauk. Ukr. – 2016. – № 10. – P. 10-14. (in Ukrainian).
3. Yegorov A.D. Approximate methods for computation of continual integral / A.D. Yegorov, P.I. Sobolevskiy, L.A. Yanovich. – Minsk: Nauka i Tehnika, 1985. (in Russian).
4. Makarov V.L. Interpolation of operators / V.L. Makarov, V.V. Khlobystov, L.A. Yanovich. – Kiev: Nauk. Dumka, 2000. (in Russian).
5. Trenogin V.A. Functional analysis / V.A. Trenogin. – Moscow: Nauka, 1980. (in Russian).
6. Ulm S. On the construction of generalized separated differences / S. Ulm, V. Polly // Izv. AN ESSR Sir phys.-mat. sciences. – 1969. – 18, № 1. – P. 100-102. (in Russian).
7. Sobolevskiy P.I. Interpolation of functionals and some approximate formulas for integrals with the Gaussian measure / P.I. Sobolevskiy // Izv. Academy of Sciences of the BSSR phys.-mate sciences. – 1975. – № 2. – P. 5-12. (in Russian).
8. Filipsson L. Kergin interpolation in Banach space / L. Filipsson // J. Approx. Theory. – 2004. – № 127. – P. 108-123.

9. Makarov V.L. Interpolation of the many-variable functionals / V.L. Makarov, V.V. Khlobystov, I.I. Demkiv // Dopov. Nac. Akad. Nauk. Ukr. – 2009. – № 5. – P. 29-35. (in Ukrainian).
10. Berezin I.S. Methods of computations / I.S. Berezin, N.P. Zhidkov. – Moscow: Fizmatgiz, 1962. – Vol. 1. (in Russian).

O.F. Kashpur,
Taras Shevchenko National University of Kyiv,
4D, Glushkova Str., Kyiv, 03187, Ukraine;
V. V. Khlobystov,
Institute of Mathematics,
National Academy of Sciences of Ukraine,
3, Tereshchenkovskaya Str., 01601, Kyiv, Ukraine.

Received 02.09.2018; revised 24.09.2018