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ALGEBRAIC AND TRIGONOMETRIC GENERALIZED INTERPOLATION OF HERMITE-BIRKHOFF TYPE FOR OPERATORS DEFINED ON FUNCTIONAL SPACES AND FUNCTIONS OF MATRIX VARIABLE, AND THEIR APPLICATIONS

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Резюме. У роботі побудовано алгебраїчну формулу типу Ерміта для операторів, визначених у функціональних просторах. Інтерполяційна формула подібного виду, яка містить значення диференціалів Гато довільного порядку, побудована на множині матриць. Отримано матрицю, аналогічну до формули Лейбніца. Сконструйовано формулу апроксимації диференціалів Гато довільного порядку з матричними аргументами. На основі матричної інтерполяційної формули типу Ерміта побудовано чисельний метод для розв'язування задачі Коші для матрично-диференціального рівняння. Продемонстровано приклад чисельного розв'язування задачі Коші для матрично-диференціального рівняння першого порядку. Побудовано і досліджено параметричне сімейство тригонометричних матричних інтерполяційних поліномів типу Ерміта-Біркгофа.

Abstract. For operators defined in function spaces, the algebraic interpolation formula of Hermite type is constructed. The interpolation formula of similar type, containing the value of the Gateaux differential of an arbitrary order, is constructed for operators on the set of matrices. Matrix analogues of the Leibniz formula are obtained. The formula for approximate calculation of the Gateaux differential of an arbitrary order of the matrix argument function is constructed. Based on the matrix interpolation formula of the Hermite type, the approximate method for solving the Cauchy problem for the matrix-differential equation is obtained. The illustrative example of approximate solving the Cauchy problem for a first-order matrix-differential equation is constructed. A parametric family of trigonometric matrix interpolation polynomials of Hermite-Birkhoff type is constructed and investigated.

1. Introduction

The fundamentals of the theory of operator interpolation are given in [1, 2]. Here, in particular, the problem of operator interpolation of Hermite-Birkhoff type is investigated. The complexity of this problem lies in the fact that even with different interpolation nodes it can either have a non-unique solution, or do not have a solution at all. Some basics of matrix interpolation are also contained in [1, 2]. The theory of matrix interpolation is quite fully given in [3]. The papers [4–6] are devoted to the construction and research of Hermite-Birkhoff generalized matrix interpolation formulas for concrete Chebyshev systems.

Key words. Generalized interpolation of Hermite-Birkhoff type, Gateaux differential, Leibniz formula, matrix argument function, Cauchy problem for the matrix-differential equation.

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In the given work the interpolation formulas for functions of a scalar argument, constructed and investigated in [7, 8], are summarized to the case of operators defined in functional spaces and on the set of matrices. When proving the theorems on the fulfillment of interpolation conditions for the respective polynomials, matrix analogues of the Leibniz formula are used, which are also obtained in this work. The parametric family of trigonometric matrix Hermite-Birkhoff polynomials is constructed.

2. Algebraic interpolation

Let X be a certain given set of functions $x = x(s)$, defined on the segment $[a, b]$, $Y = \{y(s, t), t \in T \subset \mathbb{R}^N\}$ – some function space where T is a given numerical set of N-dimensional space $\mathbb{R}^N$, and let $F(x) \equiv F(t; x(s))$ be an operator mapping X into Y. Let's assume that in the various elements $x_k = x_k(s)$ $(k = 0, 1, \dots, n)$ of the set X, such that $x_k(s) \neq x_\nu(s)$ on $[a, b]$, the values $F(x_k)$ of the operator $F(x)$, $x \in X$ are known. We choose in the set X functions $h_1(s)$, $h_2(s)$, …, $h_{n+1}(s)$ such that $h_1(s)h_2(s)\cdots h_{n+1}(s) \neq 0$ on $[a, b]$. Let the value $D_{n+1}(F;\, x_{n+1})$ of the operator of the form

$$D_{n+1}F(x) = \delta^{n+1}F[x; h_1h_2\cdots h_{n+1}],$$

where $\delta^{n+1}F[x; h_1h_2\cdots h_{n+1}]$ is the Gateaux differential of the order $n+1$ of the operator $F(x)$ at the point x in the directions h_1, h_2, …, h_{n+1}, be known in the node $x_{n+1} = x_{n+1}(s) \in X$.

We now consider further the operator polynomials $P_{n+1} : X \to Y$ of the form

$$P_{n+1}(x) = \sum_{\nu=0}^{n+1} a_\nu(t, s)x^\nu(s), \tag{1}$$

where $a_\nu(t, s)$ are some functions of the variables t and s.

We introduce the polynomials $l_{n,k}(x) = (x - x_0)(x - x_1)\cdots(x - x_{k-1})\times$ $\times(x - x_{k+1})\cdots(x - x_n)$, $\omega_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$.

Theorem 1. *The interpolation polynomial*

$$\tilde{L}_{n+1}(x) = L_n(x) + \frac{\omega_n(x)D_{n+1}F(x_{n+1})}{(n+1)!h_1h_2\cdots h_{n+1}},$$

where

$$L_n(x) = \sum_{k=0}^{n} \frac{l_{n,k}(x)F(x_k)}{l_{n,k}(x_k)}, \tag{2}$$

satisfies the interpolation conditions

$$\tilde{L}_{n+1}(x_k) = F(x_k) \ (k = 0, 1, \dots, n);$$

$$D_{n+1}\left(\tilde{L}_{n+1}; x_{n+1}\right) = D_{n+1}(F; x_{n+1}). \tag{3}$$

The formula (2) *is exact for the operator polynomials of the type* (1) *of the degree not higher than* $n+1$.

Proof. Since $l_{n,k}(x_i) = \delta_{ki} l_{n,k}(x_k)$, where δ_{ki} is the Kronecker symbol, and $\omega_n(x_k) = 0$, $k, i = 0, 1, \ldots, n$, then the fulfillment of the first group of interpolation conditions in (3) is obvious.

Since $\delta^{n+1} P_n[x; h_1 h_2 \cdots h_{n+1}] \equiv 0$, where $P_n(x)$ is an arbitrary operator algebraic polynomial of a degree not higher than n, then $\delta^{n+1} L_n[x; h_1 h_2 \cdots h_{n+1}] \equiv \equiv 0$. It is also obvious that $\delta^{n+1} \omega_n[x; h_1 h_2 \cdots h_{n+1}] = (n+1)! h_1 h_2 \cdots h_{n+1}$. Taking into account the structure of the polynomial (2), we will obtain that the last condition in (3) also holds.

We now prove the invariance of the formula (2) with respect to the polynomials of the form (1) of the degree not higher than $n+1$. If $F(x) = P_n(x)$, where $P_n(x)$ is a polynomial of the form (1) of the degree not higher than n, then as is known in [2, p. 361], $L_n(P_n;\ x) \equiv P_n(x)$. And since in this case $D_{n+1} P_n(x) \equiv 0$, then $\tilde{L}_{n+1}(P_n; x) \equiv P_n(x)$. Let further suppose $F(x) = \tilde{P}_{n+1}(x) = = x^{n+1}(s)$, then $D_{n+1} \tilde{P}_{n+1}(x) = (n+1)! h_1 h_2 \cdots h_{n+1}$, and

$$\tilde{L}_{n+1}(\tilde{P}_{n+1}; x) = L_n(\tilde{P}_{n+1}; x) + \omega_n(x).$$

By analogy with to the scalar case [7, p. 6], $\tilde{L}_{n+1}\left(\tilde{P}_{n+1};\ x\right) \equiv \tilde{P}_{n+1}(x)$. Thus, the formula (2) is exact for operator polynomials of the form (1) of the degree not higher than $n+1$. □

We now consider the problem of interpolating operators on the set of matrices. Let X be the set of functional or stationary square matrices $A = A(t)$, $t \in T \subset \subset \mathbb{R}$. Let's introduce differential operator of type

$$D^n F(A) = \left.\frac{d^n F(z)}{dz^n}\right|_{z=A}, \quad D = \frac{d}{dz},\ z \in \mathbb{C},\ A \in X, \tag{4}$$

where $F(z)$ is the entire function.

The value of the operator (4) for the matrix function of the type $B_1 F(A) B_2$, where B_1 and B_2 are some fixed matrices from X, is calculated by the formula $D^n\left(B_1 F(A) B_2\right) = B_1 D^n F(A) B_2$. The operator D, which is included in (4), for the function of the type $F(cA+B)$, where $c \in \mathbb{C}$, and B is a certain fixed matrix of X, defined by the equality $DF(cA+B) = cF'(z)|_{z=cA+B}$, and for the product $U(A)V(A)$ by the formula $D\left(U(A)V(A)\right) = DU(A)V(A) + U(A)DV(A)$. In the last expression, it is important in what order the multipliers in matrix products are taken. For example, $D\left(V(A)U(A)\right) = DV(A)U(A) + V(A)DU(A)$, and in the general case, $D\left(U(A)V(A)\right) \neq D\left(V(A)U(A)\right)$. Similarly, the values of higher-order operators are calculated, as well as operators from the products of functions with a number of multipliers more than two.

In mathematical analysis, the Leibniz formula for the derivative of n-th order ($n \in \mathbb{N}$) of the product of two scalar functions is known [9]

$$(u(z) \cdot v(z))^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)}(z) v^{(k)}(z), \text{ where } C_n^k = \frac{n!}{k!(n-k)!}, \tag{5}$$

which holds if the functions $u(z)$ and $v(z)$ are n times differentiable at the point $z \in \mathbb{C}$. We generalize this formula to the case of functions of the matrix argument and operator of the type (4).

Theorem 2. *If the functions $U(z)$ and $V(z)$, $z \in \mathbb{C}$, are differentiable n times, then the formula*

$$D^n \left(U(A)V(A)\right) = \sum_{k=0}^{n} C_n^k D^k U(A) D^{n-k} V(A), \; A \in X, \tag{6}$$

is valid.

Proof. We apply the method of mathematical induction. When $n = 1$ we will have

$$D^1 \left(U(A)V(A)\right) = DU(A)V(A) + U(A)DV(A) =$$
$$= C_1^0 D^1 U(A)V(A) + C_1^1 U(A) D^1 V(A).$$

Let's assume that the formula (6) is exact for $n = k$. We prove that it also holds for $n = k + 1$.

$$D^{k+1} \left(U(A)V(A)\right) = D\left[\sum_{k=0}^{n} C_n^k D^k U(A) D^{n-k} V(A)\right] =$$

$$= \sum_{k=0}^{n} C_n^k \left[D^{k+1} U(A) D^{n-k} V(A) + D^k U(A) D^{n-k+1} V(A)\right] =$$

$$= C_n^0 D^0 U(A) D^{n+1} V(A) + \sum_{k=1}^{n} \left(C_n^{k-1} + C_n^k\right) D^k U(A) D^{n-k+1} V(A) +$$
$$+ C_n^n D^{n+1} U(A) D^0 V(A).$$

Since $C_n^{k-1} + C_n^k = C_{n+1}^k$, $C_n^0 = C_{n+1}^0 = 1$, $C_n^n = C_{n+1}^{n+1} = 1$, then

$$D^{k+1} \left(U(A)V(A)\right) = \sum_{k=0}^{n+1} C_{n+1}^k D^k U(A) D^{n+1-k} V(A).$$

□

We now introduce the differential operator of the form

$$\tilde{D}_{n+1} F(A) \equiv \tilde{D}_{n+1} F(A;\, H_{n+1} H_n \cdots H_1) = \delta^{n+1} F[A;\, H_{n+1} H_n \cdots H_1], \tag{7}$$

where $\delta^{n+1} F[A;\, H_{n+1} H_n \cdots H_1]$ is Gateaux differential of order $n + 1$ at the point $A \in X$ in the directions $H_1, H_2, \ldots, H_{n+1}$ from X. We assume that $\tilde{D}_0 F(A) \equiv F(A)$.

Theorem 3. *If the functions $U(A)$ and $V(A)$ are Gateaux differentiable n times at the point $A \in X$, then the formula*

$$\tilde{D}_n \left(U(A)V(A);\, H_n H_{n-1} \cdots H_1\right) = \tag{8}$$

$$= \sum_{k=0}^{n} \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_{n-k}}} \tilde{D}_k U(A;\, H_{i_k} H_{i_{k-1}} \cdots H_{i_1}) \tilde{D}_{n-k} V(A;\, H_{j_{n-k}} H_{j_{n-k-1}} \cdots H_{j_1})$$

holds true.

Here, for each value of k $(0 \le k \le n)$ the summation is over for all disjoint sets $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_{n-k})$ such that $1 \le i_1 < i_2 < \ldots < i_k \le n$; $1 \le j_1 < j_2 < \ldots < j_{n-k} \le n$.

Proof. We use, as in the proof of theorem 2, the method of mathematical induction. If $n = 1$ by the definition of the Gateaux differential we will have

$$\tilde{D}_1\left(U(A)V(A);\, H_1\right) = \delta\left[U(A)V(A);\, H_1\right] = \lim_{\lambda\to 0}\left(\frac{U(A+\lambda H_1)V(A+\lambda H_1)}{\lambda} - \right.$$

$$\left. -\frac{U(A)V(A)}{\lambda}\right) = \lim_{\lambda\to 0}\left(\frac{U(A+\lambda H_1)V(A+\lambda H_1) - U(A)V(A+\lambda H_1)}{\lambda} + \right.$$

$$\left. + \frac{U(A)V(A+\lambda H_1) - U(A)V(A)}{\lambda}\right) = \delta U[A;\, H_1]V(A) + U(A)\delta V[A;\, H_1] =$$

$$= \tilde{D}_1 U(A;\, H_1)V(A) + U(A)\tilde{D}_1 V(A;\, H_1). \tag{9}$$

Hereinafter the expression of the form $\delta\left[U(A)V(A);\, H_1\right]$ should be understood as the Gateaux differential $\delta W[A;\, H_1]$, respectively, of the function $W(A) = = U(A)V(A)$ at the point A in the direction H_1.

Let's suppose that formula (8) is true when $n = m$. We show that it holds for $n = m + 1$. From (7) – (9) we have

$$\tilde{D}_{m+1}\left(U(A)V(A);\, H_{m+1}\cdots H_1\right) = \delta\left[\tilde{D}_m\left(U(A)V(A);\, H_m\cdots H_1\right);\, H_{m+1}\right] =$$

$$= \sum_{k=0}^{n}\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_{n-k}}}\left(\tilde{D}_{k+1}U\left(A;\, H_{n+1}H_{i_k}\cdots H_{i_1}\right)\tilde{D}_{n-k}V\left(A;\, H_{j_{n-k}}\cdots H_{j_1}\right) + \right.$$

$$\left. + \tilde{D}_k U\left(A;\, H_{i_k}\cdots H_{i_1}\right)\tilde{D}_{n+1-k}V\left(A;\, H_{n+1}H_{j_{n-k}}\cdots H_{j_1}\right)\right) =$$

$$= \sum_{k=0}^{n+1}\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_{n+1-k}}}\tilde{D}_k U\left(A;\, H_{i_k}\cdots H_{i_1}\right)\tilde{D}_{n+1-k}V\big(A;\, H_{j_{n+1-k}}\cdots H_{j_1}\big).$$

Here the summation is carried out in the same way as in the formulation of the theorem, while $1 \le i_1 < i_2 < \ldots < i_k \le n+1$; $1 \le j_1 < j_2 < \ldots < < j_{n+1-k} \le n+1$. $\square$

In the special case, for example, for $n = 3$ the formula (8) has the form

$$\tilde{D}_3\left(U(A)V(A); H_3H_2H_1\right) = \tilde{D}_3 U\left(A; H_3H_2H_1\right)V(A) + \tilde{D}_2 U\left(A; H_3H_2\right)\times$$

$$\times\tilde{D}_1 V\left(A; H_1\right) + \tilde{D}_2 U\left(A; H_3H_1\right)\tilde{D}_1 V\left(A; H_2\right) + \tilde{D}_2 U\left(A; H_2H_1\right)\times$$

$$\times\tilde{D}_1 V\left(A; H_3\right) + \tilde{D}_1 U\left(A; H_1\right)\tilde{D}_2 V\left(A; H_3H_2\right) + \tilde{D}_1 U\left(A; H_2\right)\times$$

$$\times\tilde{D}_2 V\left(A; H_3H_1\right) + \tilde{D}_1 U\left(A; H_3\right)\tilde{D}_2 V\left(A; H_2H_1\right) + U(A)\tilde{D}_3 V\left(A; H_3H_2H_1\right).$$

We suppose that in the elements $A_k(t)$ of the set X such that $A_k(t) - A_\nu(t)$ are invertible matrices, $t \in T$, $k, \nu = 0, 1, \ldots, n$, $k \ne \nu$, the values of the operator $F(A)$ are known, as well as at the node $A_{n+1}(t)$ the value $\tilde{D}_m F(A_{n+1}) \equiv \tilde{D}_m F(A_{n+1}; H_m H_{m-1}\cdots H_1)$ of the operator (7) from $F(A)$, where $1 \le m \le n$, $H_k \in X$ $(k = 1, 2, \ldots, m)$ is known. Let's introduce the notations $\omega(A) = (A - A_0)(A - A_1)\cdots(A - A_n)$, $l_k(A) = (A - A_0)\cdots(A - A_{k-1})(A - -A_{k+1})\cdots(A - A_n)$, $B_k = \tilde{D}_m l_k(A_{n+1})$, $\tilde{A}_k = B_k A_{n+1} + B_k^{-1}\sum\limits_{i=1}^{m}\tilde{D}_{m-1} l_k(A_{n+1};$

$H_m \cdots H_{i+1} H_{i-1} \cdots H_1) B_k H_i$ $(k = 0, 1, \ldots, n)$. We will assume that the matrices B_k, $l_k(A_k)$, $B_k A_k - \tilde{A}_k$ $(k = 0, 1, \ldots, n)$ and $\tilde{D}_m \omega(A_{n+1})$ are invertible.

Theorem 4. *The matrix polynomial of the degree not higher than* $n+1$

$$\tilde{L}_{n+1}(F;A) = \sum_{k=0}^{n} l_k(A)(B_k A - \tilde{A}_k) \left[l_k(A_k)(B_k A_k - \tilde{A}_k) \right]^{-1} F(A_k) +$$

$$+\omega(A) \left[\tilde{D}_m \omega(A_{n+1}) \right]^{-1} \tilde{D}_m F(A_{n+1}) \tag{10}$$

satisfies the interpolation conditions

$$\tilde{L}_{n+1}(A_k) = F(A_k) \ (k = 0, 1, \ldots, n); \ \ \tilde{D}_m \tilde{L}_{n+1}(A_{n+1}) = \tilde{D}_m F(A_{n+1}). \tag{11}$$

Proof. Since $l_k(A_i) = \delta_{ki} l_k(A_k)$ $(k, i = 0, 1, \ldots, n)$, where δ_{ki} is the Kronecker symbol, and $\omega(A_k) = 0$ for the same values of k, then the first group of the conditions in (11) is satisfied. By the formula (8)

$$\tilde{D}_m \Big(l_k(A)(B_k A - \tilde{A}_k); H_m \cdots H_1 \Big) = \tilde{D}_m l_k(A; H_m \cdots H_1)(B_k A - \tilde{A}_k) +$$

$$+ \sum_{i=1}^{m} \tilde{D}_{m-1} l_k(A; H_m \cdots H_{i+1} H_{i-1} \cdots H_1) \tilde{D}_1(B_k A - \tilde{A}_k; H_i).$$

Due to the fact that $\tilde{D}_1(B_k A - \tilde{A}_k; H_i) = B_k H_i$, then for $A = A_{n+1}$

$$\tilde{D}_m \Big(l_k(A)(B_k A - \tilde{A}_k); H_m \cdots H_1 \Big) \Big|_{A=A_{n+1}} = B_k(B_k A_{n+1} - \tilde{A}_k) +$$

$$+ \sum_{i=1}^{m} \tilde{D}_{m-1} l_k(A; H_m \cdots H_{i+1} H_{i-1} \cdots H_1) B_k H_i = 0.$$

Taking into account the structure of the formula (10), we will obtain that the last condition in equation (11) also holds. □

Using the interpolation polynomial (10), we can construct a formula for approximate calculation of the Gateaux differential of the m-th $(1 \le m \le n)$ order from the function of the matrix argument $F(A)$ by its values at the nodes $A_0, A_1, \ldots, A_n$. Indeed, the relation

$$F(A) = \sum_{k=0}^{n} l_k(A)(B_k A - \tilde{A}_k) \left[l_k(A_k)(B_k A_k - \tilde{A}_k) \right]^{-1} F(A_k) +$$

$$+\omega(A) \left[\tilde{D}_m \omega(A_{n+1}) \right]^{-1} \tilde{D}_m F(A_{n+1}) + R_n(F;A),$$

where $R_n(F;A)$ is the remainder term of the formula (10), holds true. Then, expressing from the last equality $\tilde{D}_m F(A_{n+1})$, we will have

$$\tilde{D}_m F(A_{n+1}) = \tilde{D}_m \omega(A_{n+1}) \omega^{-1}(A) \Bigg(F(A) - \sum_{k=0}^{n} l_k(A)(B_k A - \tilde{A}_k) \times$$

$$\times \left[l_k(A_k)(B_k A_k - \tilde{A}_k) \right]^{-1} F(A_k) - R_n(F;A) \Bigg). \tag{12}$$

Discarding in (12) the remainder term $R_n(F;A)$ of the formula (10), we will obtain the required approximate formula for calculating the Gateaux differential

$$\delta^m F[A;H_m H_{m-1}\cdots H_1] \cong \tilde{D}_m \omega(A_{n+1})\omega^{-1}(A)\times$$

$$\times\left(F(A)-\sum_{k=0}^{n} l_k(A)(B_kA-\tilde{A}_k)\left[l_k(A_k)(B_kA_k-\tilde{A}_k)\right]^{-1}F(A_k)\right). \quad (13)$$

Here, the matrix A must be such that the matrices entering into the formula are invertible.

3. The solving matrix-differential equations

Let X be the set of square stationary matrices of fixed size. We consider the matrix equation containing the first-order Gateaux differential of the matrix function

$$\delta U[A;H]=F(U,A),\ U(A_0)=U_0,\ A,H\in X, \quad (14)$$

where $U(A)$ is a function of the matrix argument, F is some generally non-linear function of two arguments, $\delta U[A;H]$ is the Gateaux differential at the point A in the direction H satisfying the specified in (14) initial condition.

For the approximate solving the Cauchy problem (14), we use the formula (13) for approximating the Gateaux differential of the matrix argument function. In our case it takes the form

$$\delta U[A;H]=\delta\omega[A;H]\omega^{-1}(A_{n+1})\times \quad (15)$$

$$\times\left(U(A_{n+1})-\sum_{k=0}^{n} l_k(A_{n+1})(B_kA_{n+1}-\tilde{A}_k)\left[l_k(A_k)(B_kA_k-\tilde{A}_k)\right]^{-1}U(A_k)\right),$$

where $B_k=B_k(A)=\delta l_k[A;H]$, $\tilde{A}_k=\tilde{A}_k(A)=B_k(A)A+B_k^{-1}(A)l_k(A)\times$ $\times B_k(A)H$. Here $A_0,A_1,\ldots,A_n$ are the matrices from X such that the inverse matrices in (15) exist.

Substituting (15) into (14), we obtain

$$\delta\omega[A;H]\omega^{-1}(A_{n+1})\left(Y_{n+1}-\sum_{k=0}^{n} l_k(A_{n+1})(B_kA_{n+1}-\tilde{A}_k)\times\right.$$

$$\left.\times\left[l_k(A_k)(B_kA_k-\tilde{A}_k)\right]^{-1}Y_k\right)=F(Y,A),\ Y_0=U_0, \quad (16)$$

where $Y_0,Y_1,\ldots,Y_{n+1}$ is approximate solution of the problem (14) in the matrix nodes $A_0,A_1,\ldots,A_{n+1}$. If now we substitute the matrix nodes A_k ($k=1,2,\ldots,n+1$) instead of A in (16), then we obtain the system (in the general case, non-linear) matrix equations. Solving this system by some direct or iterative method, we obtain the required approximate solution of the problem (14).

Example. Let X be the set of square matrices of size 2. We consider the Cauchy problem for the function of the matrix variable $U(A), A\in X$

$$\delta U[A;H]=3U(A)+2A,\ U(A_0)=U_0, \quad (17)$$

where $A_0 = \begin{pmatrix} 0.312 & 0.467 \\ 0.457 & 0.02 \end{pmatrix}$, $U_0 = \begin{pmatrix} 0.316 & 0.338 \\ 0.23 & 0.002 \end{pmatrix}$, $H = \begin{pmatrix} 0.021 & 0.43 \\ 0.405 & 0.223 \end{pmatrix}$.
Let's introduce the matrix nodes $A_1 = \begin{pmatrix} 0.11 & 0.032 \\ 0.223 & 0.155 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0.004 & 0.085 \\ 0.5 & 0.305 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0.234 & 0.028 \\ 0.2 & 0.004 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0.051 & 0.291 \\ 0.176 & 0.498 \end{pmatrix}$.

For the approximate solving of the problem (14) we use the formula (16) for $n = 3$. We construct a system of matrix equations. In this case, it is linear. We have

$$Y_0 = U_0 = \begin{pmatrix} 0.316 & 0.338 \\ 0.23 & 0.002 \end{pmatrix}, \; \delta\omega[A_i; H]\omega^{-1}(A_4) \left(Y_4 - \sum_{k=0}^{3} l_k(A_4) \times \right.$$
$$\left. \times \left(B_k(A_i)A_4 - \tilde{A}_k(A_i) \right) \left[l_k(A_k) \left(B_k(A_i)A_k - \tilde{A}_k(A_i) \right) \right]^{-1} Y_k \right) =$$
$$= 3Y_i + 2A_i, \; i = 1, 2, 3, 4. \tag{18}$$

Let's present numerically the system of the matrix equations (18) to within 3 significant digits to determine the unknowns Y_0, Y_1, Y_2, Y_3, Y_4

$$\begin{gathered} Y_0 = U_0, \; -\begin{pmatrix} 0.992 & 0.186 \\ 0.180 & 0.0380 \end{pmatrix} Y_0 - \begin{pmatrix} 292 & 302 \\ 47.5 & 51.9 \end{pmatrix} Y_1 + \begin{pmatrix} 0.142 & 4.05 \\ 0.268 & 6.00 \end{pmatrix} Y_2 + \\ + \begin{pmatrix} 2.49 & -15.5 \\ 2.00 & -12.3 \end{pmatrix} Y_3 + \begin{pmatrix} 3.33 & 4.20 \\ 0.815 & 0.606 \end{pmatrix} Y_4 = \begin{pmatrix} 0.22 & 0.064 \\ 0.446 & 0.31 \end{pmatrix}, \\ \begin{pmatrix} 2.48 & 14.1 \\ -2.12 & -12.1 \end{pmatrix} Y_0 - \begin{pmatrix} 1368 & 2630 \\ -1190 & -2289 \end{pmatrix} Y_1 - \begin{pmatrix} 246 & 297 \\ -235 & -285 \end{pmatrix} Y_2 + \\ + \begin{pmatrix} -50.8 & 6.08 \\ 52.1 & -6.20 \end{pmatrix} Y_3 + \begin{pmatrix} -8.96 & -14.4 \\ 7.56 & 12.5 \end{pmatrix} Y_4 = \begin{pmatrix} 0.008 & 0.17 \\ 1.0 & 0.61 \end{pmatrix}, \\ \begin{pmatrix} 8.20 & -2.04 \\ 1.83 & -0.441 \end{pmatrix} Y_0 - \begin{pmatrix} 211 & 135 \\ 49.2 & 32.5 \end{pmatrix} Y_1 + \begin{pmatrix} 13.7 & 21.9 \\ 2.06 & 3.15 \end{pmatrix} Y_2 + \\ + \begin{pmatrix} -10.2 & -34.7 \\ 1.20 & 8.53 \end{pmatrix} Y_3 - \begin{pmatrix} 7.12 & 12.0 \\ 1.92 & 2.75 \end{pmatrix} Y_4 = \begin{pmatrix} 0.468 & 0.056 \\ 0.4 & 0.008 \end{pmatrix}, \\ \begin{pmatrix} 0.149 & 0.662 \\ -0.286 & -0.975 \end{pmatrix} Y_0 + \begin{pmatrix} 230 & 340 \\ -363 & -539 \end{pmatrix} Y_1 + \begin{pmatrix} 2.60 & 3.26 \\ -1.86 & -2.36 \end{pmatrix} Y_2 + \\ + \begin{pmatrix} -0.991 & 0.424 \\ 0.727 & -0.138 \end{pmatrix} Y_3 + \begin{pmatrix} -14.4 & -15.6 \\ 15.9 & 21.2 \end{pmatrix} Y_4 = \begin{pmatrix} 0.102 & 0.582 \\ 0.352 & 0.996 \end{pmatrix}. \end{gathered} \tag{19}$$

The system of the matrix equations (19) can be written element-by-element, having obtained a system of 20 linear algebraic equations with respect to 20 unknowns (elements of matrices Y_0, Y_1, Y_2, Y_3, Y_4). Immediately excluding Y_0 from the remaining matrix equations in (19), we will obtain the system of 16 linear algebraic equations that can be solved, for example, by the Gauss method. According to this method, the solution of the system (19) has the form

$$Y_0 = U_0, \; Y_1 = \begin{pmatrix} 0.00221 & 0.00618 \\ -0.00177 & -0.00416 \end{pmatrix}, \; Y_2 = \begin{pmatrix} -0.0393 & 0.00504 \\ 0.0264 & -0.0223 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0.133 & 0.132 \\ -0.0130 & -0.0395 \end{pmatrix}, \; Y_4 = \begin{pmatrix} -0.171 & -0.546 \\ 0.148 & 0.455 \end{pmatrix}.$$

The solution of the problem (17) obtained in the matrix nodes can be restored using the matrix interpolation formula [2, p. 459] of the form $L_{n0}(A) = \sum_{k=0}^{n} l_k(A)l_k^{-1}(A_k)F(A_k)$, where, as before, $l_k(A) = (A - A_0)\cdots(A - A_{k-1})\times \times(A - A_{k+1})\cdots(A - A_n)$ $(k = 0, 1, \ldots, n)$, satisfying the interpolation conditions $L_{n0}(A_k) = F(A_k)$ for $k = 0, 1, \ldots, n$. In our case, $n = 4$, $F(A_k) = Y_k$ $(k = 0, 1, 2, 3, 4)$ and $U(A) \approx Y(A) = L_{4,0}(A)$.

We introduce the matrices of the form $\bar{A}_i = (A_{i-1} + A_i)/2$ $(i = 1, 2, 3, 4)$ and define the norms of the residual matrices between the left and right sides of the matrix-differential equation of the problem (14). We calculate the Gateaux differential $\delta Y[A; H] = \delta L_{4,0}[A; H]$ by the known [10] formula $\delta Y\left[\bar{A}_i; H\right] = = \lim\limits_{\lambda\to 0}\left\{\lambda^{-1}\left[Y(\bar{A}_i + \lambda H) - Y(\bar{A}_i)\right]\right\}$.

We denote by $R_i = \|\delta Y\left[\bar{A}_i; H\right] - 3Y\left(\bar{A}_i\right) - 2\bar{A}_i\|_2$, $i = 1, 2, 3, 4$, where $\|\cdot\|_2$ is the spectral norm of the corresponding matrix [11]. In our case, these norms are equal to $R_1 = 0.699$, $R_2 = 0.528$, $R_3 = 0.959$, $R_4 = 0.250$. The numerical experiment shows that the discrepancy between the left and right sides of the equation (14) is small, however, the accuracy of the approximation is not high. To obtain a higher accuracy of the solution it is necessary to involve more nodes or to use other methods of approximating the matrix-differential operator.

Analogous methods for solving matrix-differential equations can be obtained using the formulas of trigonometric, exponential, and other types of matrix generalized Hermite-Birkhoff interpolation.

4. Trigonometric interpolation

In [7] for 2π-periodic scalar functions the parametric family of trigonometric interpolation polynomials of degree not higher than $n + 1$ of the form

$$T_{n+1}^{\alpha,\beta}(x) = H_n(x) + \frac{\Omega_{n+1}^{\alpha,\beta}(x)D_{2n+1}(f; x_j)}{D_{2n+1}(\Omega_{n+1}^{\alpha,\beta}; x_j)}, \tag{20}$$

where $\Omega_{n+1}^{\alpha,\beta}(x) = \left(\alpha \sin\frac{x}{2} + \beta\cos\frac{x}{2}\right)\prod_{k=0}^{2n}\sin\frac{x - x_k}{2}$, $\alpha^2 + \beta^2 \neq 0$, $H_n(x)$ is a trigonometric interpolation polynomial of degree not higher than n of Lagrange type, and the differential operator $D_{2n+1}f(x)$ is defined by the formula

$$D_{2n+1}f(x) = (D^2 + n^2)\cdots(D^2 + 1^2)Df(x), \; D = \frac{d}{dx},$$

is constructed. The polynomial (20) satisfies the interpolation conditions

$$T_{n+1}^{\alpha,\beta}(x_i) = f(x_i) \; (i = 0, 1, \ldots, 2n); \;\; D_{2n+1}(T_{n+1}^{\alpha,\beta}; x_j) = D_{2n+1}(f; x_j).$$

We generalize the formula (20) in the case of functions of the matrix argument. Let X be the set of square matrices, $F(z)$ be an entire 2π-periodic function, $z \in \mathbb{C}$. In different matrix nodes A_k such that the matrices $A_k - A_\nu$

$(k, \nu = 0, 1, \ldots, 2n)$ are invertible, the values $F(A_k)$ of the function $F(A)$, $A \in X$, are known. The value $D_{2n+1}(F; A_j)$ of the matrix-differential operator

$$D_{2n+1}F(A) = (D^2 + n^2) \cdots (D^2 + 1^2) DF(z)\big|_{z=A}, \; D = \frac{d}{dz}, \qquad (21)$$

is also known in one of the nodes A_j.

Let's consider the differential operator of even order

$$D_{2n}F(A) = \left(D^2 + (n-1)^2\right) \cdots \left(D^2 + 1^2\right) D^2 F(z)\big|_{z=A}. \qquad (22)$$

The values of the operator for functions of the forms $B_1 F(A) B_2$, $F(cA + B)$ and $U(A)V(A)$ are calculated similarly, as are the values of the operator (4) for functions of this type. We assume that $D_0 F(A) \equiv F(A)$.

Let's generalize the Leibniz formula (5) to the case of functions of the matrix argument, and when the differential operators (21) and (22) are taken instead of the derivatives. Is valid

Theorem 5. *If the functions $U(z)$ and $V(z)$, $z \in \mathbb{C}$, are differentiable m times, then the formula*

$$D_m \left(U(A)V(A)\right) = D_{2p+1} \left(U(A)V(A)\right) = \sum_{k=0}^{m} C_m^k D_{m-k} U(A) D_k V(A), \qquad (23)$$

$$D_m \left(U(A)V(A)\right) = D_{2p+2} \left(U(A)V(A)\right) = \sum_{k=0}^{m} C_m^k D_{m-k} U(A) D_k V(A) -$$

$$- \frac{m(m-1)}{4} \sum_{k=1,3,\ldots}^{m-3} C_{m-2}^k D_{m-k-2} U(A) D_k V(A), \; A \in X, \; p = 0, 1, \ldots,$$

is valid.

The proof of the theorem 5 repeats the proof of the analogous theorem for the scalar case [8, p. 18-21]. In this case, the order of the multipliers in the matrix products must be strictly preserved: the values of the operators (21), (22) from the function $U(A)$ should be located to the left of the values of these operators from the function $V(A)$.

Lemma 1. *For trigonometric polynomials of the form*

$$P_n(A) = \sin\frac{A - B_1}{2} \sin\frac{A - B_2}{2} \cdots \sin\frac{A - B_{2n}}{2},$$

where $B_1, B_2, \ldots, B_{2n}$ are some matrices from X, the following identities are valid

$$D_j P_n(A) \equiv 0, \; j = 2n+1, 2n+2, \ldots \qquad (24)$$

Proof. Let's apply the method of mathematical induction. When $n = 1$

$$P_1(A) = \sin\frac{A - B_1}{2} \sin\frac{A - B_2}{2},$$

and by the formula (23) for $m = 3$ we have

$$D_3 P_1(A) = D_3 \sin\frac{A - B_1}{2} \cdot \sin\frac{A - B_2}{2} + 3 D_2 \sin\frac{A - B_1}{2} \cdot D_1 \sin\frac{A - B_2}{2} +$$

$$+3D_1\sin\frac{A-B_1}{2}\cdot D_2\sin\frac{A-B_2}{2}+\sin\frac{A-B_1}{2}\cdot D_3\sin\frac{A-B_2}{2}.$$

Since

$$D_1\sin\frac{A-B_k}{2}=D\sin\frac{A-B_k}{2}=\frac{1}{2}\cos\frac{A-B_k}{2},$$

$$D_2\sin\frac{A-B_k}{2}=D^2\sin\frac{A-B_k}{2}=-\frac{1}{4}\sin\frac{A-B_k}{2},$$

$$D_3\sin\frac{A-B_k}{2}=\big(D^3+D\big)\sin\frac{A-B_k}{2}=\frac{3}{8}\cos\frac{A-B_k}{2}\quad(k=1,2),$$

then $D_3P_1(A)\equiv 0$.

For the operator (21), (22) the properties $D_{2n+2}F(A)=DD_{2n+1}F(A)$, $D_{2n+3}F(A)=\big(D^2+(n+1)^2\big)D_{2n+1}F(A)$, $n\in\mathbb{N}$, where $F(A)$ is some matrix function for which the values of the operators (21) and (22) at the point $A\in X$ exist, are hold. Then it is obvious that $D_jP_1(A)\equiv 0$ when $j=4,5,\dots$

Let's suppose that the relations (24) hold when $n=k$. We will show that they are true when $n=k+1$. By the formula (23) for $m=2k+3$ we have

$$D_{2k+3}P_{k+1}(A)=D_{2k+3}\Big(P_k(A)\tilde{P}_1(A)\Big)=\sum_{i=0}^{2k+3}C^i_{2k+3}D_{2k+3-i}P_k(A)\cdot D_i\tilde{P}_1(A),$$

where

$$\tilde{P}_1(A)=\sin\frac{A-B_{2k+1}}{2}\sin\frac{A-B_{2k+2}}{2}.$$

For $i\le 2$, by assumption, the identities $D_{2k+3-i}P_k(A)\equiv 0$ hold, and when $i>2$ the identities $D_i\tilde{P}_1(A)\equiv 0$ are valid. Therefore $D_{2k+3}P_{k+1}(A)\equiv 0$. □

Let α and β be some fixed matrices from X that are not simultaneously zero.

Theorem 6. *The trigonometric polynomial*

$$T_{n+1}(A)\equiv T_{n+1}(A;\alpha,\beta)=$$

$$=H_n(A)+\Omega_{n+1}(A)\big[D_{2n+1}(\Omega_{n+1};A_{n+1})\big]^{-1}D_{2n+1}(F;A_{n+1}),\qquad(25)$$

where

$$H_n(A)=\sum_{k=0}^{2n}\Psi_k(A)\Psi_k^{-1}(A_k)F(A_k),\qquad(26)$$

$$\Psi_k(A)=\sin\frac{A-A_0}{2}\cdots\sin\frac{A-A_{k-1}}{2}\sin\frac{A-A_{k+1}}{2}\cdots\sin\frac{A-A_{2n}}{2},$$

$$\Omega_{n+1}(A)\equiv\Omega_{n+1}(A;\alpha,\beta)=\left(\alpha\sin\frac{A}{2}+\beta\cos\frac{A}{2}\right)\prod_{k=0}^{2n}\sin\frac{A-A_k}{2},$$

satisfies the interpolation conditions

$$T_{n+1}(A_k)=F(A_k)\ (k=0,1,\dots,2n);$$

$$D_{2n+1}(T_{n+1};A_{2n+1})=D_{2n+1}(F;A_{2n+1}).\qquad(27)$$

Proof. Since $\Psi_k(A_i) = \delta_{ki}\Psi_k(A_k)$, where δ_{ki} is the Kronecker symbol $(k, i = 0, 1, \ldots, 2n)$, then the polynomial (26) coincides with the operator $F(A)$ at the interpolation nodes $A_0, A_1, \ldots, A_{2n}$. It's obvious that $\Omega_{n+1}(A_k) = 0$ when $k = \overline{0, 2n}$. Therefore, the polynomial (25) coincides with $F(A)$ at the above-mentioned interpolation nodes.

We show that the last condition in (27) also holds. By the lemma $D_{2n+1}\Psi_k(A) = 0$ for $k = 0, 1, \ldots, 2n$, so $D_{2n+1}H_n(A) = 0$. Taking into account the structure of the formula (25), we obtain that the condition stated above for the polynomial $T_{n+1}(A)$ is satisfied. □

5. Conclusion

In this work we obtained the following new results: interpolation formulas for functions of a scalar argument are generalized to the case of operators defined in functional spaces and on the set of matrices. The algebraic operator and matrix interpolation Hermite–Birkhoff polynomials are constructed, as well as the parametric family of trigonometric matrix interpolation polynomials of Hermite type. Theorems on the fulfillment of the interpolation conditions are proved. For the operator interpolation formula, a class of polynomials for which it is exact is found. Matrix analogues of the Leibniz formula for linear matrix-differential operators of a special form are constructed. Based on the matrix algebraic interpolation polynomial, the formula for the approximation of the Gateaux differential of an arbitrary order of the matrix argument function is obtained. This formula is used in the construction of the approximate method for solving the Cauchy problem with a matrix-differential equation of the first order. In the computer algebra system, the illustrative example of a numerical solving the Cauchy problem of the indicated type is realized.

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