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METHOD OF TWO-SIDED APPROXIMATIONS FOR FINDING POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SEMILINEAR ELLIPTIC SYSTEMS: THE USE OF THE GREEN-RVACHEV'S QUASI-FUNCTION

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Резюме. Розглядається однорідна задача Діріхле для системи напівлінійних еліптичних рівнянь. Для побудови двобічних наближень до додатного розв'язку цієї системи використовується перехід до еквівалентної системи нелінійних інтегральних рівнянь (за допомогою квазіфункції Гріна-Рвачова) з подальшим її аналізом методами теорії напівупорядкованих просторів. Робота і ефективність розробленого метода продемонстрована обчислювальним експериментом для тестової системи з експоненціальною нелінійністю.

Abstract. A homogeneous Dirichlet problem for a system of semilinear elliptic equations is investigated. To construct two-sided approximations to a positive solution of this system, the transition to an equivalent system of nonlinear integral equations (with the help of the Green-Rvachev's quasi-function) with its subsequent analysis by methods of the theory of semiordered spaces is used. The work and efficiency of the developed method are demonstrated by a computational experiment for a test system with exponential nonlinearity.

1. Introduction

Let us consider the problem of finding a positive solution of a system of n semilinear elliptic equations with a homogeneous Dirichlet condition:

$$-\Delta u_i = f_i(\mathbf{x}, u_1, ..., u_n), \quad \mathbf{x} \in \Omega, \tag{1}$$

$$u_i(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega, \tag{2}$$

$$u_i|_{\partial\Omega} = 0, \quad i = 1, ..., n, \tag{3}$$

or in a vector form

$$-\Delta \mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x} \in \Omega,$$

$$\mathbf{u} > \boldsymbol{\theta}, \quad \mathbf{x} \in \Omega,$$

$$\mathbf{u}|_{\partial\Omega} = \boldsymbol{\theta},$$

where Ω is a bounded Jordan-measurable domain from $\mathbb{R}^m$ with piecewise smooth boundary $\partial\Omega$ ($\bar{\Omega} = \Omega \cup \partial\Omega$), $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{u} = (u_1, ..., u_n)$, $-\Delta\mathbf{u} = (-\Delta u_1, ..., -\Delta u_n)$, $\mathbf{f} = (f_1, ..., f_n)$, $\boldsymbol{\theta} = (0, ..., 0)$, Δ is the Laplace operator, $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_m^2}$.

Let us assume that the functions $f_i(\mathbf{x}, u_1, ..., u_n)$ are continuous and positive for $\mathbf{x} \in \bar{\Omega}$, $u_1, ..., u_n > 0$, for all $i = 1, 2, ..., n$.

Key words. Positive solution; semilinear elliptic systems; heterotone operator; two-sided approach; Green-Rvachev's quasi-function.

The problem (1) – (3) is a mathematical model of many stationary processes, which are considered in chemical kinetics, biology, combustion theory, etc. [12], and the condition of positivity (2) naturally arises from the physical meaning of the functions $u_1, \ldots, u_n$ as the substance concentration, population size, temperature, etc. Many studies are devoted to the investigation of problem (1) – (3) [1, 2, 6, 9, 10, 12, 19, etc.], but the focus in these papers was mainly on elucidating the conditions of the existence and uniqueness of a positive solution of the problem or on the conditions of the presence a solution with radial symmetry for the case when Ω is the unit ball. In the paper [17] for numerical analysis of the problem (1) – (3) a method of two-sided approximations, which consists in the transition to an equivalent system of Hammerstein integral equations with its subsequent investigations by methods of the theory of nonlinear operators in semiordered spaces, in particular, using the theory of heterotone operators developed by V. I. Opoĭcev, was proposed. The method showed effectiveness in solving the test problem, but it has some limitations in practical application. They are related to the fact that an analytic expression for the Green's function must be known. This significantly limits the range of regions Ω, in which a numerical solution can be found, to the cases presented in the reference literature [15].

The purpose of the paper is to develop iterative methods for solving the boundary value problem (1) – (3), which have a two-sided nature of convergence to the desired solution and would not be tied to the presence of a known Green's function. Two-sided approximate methods for solving nonlinear operator equations based on the theory of nonlinear operators in semiordered spaces were developed in [4, 5, 7, 8, 13, 14]. This paper continues the research begun in [17, 18], and extends them to areas of arbitrary geometry.

2. Some information from the theory of nonlinear operators in spaces with cones

Let us give from the theory of nonlinear operators in semiordered spaces some concepts and facts, which will be used further [7, 13, 14].

Let $\mathcal{E}$ be a real Banach space, θ is a zero element of space $\mathcal{E}$. A closed convex set $\mathcal{K} \subset \mathcal{E}$ is called a cone, if from the fact that $u \in \mathcal{E}$, $u \neq \theta$, follows $\alpha u \in \mathcal{K}$ with $\alpha \geq 0$ and $-u \notin \mathcal{K}$.

Any cone $\mathcal{K} \subset \mathcal{E}$ allows to enter in space $\mathcal{E}$ a semiordering by the rule: $v \leqslant w$, if $w - v \in \mathcal{K}$. The elements $u \geqslant \theta$ (i.e. $u \in \mathcal{K}$) are called positive. The set of elements $< v, w >$ of a semiordered space, which consists of those $u \in \mathcal{E}$ for which $v \leqslant u \leqslant w$, is called a cone segment.

An important class of cones for the applications of the theory of semiordered spaces in computational mathematics is normal cones. A cone $\mathcal{K}$ is called normal if there exists a number $N(\mathcal{K}) > 0$, that from $\theta \leqslant x \leqslant y$ follows $\|x\| \leq N(\mathcal{K}) \|y\|$. In this case, it is said that the norm is semimonotonic. If $N(\mathcal{K}) = 1$, then the cone is called acute and it is said that the norm is monotonous.

The operator $T : \mathcal{E} \to \mathcal{E}$ is called positive if it leaves invariant the cone $\mathcal{K}$, i.e. $T(u) \in \mathcal{K}$ for any $u \in \mathcal{K}$.

The operator $T : \mathcal{E} \to \mathcal{E}$ is called heterotone (or mixed monotone [3,11, etc.]), if it allows a diagonal representation $T(u) \equiv \hat{T}(u,u)$, where the companion operator $\hat{T} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ monotonically increases with respect to the first argument and decreases with respect to the second one, i.e.

a) if $v_1 \leqslant v_2$, then $\hat{T}(v_1, w) \leqslant \hat{T}(v_2, w)$ for all $w \in \mathcal{E}$;
b) if $w_1 \leqslant w_2$, then $\hat{T}(v, w_1) \geqslant \hat{T}(v, w_2)$ for all $v \in \mathcal{E}$.

A cone segment $< v^0, w^0 >$ is called strongly invariant for a heterotone operator T, if

$$\hat{T}(v^0, w^0) \geq v^0, \quad \hat{T}(w^0, v^0) \leq w^0. \tag{4}$$

For the equation $u = T(u)$ with the heterotone operator T, let us form two iterative processes

$$v^{(k+1)} = \hat{T}(v^{(k)}, w^{(k)}), \quad w^{(k+1)} = \hat{T}(w^{(k)}, v^{(k)}), \quad k = 0, 1, 2, ..., \tag{5}$$

starting from the point (v^0, w^0) formed by the ends of the strongly invariant cone segment $< v^0, w^0 >$.

From the heterotony of the operator T for which the operator $\hat{T}$ is a companion one, it follows that the sequence $\{v^{(k)}\}$ does not increase, and the sequence $\{w^{(k)}\}$ does not decrease with respect to the cone $\mathcal{K}$. If the cone $\mathcal{K}$ is normal and the operator $\hat{T}$ is completely continuous, then the limits v^* and w^* of these sequences exist. Thus, the chain of inequalities holds:

$$v^0 = v^{(0)} \leqslant v^{(1)} \leqslant ... \leqslant v^{(k)} \leqslant ... \leqslant v^* \leqslant w^* \leqslant ... \leqslant \\ \leqslant w^{(k)} \leqslant ... \leqslant w^{(1)} \leqslant w^{(0)} = w^0.$$

In this case, two cases are possible: $v^* < w^*$ and $v^* = w^*$. In the second case, $u^* := v^* = w^*$ is the unique on $< v^0, w^0 >$ fixed point of the operator T, that is, it is the unique on $< v^0, w^0 >$ solution of the equation $u = T(u)$.

The elements v^* and w^* are a solution of the system

$$v^{(k+1)} = \hat{T}(v^{(k)}, w^{(k)}), \quad w^{(k+1)} = \hat{T}(w^{(k)}, v^{(k)}), \quad k = 0, 1, 2, ... \tag{6}$$

The equality $v^* = w^*$ will hold if the system (6) does not have on $< v^0, w^0 >$ such solutions (v, w) that $v \neq w$.

Then the results of [7] imply the following fact.

Theorem 1. *Let the cone segment $< v^0, w^0 >$ be strongly invariant for the heterotone operator T for which the operator $\hat{T}$ is a companion one, the cone $\mathcal{K}$ be normal, and the operator $\hat{T}$ be completely continuous. Then the successive approximations, which are formed according to scheme (5), where $v^{(0)} = v^0$, $w^{(0)} = w^0$, converge to the unique on $< v^0, w^0 >$ fixed point u^* of the operator T and the following inequalities*

$$v^0 = v^{(0)} \leqslant v^{(1)} \leqslant ... \leqslant v^{(k)} \leqslant ... \leqslant u^* \leqslant ... \leqslant \\ \leqslant w^{(k)} \leqslant ... \leqslant w^{(1)} \leqslant w^{(0)} = w^0 \tag{7}$$

are satisfied.

The chain of inequalities (7) characterizes the iterative process (5) as a method of two-sided approximations.

The condition that the system (6) does not have on $< v^0, w^0 >$ such solutions (v, w) that $v \neq w$, can be complicated for practical employment. A sufficient condition of the fulfilment of the equality $v^* = w^*$ is the existence of such $\gamma \in (0; 1)$ that

$$\left\| \hat{T}(v, w) - \hat{T}(w, v) \right\| \leq \gamma \left\| v - w \right\| \tag{8}$$

for all $v, w \in < v^0, w^0 >$ [3].

If the condition (8) is satisfied, it is obtained the estimate

$$\left\| w^{(k)} - v^{(k)} \right\| = \left\| \hat{T}(w^{(k-1)}, v^{(k-1)}) - \hat{T}(v^{(k-1)}, w^{(k-1)}) \right\| \leq$$
$$\leq \gamma \left\| w^{(k-1)} - v^{(k-1)} \right\| \leq ... \leq \gamma^k \left\| w^0 - v^0 \right\|.$$

Then, if

$$u^{(k)} = \frac{1}{2}(w^{(k)} + v^{(k)}) \tag{9}$$

is taken as the approximate solution of the operator equation $u = T(u)$ on the k-th iteration, then the following error estimate holds:

$$\left\| u^* - u^{(k)} \right\| \leq \frac{\gamma^k}{2} \left\| w^0 - v^0 \right\|. \tag{10}$$

Thus, the following theorem holds.

Theorem 2. *Let the cone segment $< v^0, w^0 >$ be strongly invariant for the heterotone operator T for which the operator $\hat{T}$ is a companion one, the cone $\mathcal{K}$ be normal, and the operator $\hat{T}$ be completely continuous. Then, if condition (8) is satisfied, the successive approximations that are formed according to the scheme (5), where $v^{(0)} = v^0$, $w^{(0)} = w^0$, two-sided in the sense of (7) converge to the unique on $< v^0, w^0 >$ fixed point u^* of the operator T and for the approximate solution of the form (9) on the k-th iteration the estimate (10) holds.*

From estimation (10) it follows that for a faster convergence of iterations (5) it is necessary to choose a strongly invariant cone segment $< v^0, w^0 >$ of as short as possible length $\left\| w^0 - v^0 \right\|$.

If the accuracy $\varepsilon > 0$ with which it is necessary to find an approximate solution of the equation $u = T(u)$, is given, then, using the estimate (10), from the inequality $\left\| u^* - u^{(k)} \right\| < \varepsilon$, it is obtained that to achieve the specified accuracy it is necessary to do

$$k_0(\varepsilon) = \left[\frac{\ln \frac{\left\| w^0 - v^0 \right\|}{2\varepsilon}}{\ln \frac{1}{\gamma}} \right] + 1 \tag{11}$$

iterations, where the square brackets denote the integer part of the number.

3. Construction of two-sided approximations

To analyze the problem (1) – (3) and construct two-sided approximations to its positive solution, let us use the methods of the theory of nonlinear operators in semiordered spaces [7,13,14] and the Green-Rvachev's quasi-function [16,18].

Let the boundary $\partial\Omega$ of the domain consists of a finite number of pieces of lines $\sigma_i(\mathbf{x}) = 0$, $i = 1, 2, ..., r$, where each $\sigma_i(\mathbf{x})$ is an elementary function. Then with the help of the R-functions method [15] one can construct in the form of a single analytic expression an elementary function $\omega(\mathbf{x})$, which describes the geometry of the region Ω, that is:

a) $\omega(\mathbf{x}) > 0$ in Ω;
b) $\omega(\mathbf{x}) = 0$ on $\partial\Omega$;
c) $|\nabla\omega(\mathbf{x})| \neq 0$ on $\partial\Omega$.

Also, the function $\omega(\mathbf{x})$ can have certain properties of differentiation due to the use of various sufficiently complete systems of R-functions [16].

Definition 7. Let $g_m(r)$ be a fundamental solution of the equation $\Delta u = 0$ in $\mathbb{R}^m$. The Green-Rvachev's quasi-function of the first boundary value problem for the Laplace operator in $\mathbb{R}^m$ is the function

$$Q_m(\mathbf{x}, \boldsymbol{\xi}) = g_m(r) - \tilde{g}_m(\mathbf{x}, \boldsymbol{\xi}), \tag{12}$$

where $\mathbf{x} = (x_1, ..., x_m)$, $\boldsymbol{\xi} = (\xi_1, ..., \xi_m)$,

$$r = |\mathbf{x} - \boldsymbol{\xi}| = \sqrt{\sum_{i=1}^{m} (x_i - \xi_i)^2}, \quad \tilde{g}_m(\mathbf{x}, \boldsymbol{\xi}) = g_m\left(\sqrt{r^2 + 4\omega(\mathbf{x})\omega(\boldsymbol{\xi})}\right),$$

$\omega(\mathbf{x})$ is the function that describes the geometry of the domain Ω.

Let us note [16] that for the case when Ω is a ball of radius R in $\mathbb{R}^m$, and $\omega(\mathbf{x}) = \frac{1}{2R}(R^2 - x_1^2 - ... - x_m^2)$, the Green-Rvachev's quasi-function (12) turns into the exact Green's function of the first boundary value problem for the Laplace operator considered in a ball Ω.

The fundamental solutions of the Laplace equation have the form

$$g_2(r) = \frac{1}{2\pi} \ln \frac{1}{r},$$

$$g_3(r) = \frac{1}{4\pi} \cdot \frac{1}{r},$$

$$g_m(r) = \frac{1}{|S_1|\,(m-2)} \cdot \frac{1}{r^{m-2}}, \quad m > 3,$$

where $|S_1|$ is the area of a single sphere in $\mathbb{R}^m$, consequently, the Green-Rvachev's quasi-function acquires the form

$$Q_2(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{2\pi} \ln \sqrt{1 + \frac{4\omega(\mathbf{x})\omega(\boldsymbol{\xi})}{r^2}} \text{ in } \mathbb{R}^2, \tag{13}$$

$$Q_3(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{4\pi} \cdot \frac{\sqrt{r^2 + 4\omega(\mathbf{x})\omega(\boldsymbol{\xi})} - r}{r\sqrt{r^2 + 4\omega(\mathbf{x})\omega(\boldsymbol{\xi})}} \text{ in } \mathbb{R}^3, \tag{14}$$

$$Q_m(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{|S_1|\,(m-2)} \cdot \frac{(r^2+4\omega(\mathbf{x})\omega(\boldsymbol{\xi}))^{\frac{m}{2}-1} - r^{m-2}}{r^{m-2}(r^2+4\omega(\mathbf{x})\omega(\boldsymbol{\xi}))^{\frac{m}{2}-1}} \text{ in } \mathbb{R}^m, \quad m>3. \quad (15)$$

From (13) – (15) and Definition 7 the following lemma on the properties of the Green-Rvachev's quasi-function follows.

Lemma 1. *The Green-Rvachev's quasi-function (12) has the following properties:*

a) $Q(\mathbf{x},\boldsymbol{\xi}) = 0$ *on* $\partial\Omega$*;*
b) *is a symmetric function:* $Q(\mathbf{x},\boldsymbol{\xi}) = Q(\boldsymbol{\xi},\mathbf{x})$*;*
c) *has the same feature for* $\mathbf{x}=\boldsymbol{\xi}$ *as the usual Green's function;*
d) *is positive in the area* Ω*:* $Q(\mathbf{x},\boldsymbol{\xi}) > 0$, $\mathbf{x},\boldsymbol{\xi} \in \Omega$, $\mathbf{x} \neq \boldsymbol{\xi}$.

According to [16, 18], from each of the equations (1) let us proceed to an integral equation of the form

$$\begin{aligned} u_i(\mathbf{x}) = &\int_\Omega K_m(\mathbf{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi})d\boldsymbol{\xi} + \\ &+ \int_\Omega Q_m(\mathbf{x},\boldsymbol{\xi})f_i(\boldsymbol{\xi},u_1(\boldsymbol{\xi}),...,u_n(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad i=1,...,n, \end{aligned} \quad (16)$$

where $K_m(\mathbf{x},\boldsymbol{\xi}) = -\frac{\partial^2}{\partial\xi_1^2}\tilde{g}_m(\mathbf{x},\boldsymbol{\xi}) - ... - \frac{\partial^2}{\partial\xi_m^2}\tilde{g}_m(\mathbf{x},\boldsymbol{\xi})$.

The system of equations (16) can be written in the form of a vector equation of Urysohn

$$\mathbf{u}(\mathbf{x}) = \int_\Omega \mathbf{P}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}(\boldsymbol{\xi}))d\boldsymbol{\xi},$$

where

$$\mathbf{P}(\mathbf{x},\boldsymbol{\xi},\mathbf{u}(\boldsymbol{\xi})) = (P_1(\mathbf{x},\boldsymbol{\xi},u_1(\boldsymbol{\xi}),...,u_n(\boldsymbol{\xi})),...,P_n(\mathbf{x},\boldsymbol{\xi},u_1(\boldsymbol{\xi}),...,u_n(\boldsymbol{\xi}))),$$

$$P_i(\mathbf{x},\boldsymbol{\xi},u_1(\boldsymbol{\xi}),...,u_n(\boldsymbol{\xi})) = K_m(\mathbf{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi}) + Q(\mathbf{x},\boldsymbol{\xi})f_i(\boldsymbol{\xi},u_1(\boldsymbol{\xi}),...,u_n(\boldsymbol{\xi}))),$$
$$i=1,...,n.$$

If the boundary value problem (1) – (3) has a classical solution, then it also satisfies the system of equations (16). If the classical solution of the problem does not exist, then the system of equations (16) can be used to introduce the concept of a generalized solution of the boundary value problem (1) – (3).

The system of equations (16) will be considered in a Banach space $\mathbf{C}_n(\bar{\Omega}) = \{\mathbf{u} = (u_1,...,u_n) : u_i \in C(\bar{\Omega}), i=1,...,n\}$ of vector functions continuous in $\bar{\Omega}$ with the norm $\|\mathbf{u}\|_n = \max\{\|u_1\|,...,\|u_n\|\}$, where $\|u_i\| = \max_{\mathbf{x}\in\bar{\Omega}} |u_i(\mathbf{x})|$, $i=1,...,n$. Let us select in $\mathbf{C}^n(\bar{\Omega})$ the cone $\boldsymbol{\mathcal{K}}_+ = \{\mathbf{u} = (u_1,...,u_n) \in \mathbf{C}^n(\bar{\Omega}) : u_i(\mathbf{x}) \geq 0, \mathbf{x} \in \bar{\Omega}, i=1,...,n\}$ of vector functions with non-negative coordinates. Note that the cone $\boldsymbol{\mathcal{K}}_+$ in $\mathbf{C}^n(\bar{\Omega})$ is normal (and even acute).

With the help of the cone $\boldsymbol{\mathcal{K}}_+$ in the space $\mathbf{C}^n(\bar{\Omega})$ let us introduce a semiordering by the rule:

$$\text{for } \mathbf{u},\mathbf{v} \in \mathbf{C}^n(\bar{\Omega}) \quad \mathbf{u} \leqslant \mathbf{v}, \text{ if } \mathbf{v}-\mathbf{u} \in \boldsymbol{\mathcal{K}}_+,$$

that is,

$$\mathbf{u} \leqslant \mathbf{v}, \text{ if } u_i(\mathbf{x}) \leq v_i(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega} \text{ and for all } i = 1, ..., n.$$

Definition 8. By a solution (generalized) of the problem (1) – (3) will be meant a vector-valued function $\mathbf{u}^* \in \mathcal{K}_+$, which is a solution of the system of integral equations (16).

Let us construct a process of two-sided approximations for finding the solution of the integral equations system (16) (and consequently, the solution of the boundary value problem (1) – (3)).

Let us introduce a nonlinear integral operator $\mathbf{T}$ acting in $\mathbf{C}_n(\bar{\Omega})$ by the rule, which is determined by the right-hand side of the equations system (16)

$$\mathbf{T}(\mathbf{u})(\mathbf{x}) = \int\limits_{\Omega} \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{u}(\boldsymbol{\xi}))d\boldsymbol{\xi} = (T_1(\mathbf{u})(\boldsymbol{\xi}), ..., T_n(\mathbf{u})(\boldsymbol{\xi})), \tag{17}$$

where

$$T_i(\mathbf{u})(\mathbf{x}) = \int\limits_{\Omega} P_i(\mathbf{x}, \boldsymbol{\xi}, u_1(\boldsymbol{\xi}), ..., u_n(\boldsymbol{\xi}))d\boldsymbol{\xi} =$$

$$= \int\limits_{\Omega} K_m(\mathbf{x}, \boldsymbol{\xi})u_i(\boldsymbol{\xi})d\boldsymbol{\xi} + \int\limits_{\Omega} Q_m(\mathbf{x}, \boldsymbol{\xi})f_i(\boldsymbol{\xi}, u_1(\boldsymbol{\xi}), ..., u_n(\boldsymbol{\xi}))d\boldsymbol{\xi}. \tag{18}$$

The operator $\mathbf{T}$ of the form (17) can be represented as the sum of a linear integral operator $\mathbf{T}_1$ acting in $\mathbf{C}_n(\bar{\Omega})$ by the rule

$$\mathbf{T}_1(\mathbf{u})(\mathbf{x}) = \left(\int\limits_{\Omega} K_1(\mathbf{x}, \boldsymbol{\xi})u_1(\boldsymbol{\xi})d\boldsymbol{\xi}, ..., \int\limits_{\Omega} K_n(\mathbf{x}, \boldsymbol{\xi})u_n(\boldsymbol{\xi})d\boldsymbol{\xi} \right),$$

and a nonlinear Hammerstein operator $\mathbf{T}_2$ acting in $\mathbf{C}_n(\bar{\Omega})$ by the rule

$$\mathbf{T}_2(\mathbf{u})(\mathbf{x}) = \left(\int\limits_{\Omega} Q_m(\mathbf{x}, \boldsymbol{\xi})f_1(\boldsymbol{\xi}, u_1(\boldsymbol{\xi}), ..., u_n(\boldsymbol{\xi}))d\boldsymbol{\xi}, ..., \right.$$

$$\left. \int\limits_{\Omega} Q_m(\mathbf{x}, \boldsymbol{\xi})f_n(\boldsymbol{\xi}, u_1(\boldsymbol{\xi}), ..., u_n(\boldsymbol{\xi}))d\boldsymbol{\xi} \right).$$

From the item d) of Lemma 1 it follows that the operator $\mathbf{T}_2$ is a positive operator, because it leaves the cone $\mathcal{K}_+$ invariant, but because there is no assurance in the sign of the function $K_m(\mathbf{x}, \boldsymbol{\xi})$ for $\mathbf{x}, \boldsymbol{\xi} \in \Omega$ ($\mathbf{x} \neq \boldsymbol{\xi}$), the question of the positivity of the operator $\mathbf{T}_1$ is an open one. Therefore, we can not say that the operator $\mathbf{T}$ is positive. However, the operator $\mathbf{T}$ of the form (17) can be represented as a difference of positive operators.

Let us denote

$$K_m^+(\mathbf{x}, \boldsymbol{\xi}) = \max\{0, K_m(\mathbf{x}, \boldsymbol{\xi})\}, \quad K_m^-(\mathbf{x}, \boldsymbol{\xi}) = \max\{0, -K_m(\mathbf{x}, \boldsymbol{\xi})\}.$$

It is clear that $K_m^+(\mathbf{x}, \boldsymbol{\xi}) \geq 0$ and $K_m^-(\mathbf{x}, \boldsymbol{\xi}) \geq 0$ for $\mathbf{x}, \boldsymbol{\xi} \in \Omega$ ($\mathbf{x} \neq \boldsymbol{\xi}$).

Then

$$K_m(\mathbf{x},\boldsymbol{\xi}) = K_m^+(\mathbf{x},\boldsymbol{\xi}) - K_m^-(\mathbf{x},\boldsymbol{\xi}), \quad |K_m(\mathbf{x},\boldsymbol{\xi})| = K_m^+(\mathbf{x},\boldsymbol{\xi}) + K_m^-(\mathbf{x},\boldsymbol{\xi})$$

and the operators T_i, $i = 1, ..., n$, of the form (17) will be written in the form

$$T_i(\mathbf{u})(\mathbf{x}) = \int_\Omega K_m^+(\mathbf{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi})d\boldsymbol{\xi} - \int_\Omega K_m^-(\mathbf{x},\boldsymbol{\xi})u_i(\boldsymbol{\xi})d\boldsymbol{\xi} +$$
$$+ \int_\Omega Q_m(\mathbf{x},\boldsymbol{\xi})f_i(\boldsymbol{\xi}, u_1(\boldsymbol{\xi}), ..., u_n(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad i = 1, ..., n. \tag{19}$$

Suppose that the vector-valued function $\mathbf{f}(\mathbf{x},\mathbf{u})$ allows a diagonal representation $\mathbf{f}(\mathbf{x},\mathbf{u}) = \hat{\mathbf{f}}(\mathbf{x},\mathbf{u},\mathbf{u}) = (\hat{f}_1(\mathbf{x},\mathbf{u},\mathbf{u}), ..., \hat{f}_n(\mathbf{x},\mathbf{u},\mathbf{u}))$, besides, continuous on the sets of variables $\mathbf{x}$, $\mathbf{v}$, $\mathbf{w}$ functions $\hat{f}_i(\mathbf{x},\mathbf{v},\mathbf{w}) = \hat{f}_i(\mathbf{x}, v_1, ..., v_n, w_1, ..., w_n)$ monotonically increase with respect to all v_i and monotonically decrease with respect to all w_i, $i = 1, ..., n$, for all $\mathbf{x} \in \Omega$. Then the operator $\mathbf{T}$ of the form (17) will be heterotone with the companion operator

$$\hat{\mathbf{T}}(\mathbf{v},\mathbf{w})(\mathbf{x}) = (\hat{T}_1(\mathbf{v},\mathbf{w})(\mathbf{x}), ..., \hat{T}_n(\mathbf{v},\mathbf{w})(\mathbf{x})), \tag{20}$$

where

$$\hat{T}_i(\mathbf{v},\mathbf{w})(\mathbf{x}) = \int_\Omega K_m^+(\mathbf{x},\boldsymbol{\xi})v_i(\boldsymbol{\xi})d\boldsymbol{\xi} - \int_\Omega K_m^-(\mathbf{x},\boldsymbol{\xi})w_i(\boldsymbol{\xi})d\boldsymbol{\xi} +$$
$$+ \int_\Omega Q_m(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, v_1(\boldsymbol{\xi}), ..., v_n(\boldsymbol{\xi}), w_1(\boldsymbol{\xi}), ..., w_n(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad i = 1, ..., n. \tag{21}$$

It is clear that the operators $\mathbf{T}$ and $\hat{\mathbf{T}}$ are completely continuous, and the operator T_i of the form (18) will be heterotone with the companion operator $\hat{T}_i$ of the form (21).

In the cone $\boldsymbol{\mathcal{K}}_+$ let us select a strongly invariant cone segment $< \mathbf{v}^0, \mathbf{w}^0 >$, $\mathbf{v}^0 = (v_1^0, ..., v_n^0)$, $\mathbf{w}^0 = (w_1^0, ..., w_n^0)$, by conditions (4), which for the operator $\hat{\mathbf{T}}$ that is defined by (20), will have the form: for all $\mathbf{x} \in \bar{\Omega}$

$$\int_\Omega K_m^+(\mathbf{x},\boldsymbol{\xi})v_i^0(\boldsymbol{\xi})d\boldsymbol{\xi} - \int_\Omega K_m^-(\mathbf{x},\boldsymbol{\xi})w_i^0(\boldsymbol{\xi})d\boldsymbol{\xi} + \tag{22}$$
$$+ \int_\Omega Q_m(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, v_1^0(\boldsymbol{\xi}), ..., v_n^0(\boldsymbol{\xi}), w_1^0(\boldsymbol{\xi}), ..., w_n^0(\boldsymbol{\xi}))d\boldsymbol{\xi} \geq v_i^0(\mathbf{x}), \quad i = 1, ..., n,$$

$$\int_\Omega K_m^+(\mathbf{x},\boldsymbol{\xi})w_i^0(\boldsymbol{\xi})d\boldsymbol{\xi} - \int_\Omega K_m^-(\mathbf{x},\boldsymbol{\xi})v_i^0(\boldsymbol{\xi})d\boldsymbol{\xi} + \tag{23}$$
$$+ \int_\Omega Q_m(\mathbf{x},\boldsymbol{\xi})\hat{f}_i(\boldsymbol{\xi}, w_1^0(\boldsymbol{\xi}), ..., w_n^0(\boldsymbol{\xi}), v_1^0(\boldsymbol{\xi}), ..., v_n^0(\boldsymbol{\xi}))d\boldsymbol{\xi} \leq w_i^0(\mathbf{x}), \quad i = 1, ..., n.$$

Let us form an iterative process by the scheme (5):

$$v_i^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} K_m^{+}(\mathbf{x},\boldsymbol{\xi}) v_i^{(k)}(\boldsymbol{\xi}) d\xi - \int\limits_{\Omega} K_m^{-}(\mathbf{x},\boldsymbol{\xi}) w_i^{(k)}(\boldsymbol{\xi}) d\boldsymbol{\xi} +$$

$$+\int\limits_{\Omega} Q_m(\mathbf{x},\boldsymbol{\xi}) \hat{f}_i(\boldsymbol{\xi}, v_1^{(k)}(\boldsymbol{\xi}),...,v_n^{(k)}(\boldsymbol{\xi}), w_1^{(k)}(\boldsymbol{\xi}),...,w_n^{(k)}(\boldsymbol{\xi}))d\boldsymbol{\xi}, \qquad (24)$$

$$w_i^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} K_m^{+}(\mathbf{x},\boldsymbol{\xi}) w_i^{(k)}(\boldsymbol{\xi}) d\xi - \int\limits_{\Omega} K_m^{-}(\mathbf{x},\xi) v_i^{(k)}(\boldsymbol{\xi}) d\boldsymbol{\xi} +$$

$$+\int\limits_{\Omega} Q_m(\mathbf{x},\boldsymbol{\xi}) \hat{f}_i(\boldsymbol{\xi}, w_1^{(k)}(\boldsymbol{\xi}),...,w_n^{(k)}(\boldsymbol{\xi}), v_1^{(k)}(\boldsymbol{\xi}),...,v_n^{(k)}(\boldsymbol{\xi}))d\boldsymbol{\xi}, \qquad (25)$$

$$i = 1,...,n, \quad k = 0,1,2,...; \qquad (26)$$

$$v_i^{(0)}(\mathbf{x}) = v_i^0(\mathbf{x}), \quad w_i^{(0)}(\mathbf{x}) = w_i^0(\mathbf{x}), \quad i = 1,...,n. \qquad (27)$$

Taking into account Theorem 1, such conditions for the existence of a unique solution of the problem (1) – (3) and the convergence of successive approximations (24) – (27) to it can be given.

Theorem 3. *Let $< \mathbf{v}^0, \mathbf{w}^0 >$ be a strongly invariant cone segment for the heterotone operator $\mathbf{T}$ of the form (17) with the companion operator $\hat{\mathbf{T}}$ of the form (20) and the system of $2n$ integral equations*

$$v_i(\mathbf{x}) = \int\limits_{\Omega} K_m^{+}(\mathbf{x},\boldsymbol{\xi}) v_i(\boldsymbol{\xi}) d\boldsymbol{\xi} - \int\limits_{\Omega} K_m^{-}(\mathbf{x},\boldsymbol{\xi}) w_i(\boldsymbol{\xi}) d\boldsymbol{\xi} +$$

$$+\int\limits_{\Omega} Q_m(\mathbf{x},\boldsymbol{\xi}) \hat{f}_i(\boldsymbol{\xi}, v_1(\boldsymbol{\xi}),...,v_n(\boldsymbol{\xi}), w_1(\boldsymbol{\xi}),...,w_n(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad i = 1,...,n,$$

$$w_i(\mathbf{x}) = \int\limits_{\Omega} K_m^{+}(\mathbf{x},\boldsymbol{\xi}) w_i(\boldsymbol{\xi}) d\boldsymbol{\xi} - \int\limits_{\Omega} K_m^{-}(\mathbf{x},\boldsymbol{\xi}) v_i(\boldsymbol{\xi}) d\boldsymbol{\xi} +$$

$$+\int\limits_{\Omega} Q_m(\mathbf{x},\boldsymbol{\xi}) \hat{f}_i(\boldsymbol{\xi}, w_1(\boldsymbol{\xi}),...,w_n(\boldsymbol{\xi}), v_1(\boldsymbol{\xi}),...,v_n(\boldsymbol{\xi}))d\boldsymbol{\xi}, \quad i = 1,...,n,$$

does not have on $< \mathbf{v}^0, \mathbf{w}^0 >$ solutions such that $\mathbf{v} \neq \mathbf{w}$. Then the iterative process (24) – (27) converges in the norm of the space $\mathbf{C}_n(\bar{\Omega})$ to the unique on $< \mathbf{v}^0, \mathbf{w}^0 >$ continuous positive solution $\mathbf{u}^*$ of the boundary value problem (1) – (3), and a chain of inequalities hold:

$$\mathbf{v}^0 = \mathbf{v}^{(0)} \leqslant \mathbf{v}^{(1)} \leqslant ... \leqslant \mathbf{v}^{(k)} \leqslant ... \leqslant \mathbf{u}^* \leqslant ... \leqslant \mathbf{w}^{(k)} \leqslant ... \leqslant \mathbf{w}^{(1)} \leqslant \mathbf{w}^{(0)} = \mathbf{w}^0.$$

Let us now use Theorem 2. Let for each i, $i = 1,...,n$, there exist such number $L_i > 0$ that the function $\hat{f}_i(\mathbf{x}, v_1,...,v_n, w_1,...,w_n)$ for all numbers v_1,

$\ldots,\ v_n,\ w_1,\ \ldots,\ w_n$ such that $0 < v_i, w_i < M_0^i$, where $M_0^i = \max\limits_{\mathbf{x}\in\bar{\Omega}} w_i^0(\mathbf{x})$, $i = 1, ..., n$, and for all $\mathbf{x} \in \Omega$ satisfies the inequality

$$\left|\hat{f}_i(\mathbf{x}, v_1, ..., v_n, w_1, ..., w_n) - \hat{f}_i(\mathbf{x}, w_1, ..., w_n, v_1, ..., v_n)\right| \leq$$
$$\leq L_i \max\{|v_1 - w_1|, ..., |v_n - w_n|\}. \tag{28}$$

Let us consider for an arbitrary i, $i = 1, ..., n$, the difference $\hat{T}_i(\mathbf{w}, \mathbf{v})(\mathbf{x}) - \hat{T}_i(\mathbf{v}, \mathbf{w})(\mathbf{x})$:

$$\hat{T}_i(\mathbf{w}, \mathbf{v})(\mathbf{x}) - \hat{T}_i(\mathbf{v}, \mathbf{w})(\mathbf{x}) = \int\limits_{\Omega} [K_m^+(\mathbf{x}, \boldsymbol{\xi}) + K_m^-(\mathbf{x}, \boldsymbol{\xi})][w_i(\boldsymbol{\xi}) - v_i(\boldsymbol{\xi})]d\boldsymbol{\xi} +$$
$$+ \int\limits_{\Omega} Q_m(\mathbf{x}, \mathbf{s})[\hat{f}_i(\boldsymbol{\xi}, w_1(\boldsymbol{\xi}), ..., w_n(\boldsymbol{\xi}), v_1(\boldsymbol{\xi}), ..., v_n(\boldsymbol{\xi})) -$$
$$- \hat{f}_i(\boldsymbol{\xi}, v_1(\boldsymbol{\xi}), ..., v_n(\boldsymbol{\xi}), w_1(\boldsymbol{\xi}), ..., w_n(\boldsymbol{\xi}))]d\boldsymbol{\xi}.$$

Then, taking into account the inequality (28), we get an estimate

$$\left\|\hat{\mathbf{T}}(\mathbf{w}, \mathbf{v}) - \hat{\mathbf{T}}(\mathbf{v}, \mathbf{w})\right\|_n = \max_{i=1,...,n} \max_{\mathbf{x}\in\bar{\Omega}} \left|\hat{T}_i(\mathbf{w}, \mathbf{v})(\mathbf{x}) - \hat{T}_i(\mathbf{v}, \mathbf{w})(\mathbf{x})\right| \leq$$
$$\leq \max_{i=1,...,n} \{M_1 + L_i M\} \cdot \max_{i=1,...,n} \max_{\mathbf{x}\in\bar{\Omega}} |w_i(\mathbf{x}) - v_i(\mathbf{x})| = (M_1 + LM)\|\mathbf{w} - \mathbf{v}\|_n,$$

where

$$M = \max_{\mathbf{x}\in\bar{\Omega}} \int\limits_{\Omega} Q_m(\mathbf{x}, \boldsymbol{\xi})d\boldsymbol{\xi}, \tag{29}$$

$$M_1 = \max_{\mathbf{x}\in\bar{\Omega}} \int\limits_{\Omega} [K_m^+(\mathbf{x}, \boldsymbol{\xi}) + K_m^-(\mathbf{x}, \boldsymbol{\xi})]d\boldsymbol{\xi}, \tag{30}$$

$$L = \max_{i=1,...,n} L_i. \tag{31}$$

Therefore,

$$\left\|\hat{\mathbf{T}}(\mathbf{w}, \mathbf{v}) - \hat{\mathbf{T}}(\mathbf{v}, \mathbf{w})\right\|_n \leq \gamma\|\mathbf{w} - \mathbf{v}\|_n,$$

where $\gamma = M_1 + LM$.

Thus, the following theorem holds.

Theorem 4. *Let $< \mathbf{v}^0, \mathbf{w}^0 >$ be a strongly invariant cone segment for the heterotone operator $\mathbf{T}$ of the form (17) with the companion operator $\hat{\mathbf{T}}$ of the form (20) and the condition (28) holds, besides, $\gamma = M_1 + LM < 1$, where the constants M, M_1 and L are defined by the equalities (29), (30) and 31) respectively. Then, the iterative process (24) – (27) two-sided converges in the norm of the space $\mathbf{C}_n(\bar{\Omega})$ to the unique on $< \mathbf{v}^0, \mathbf{w}^0 >$ continuous positive solution $\mathbf{u}^*$ of the boundary value problem (1) – (3).*

On the k-th iteration, in accordance with (9), as an approximate solution of the boundary value problem (1) – (3) the vector function

$$\mathbf{u}^{(k)}(\mathbf{x}) = \frac{1}{2}(\mathbf{w}^{(k)}(\mathbf{x}) + \mathbf{v}^{(k)}(\mathbf{x}))$$

is accepted.

Then there will be a posteriori estimate of the error of the approximation $\mathbf{u}^{(k)}(\mathbf{x})$:

$$\left\|\mathbf{u}^* - \mathbf{u}^{(k)}\right\|_n \le \frac{1}{2}\max_{i=1,\dots,n}\max_{\mathbf{x}\in\bar{\Omega}}(w_i^{(k)}(\mathbf{x}) - v_i^{(k)}(\mathbf{x})).$$

If the accuracy $\varepsilon > 0$ is given, then the iterative process should be carried out until the inequality

$$\max_{i=1,\dots,n}\max_{\mathbf{x}\in\bar{\Omega}}(w_i^{(k)}(\mathbf{x}) - v_i^{(k)}(\mathbf{x})) < 2\varepsilon \tag{32}$$

will be satisfied and then with an accuracy ε it can be expected that $\mathbf{u}^*(\mathbf{x}) \approx \mathbf{u}^{(k)}(\mathbf{x})$.

If the conditions of Theorem 4 are satisfied, then an a priori estimate of the error will be:

$$\left\|\mathbf{u}^* - \mathbf{u}^{(k)}\right\|_n \le \frac{\gamma^k}{2}\max_{i=1,\dots,n}\max_{\mathbf{x}\in\bar{\Omega}}(w_i^0(\mathbf{x}) - v_i^0(\mathbf{x})),$$

from which it is obtained that to achieve the accuracy ε it is necessary to do

$$k_0(\varepsilon) = \left[\frac{\ln\frac{\max\limits_{i=1,\dots,n}\max\limits_{\mathbf{x}\in\bar{\Omega}}(w_i^0(\mathbf{x}) - v_i^0(\mathbf{x}))}{2\varepsilon}}{\ln\frac{1}{\gamma}}\right] + 1 \tag{33}$$

iterations, where the square brackets denote the integer part of the number.

4. Numerical experiment

The construction of two-sided approximations to the solution of the boundary value problem (1) – (3) will be demonstrated on the system of two equations with exponential nonlinearities:

$$-\Delta u_1 = e^{u_2}, \quad -\Delta u_2 = e^{-u_1}, \quad \mathbf{x} \in \Omega, \tag{34}$$

$$u_1(\mathbf{x}) > 0, \quad u_2(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega, \tag{35}$$

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0, \tag{36}$$

where $\Omega = \{\mathbf{x} = (x_1, x_2) : 0 < x_1, x_2 < 1\}$.

The functions $f_1(\mathbf{x}, u_1, u_2) = e^{u_2}$, $f_2(\mathbf{x}, u_1, u_2) = e^{-u_1}$ are positive and continuous with respect to the set of variables, if $u_1, u_2 > 0$ and allow a diagonal representation with the help of functions

$$\hat{f}_1(\mathbf{x}, v_1, v_2, w_1, w_2) = e^{v_2}, \quad \hat{f}_2(\mathbf{x}, v_1, v_2, w_1, w_2) = e^{-w_1}. \tag{37}$$

The problem (34) – (36) is replaced by an equivalent system of integral equations

$$u_1(\mathbf{x}) = \int\limits_\Omega K_2(\mathbf{x}, \boldsymbol{\xi})u_1(\boldsymbol{\xi})d\boldsymbol{\xi} + \int\limits_\Omega Q_2(\mathbf{x}, \boldsymbol{\xi})e^{u_2(\boldsymbol{\xi})}d\boldsymbol{\xi}, \tag{38}$$

$$u_2(\mathbf{x}) = \int\limits_\Omega K_2(\mathbf{x}, \boldsymbol{\xi})u_2(\boldsymbol{\xi})d\boldsymbol{\xi} + \int\limits_\Omega Q_2(\mathbf{x}, \boldsymbol{\xi})e^{-u_1(\boldsymbol{\xi})}d\boldsymbol{\xi}, \tag{39}$$

where $Q_2(\mathbf{x},\boldsymbol{\xi})$ is determined by the formula (13),

$$K_2(\mathbf{x},\boldsymbol{\xi}) = -\frac{\partial^2}{\partial\xi_1^2}\tilde{g}_2(\mathbf{x},\boldsymbol{\xi}) - \frac{\partial^2}{\partial\xi_2^2}\tilde{g}_2(\mathbf{x},\boldsymbol{\xi}),$$

$$\tilde{g}_2(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{2\pi}\ln\frac{1}{\sqrt{r^2+4\omega(\mathbf{x})\omega(\boldsymbol{\xi})}},$$

$$\omega(\mathbf{x}) = [x_1(1-x_1)]\wedge_0[x_2(1-x_2)] \equiv$$
$$\equiv x_1(1-x_1) + x_2(1-x_2) - \sqrt{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2}.$$

With the system (38) – (39) let us associate a heterotone operator

$$\mathbf{T}(u_1,u_2) = \left(\int\limits_\Omega K_2(\mathbf{x},\boldsymbol{\xi})u_1(\boldsymbol{\xi})d\boldsymbol{\xi} + \int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{u_2(\boldsymbol{\xi})}d\boldsymbol{\xi},\right.$$
$$\left.\int\limits_\Omega K_2(\mathbf{x},\boldsymbol{\xi})u_2(\boldsymbol{\xi})d\boldsymbol{\xi} + \int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{-u_1(\boldsymbol{\xi})}d\boldsymbol{\xi}\right), \tag{40}$$

for which the companion operator has the form

$$\hat{\mathbf{T}}(v_1,v_2,w_1,w_2) = \left(\int\limits_\Omega K_2^+(\mathbf{x},\boldsymbol{\xi})v_1(\boldsymbol{\xi})d\boldsymbol{\xi} - \int\limits_\Omega K_2^-(\mathbf{x},\boldsymbol{\xi})w_1(\boldsymbol{\xi})d\boldsymbol{\xi} + \right.$$
$$+\int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{v_2(\boldsymbol{\xi})}d\boldsymbol{\xi}, \int\limits_\Omega K_2^+(\mathbf{x},\boldsymbol{\xi})v_2(\boldsymbol{\xi})d\boldsymbol{\xi} -$$
$$\left. - \int\limits_\Omega K_2^-(\mathbf{x},\boldsymbol{\xi})w_2(\boldsymbol{\xi})d\boldsymbol{\xi} + \int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{-w_1(\boldsymbol{\xi})}d\boldsymbol{\xi}\right),$$

where

$$K_2^+(\mathbf{x},\boldsymbol{\xi}) = \max\{0, K_2(\mathbf{x},\boldsymbol{\xi})\}, \quad K_2^-(\mathbf{x},\boldsymbol{\xi}) = \max\{0, -K_2(\mathbf{x},\boldsymbol{\xi})\}.$$

For the operator $\mathbf{T}$ of the form (40) a strongly invariant cone segment will be sought in the form $< \mathbf{v}^0, \mathbf{w}^0 >$, where $\mathbf{v}^0(\mathbf{x}) = (v_1^0(\mathbf{x}), v_2^0(\mathbf{x})) = (\alpha_1\omega(\mathbf{x}), \alpha_2\omega(\mathbf{x}))$, $\mathbf{w}^0(\mathbf{x}) = (w_1^0(\mathbf{x}), w_2^0(\mathbf{x})) = (\beta_1\omega(\mathbf{x}), \beta_2\omega(\mathbf{x}))$, $0 < \alpha_1 < \beta_1$, $0 < \alpha_2 < \beta_2$.

For the chosen vector-valued functions $\mathbf{v}^0$, $\mathbf{w}^0$ the system of inequalities (22), (23) for determining the constants α_1, α_2, β_1, β_2 has the form: for all $\mathbf{x} \in \bar{\Omega}$

$$\alpha_1\int\limits_\Omega K_2^+(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} - \beta_1\int\limits_\Omega K_2^-(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} +$$
$$+\int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{\alpha_2\omega(\boldsymbol{\xi})}d\boldsymbol{\xi} \geq \alpha_1\omega(\mathbf{x}),$$

$$\alpha_2 \int\limits_{\Omega} K_2^+(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} - \beta_2 \int\limits_{\Omega} K_2^-(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} +$$

$$+ \int\limits_{\Omega} Q_2(\mathbf{x},\boldsymbol{\xi})e^{-\beta_1\omega(\boldsymbol{\xi})}d\boldsymbol{\xi} \geq \alpha_2\omega(\mathbf{x}),$$

$$\beta_1 \int\limits_{\Omega} K_2^+(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} - \alpha_1 \int\limits_{\Omega} K_2^-(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} +$$

$$+ \int\limits_{\Omega} Q_2(\mathbf{x},\boldsymbol{\xi})e^{\beta_2\omega(\boldsymbol{\xi})}d\boldsymbol{\xi} \leq \beta_1\omega(\mathbf{x}),$$

$$\beta_2 \int\limits_{\Omega} K_2^+(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} - \alpha_2 \int\limits_{\Omega} K_2^-(\mathbf{x},\boldsymbol{\xi})\omega(\boldsymbol{\xi})d\boldsymbol{\xi} +$$

$$+ \int\limits_{\Omega} Q_2(\mathbf{x},\boldsymbol{\xi})e^{-\alpha_1\omega(\boldsymbol{\xi})}d\boldsymbol{\xi} \leq \beta_2\omega(\mathbf{x}).$$

These inequalities are satisfied, for example, by the numbers $\alpha_1 = 0,01$, $\alpha_2 = 0,01$, $\beta_1 = 0,59$, $\beta_2 = 0,55$.

Because for $0 < v_1, w_1 < \frac{\sqrt{2}-1}{2\sqrt{2}}\beta_1$, $0 < v_2, w_2 < \frac{\sqrt{2}-1}{2\sqrt{2}}\beta_2$ ($\max\limits_{\mathbf{x}\in\bar{\Omega}} \omega(\mathbf{x}) = \frac{\sqrt{2}-1}{2\sqrt{2}}$)

$$\left|\hat{f}_1(\mathbf{x}, v_1, v_2, w_1, w_2) - \hat{f}_1(\mathbf{x}, w_1, w_2, v_1, v_2)\right| = |e^{v_2} - e^{w_2}| \leq$$

$$\leq e^{\frac{\sqrt{2}-1}{2\sqrt{2}}\beta_2} |v_2 - w_2| \leq e^{\frac{\sqrt{2}-1}{2\sqrt{2}}\beta_2} \max\{|v_1 - w_1|, |v_2 - w_2|\},$$

$$\left|\hat{f}_2(\mathbf{x}, v_1, v_2, w_1, w_2) - \hat{f}_2(\mathbf{x}, w_1, w_2, v_1, v_2)\right| = \left|e^{-w_1} - e^{-v_1}\right| \leq$$

$$\leq |v_2 - w_2| \leq \max\{|v_1 - w_1|, |v_2 - w_2|\},$$

then

$$L = \max\left\{e^{\frac{\sqrt{2}-1}{2\sqrt{2}}\beta_2}, 1\right\} = \max\{1,08388; 1\} = 1,08388.$$

Further we find

$$M = \max_{\mathbf{x}\in\bar{\Omega}} \int\limits_{\Omega} Q_2(\mathbf{x},\boldsymbol{\xi})d\boldsymbol{\xi} = 0,04093,$$

$$M_1 = \max_{\mathbf{x}\in\bar{\Omega}} \int\limits_{\Omega} [K_2^+(\mathbf{x},\boldsymbol{\xi}) + K_2^-(\mathbf{x},\boldsymbol{\xi})]d\boldsymbol{\xi} = 0,70819,$$

$$\gamma = M_1 + LM = 0,753.$$

Thus, $\gamma < 1$ and by Theorem 4, the successive approximations that are formed by the scheme

$$v_1^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} K_2^+(\mathbf{x},\boldsymbol{\xi})v_1^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} - \int\limits_{\Omega} K_2^-(\mathbf{x},\boldsymbol{\xi})w_1^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} +$$

$$+ \int\limits_{\Omega} Q_2(\mathbf{x},\boldsymbol{\xi})e^{v_2^{(k)}(\boldsymbol{\xi})}d\boldsymbol{\xi},$$

$$v_2^{(k+1)}(\mathbf{x}) = \int\limits_\Omega K_2^+(\mathbf{x},\boldsymbol{\xi})v_2^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} - \int\limits_\Omega K_2^-(\mathbf{x},\boldsymbol{\xi})w_2^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} + $$
$$+ \int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{-w_2^{(k)}(\boldsymbol{\xi})}d\boldsymbol{\xi},$$

$$w_1^{(k+1)}(\mathbf{x}) = \int\limits_\Omega K_2^+(\mathbf{x},\boldsymbol{\xi})w_1^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} - \int\limits_\Omega K_2^-(\mathbf{x},\boldsymbol{\xi})v_1^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} + $$
$$+ \int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{w_2^{(k)}(\boldsymbol{\xi})}d\boldsymbol{\xi},$$

$$w_2^{(k+1)}(\mathbf{x}) = \int\limits_\Omega K_2^+(\mathbf{x},\boldsymbol{\xi})w_2^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} - \int\limits_\Omega K_2^-(\mathbf{x},\boldsymbol{\xi})v_2^{(k)}(\boldsymbol{\xi})d\boldsymbol{\xi} + $$
$$+ \int\limits_\Omega Q_2(\mathbf{x},\boldsymbol{\xi})e^{-v_2^{(k)}(\boldsymbol{\xi})}d\boldsymbol{\xi}, \quad k = 0, 1, 2, ...,$$

$$v_1^{(0)}(\mathbf{x}) = \alpha_1\omega(\mathbf{x}), \quad v_2^{(0)}(\mathbf{x}) = \alpha_2\omega(\mathbf{x}),$$
$$w_1^{(0)}(\mathbf{x}) = \beta_1\omega(\mathbf{x}), \quad w_2^{(0)}(\mathbf{x}) = \beta_2\omega(\mathbf{x}),$$

two-sided converge to the solution of problem (34) – (36).

Tabl. 1. The values of the estimate of the approximate solution error

Iteration number k	$\varepsilon_1^{(k)}$	$\varepsilon_2^{(k)}$
0	$0,42 \cdot 10^{-1}$	$0,40 \cdot 10^{-1}$
1	$0,23 \cdot 10^{-1}$	$0,22 \cdot 10^{-1}$
2	$0,12 \cdot 10^{-1}$	$0,11 \cdot 10^{-1}$
3	$0,60 \cdot 10^{-2}$	$0,56 \cdot 10^{-2}$
4	$0,29 \cdot 10^{-2}$	$0,28 \cdot 10^{-2}$
5	$0,14 \cdot 10^{-2}$	$0,13 \cdot 10^{-2}$
6	$0,70 \cdot 10^{-3}$	$0,66 \cdot 10^{-3}$
7	$0,34 \cdot 10^{-3}$	$0,32 \cdot 10^{-3}$
8	$0,17 \cdot 10^{-3}$	$0,16 \cdot 10^{-3}$
9	$0,80 \cdot 10^{-4}$	$0,76 \cdot 10^{-4}$

Let us choose $\varepsilon = 10^{-4}$. Then, in accordance with (33), to achieve this accuracy, it is necessary to make $k_0(\varepsilon) = \left[\frac{\ln \frac{\max\{\beta_1,\beta_2\}}{2\varepsilon}}{\ln \frac{1}{\gamma}}\right] + 1 = 28$ iterations. In fact, the accuracy $\varepsilon = 10^{-4}$ was achieved at the ninth iteration. As one can see, the theoretical error estimate turned out to be greatly overestimated. As an approximate solution of problem (34) – (36), the functions $u_1^{(9)}(\mathbf{x}) = \frac{v_1^{(9)}(\mathbf{x})+w_1^{(9)}(\mathbf{x})}{2}$, $u_2^{(9)}(\mathbf{x}) = \frac{v_2^{(9)}(\mathbf{x})+w_2^{(9)}(\mathbf{x})}{2}$ will be accepted.

TABL. 2. The values of the approximate solution in points $\mathbf{x}_i = (0,1i; 0,5)$, $i = 0, 1, ..., 10$

$\mathbf{x}_i = (0,1i; 0,5)$	$u_1^{(9)}(\mathbf{x}_i)$	$u_2^{(9)}(\mathbf{x}_i)$
$(0; 0,5)$	0	0
$(0,1; 0,5)$	$0,0301$	$0,0274$
$(0,2; 0,5)$	$0,0520$	$0,0471$
$(0,3; 0,5)$	$0,0666$	$0,0599$
$(0,4; 0,5)$	$0,0751$	$0,0672$
$(0,5; 0,5)$	$0,0778$	$0,0696$
$(0,6; 0,5)$	$0,0751$	$0,0672$
$(0,7; 0,5)$	$0,0666$	$0,0599$
$(0,8; 0,5)$	$0,0520$	$0,0471$
$(0,9; 0,5)$	$0,0301$	$0,0274$
$(1; 0,5)$	0	0

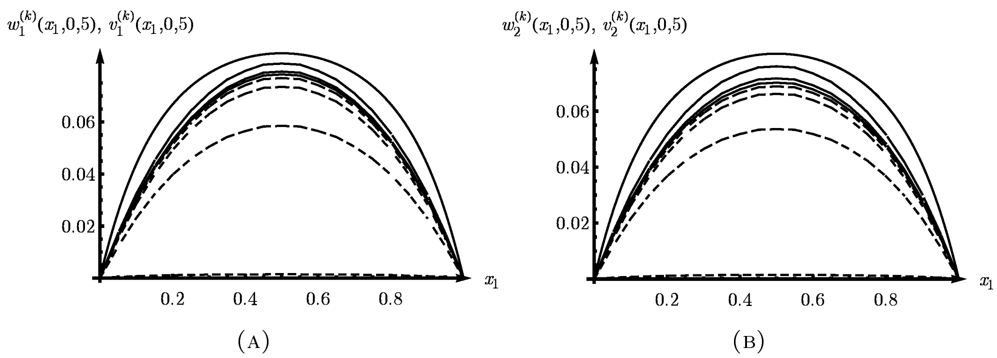

FIG. 1. Graphs of the cross-sections of upper and lower approximations $w_1^{(k)}(x_1, 0, 5)$, $v_1^{(k)}(x_1, 0, 5)$ (a) and $w_2^{(k)}(x_1, 0, 5)$, $v_2^{(k)}(x_1, 0, 5)$ (b), $k = 0, 2, 6, 8$

Table 1 gives the data how the estimate $\varepsilon_i^{(k)} = \max\limits_{\mathbf{x} \in \bar{\Omega}} \frac{1}{2}(w_i^{(k)}(\mathbf{x}) - v_i^{(k)}(\mathbf{x}))$ of the norm of the error $\left\| u_i^* - u_i^{(k)} \right\|$ of the approximate solution $u_i^{(k)}(\mathbf{x})$, $i = 1, 2$, varies depending on the iteration number k, $k = 0, 1, ..., 9$. Table 2 shows the values, found with accuracy $\varepsilon = 10^{-4}$ of the approximate solution $u_1^{(9)}(\mathbf{x})$, $u_2^{(9)}(\mathbf{x})$ at the points located on the straight line $x_2 = 0,5$ with the step $0,1$, and also it was found that $\left\| u_1^{(9)} \right\| = 0,0778$, $\left\| u_2^{(9)} \right\| = 0,0696$.

Fig. 1 shows the graphs of the cross-sections of the upper $w_1^{(k)}(\mathbf{x})$, $w_2^{(k)}(\mathbf{x})$ and the lower $v_1^{(k)}(\mathbf{x})$, $v_2^{(k)}(\mathbf{x})$ approximations at $x_2 = 0,5$ for $k = 0, 2, 6, 8$. Fig. 2, 3 show the surfaces of the approximate solutions $u_1^{(9)}(\mathbf{x})$, $u_2^{(9)}(\mathbf{x})$ and

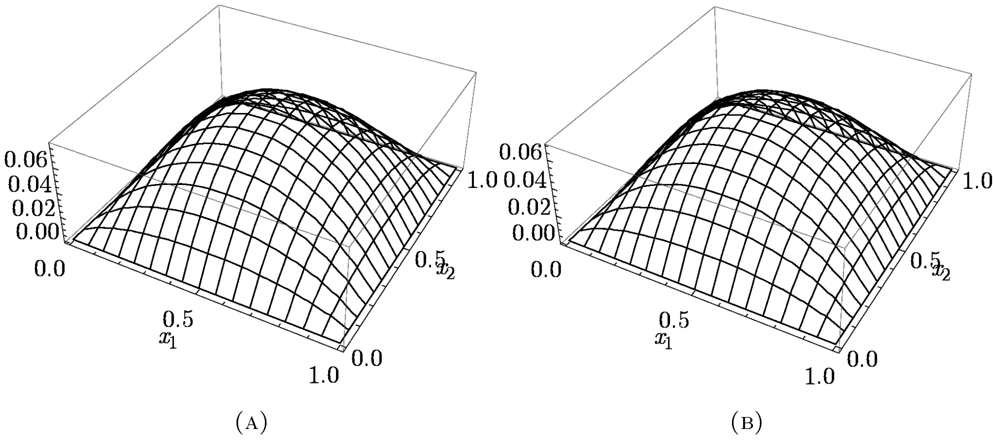

FIG. 2. Graphs of the approximate solutions $u_1^{(9)}(\mathbf{x})$ (a) and $u_2^{(9)}(\mathbf{x})$ (b)

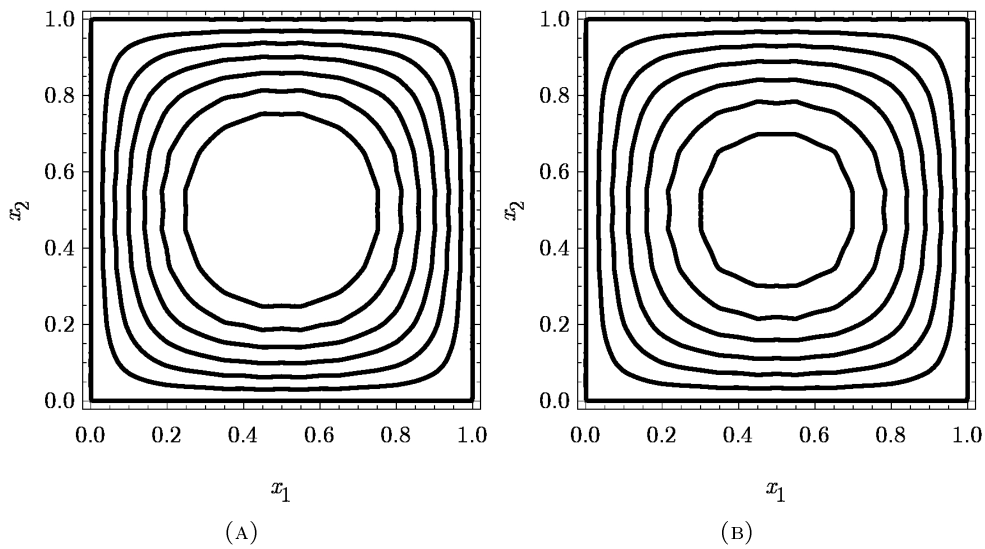

FIG. 3. Contour lines of the approximate solutions $u_1^{(9)}(\mathbf{x})$ (a) and $u_2^{(9)}(\mathbf{x})$ (b)

their contour lines (with the step $0,01$) respectively. Considering the relationship $\frac{\varepsilon_i^{(k+1)}}{\varepsilon_i^{(k)}}$, $k = 0, 1, ..., 10$, $i = 1, 2$, according to the table 1, it was received that $\frac{\varepsilon_1^{(k+1)}}{\varepsilon_1^{(k)}} \approx \frac{\varepsilon_2^{(k+1)}}{\varepsilon_2^{(k)}} \approx 0,486$, that indicates the geometric rate of convergence of the iterative sequence with the corresponding index. Let us note that the convergence exponent turned out to be less than the exponent γ estimated in accordance with Theorem 4.

5. Conclusions

In the paper a method of two-sided approximations of the solution of the homogeneous Dirichlet problem for a system of semilinear elliptic equations is proposed on the basis of the Green-Rvachev's quasi-function method. A computational experiment carried out for a system with exponential nonlinearity demonstrated the possibilities and effectiveness of the method. The proposed approach to the numerical solution of semilinear systems can be used in solving various applied problems, the mathematical model of which is the problem (1) – (3). The proposed method is more universal than the existing methods, and it allows to solve the problem in question in areas of arbitrary geometry, provided that this region can be described by the R-function method.

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