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ON THE NON-LINEAR INTEGRAL EQUATION METHOD FOR THE RECONSTRUCTION OF AN INCLUSION IN THE ELASTIC BODY

R. S. Chapko, O. M. Ivanyshyn Yaman, V. G. Vavrychuk

Резюме. Для знаходження границі об'єкту в пружній двовимірній області за відомими даними Коші на її границі застосовано метод нелінійних інтегральних рівнянь, що грунтується на пружних потенціалах. Розроблено ітераційний метод для наближеного розв'язування отриманих інтегральних рівнянь. Знайдено похідну Фреше відповідного оператора і показано розв'язність лінеаризованої системи. Повну дискретизацію здійснено методом тригонометричних квадратур. Через некоректність до отриманої системи лінійних рівнянь застосовано метод регуляризації Тіхонова. Чисельні експерименти показують, що пропонований метод дає добру точність реконструкції при економних обчислювальних затратах.

Abstract. We apply the non-linear integral equation approach based on elastic potentials for determining the shape of a bounded object in the elastostatic two-dimensional domain from given Cauchy data on its boundary. The iterative algorithm is developed for the numerical solution of obtained integral equations. We find the Fréchet derivative for the corresponding operator and show unique solviability of the linearized system. Full discretization of the system is realized by a trigonometric quadrature method. Due to the inherited ill-possedness in the system of linear equations we apply the Tikhonov regularization. The numerical results show that the proposed method gives a good accuracy of reconstructions with an economical computational cost.

1. Introduction

The idea to reduce the problem of the boundary reconstruction directly to non-linear equations and to employ a regularized iterative procedure was firstly suggested in [18]. The concept consists in the use of the reciprocity gap approach based on Green's integral theorem. This approach was successfully extended in [9, 13, 16, 18, 20] for the case of the Laplace equation and in [11, 12, 14, 15] for the Helmholtz equation. The other possible way for it is related with the Green's function [6,7,10,20]. This method is applicable for the reconstruction of an inclusion in some canonical domains for which the Green's functions are known. In this paper we would like to use the potential theory to receive a system of non-linear integral equations [5] which is equivalent to an inverse boundary problem for the Navier equation. As motivation for this research we consider the extension of the potential approach to the system of differential equations in elasticity and on the other hand the problem of the

Key words. Double connected elastostatic domain; boundary reconstruction; elastic potentials; boundary integral equations; trigonometric quadrature method; Newton method; Tikhonov regularization.

shape reconstruction in the elastic medium is of interest for the solid mechanics community.

We assume that D is a doubly connected bounded domain in $\mathbb{R}^2$ with the boundary ∂D consisting of two disjoint closed C^2 curves Γ_1 and Γ_2 such that Γ_1 is contained in the interior of Γ_2.

The corresponding direct problem is: Given a vector function g on Γ_2 consider the Dirichlet problem for a vector function $u \in C^2(D) \cap C^1(\bar{D})$ satisfying the Navier equation

$$\Delta^* u = 0 \quad \text{in } D \tag{1}$$

and the boundary conditions

$$u = 0 \quad \text{on } \Gamma_1, \tag{2}$$

$$Tu = g \quad \text{on } \Gamma_2. \tag{3}$$

Here $\Delta^* u = \mu \Delta u + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u$ and

$$Tu = \lambda \mathrm{div}\, u\, \nu + 2\mu(\nu \cdot \mathrm{grad})u + \mu \mathrm{div}(Qu)Q\nu,$$

where ν is an outward unit normal vector to the boundary and the matrix Q is given by $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Constants μ and λ ($\mu > 0, \lambda > -\mu$) are called the Lame coefficients, they characterize the physical properties of the material. Note that throughout the paper the function spaces have to be understood as vector valued.

It is well-know that the direct mixed boundary value problem has the unique solution [21, Chapter X, §10].

The inverse problem we are concerned with is: Given the Neumannn data g on Γ_2 and the Dirichlet data

$$u = f \quad \text{on } \Gamma_2, \tag{4}$$

determine the shape of the interior boundary Γ_1.

As opposed to the forward boundary value problem, the inverse problem is nonlinear and ill-posed.

The issue of uniqueness, i.e., identifiability of the unknown curve Γ_1 from the Cauchy data on Γ_2, is settled by the following theorem (see [4]).

Theorem 1. *Let Γ_1 and $\widetilde{\Gamma}_1$ be two closed curves contained in the interior of Γ_2 and denote by u and $\widetilde{u}$ the solutions to the mixed problem* (1)–(3) *for the interior boundaries Γ_1 and $\widetilde{\Gamma}_1$, respectively. Assume that $g \neq 0$ and*

$$u = \widetilde{u}$$

on an open subset of Γ_2. Then $\Gamma_1 = \widetilde{\Gamma}_1$.

2. Nonlinear integral equations and iterative schemes for its solutions

Firstly we introduce the single-layer elasticity potential. As it is well known, the fundamental solution to the Navier equation (1) is given by

$$\Phi(x, y) = \frac{c_1}{\pi} \ln \frac{1}{|x-y|} I + \frac{c_2}{\pi} J(x-y),$$

where $c_1 = \frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}$, $c_2 = \frac{\lambda+\mu}{4\mu(\lambda+2\mu)}$, I is the identity matrix and the matrix J is defined by

$$J(w) = \frac{w\, w^\top}{|w|^2}$$

in terms of a dyadic product of $w \in \mathbb{R}^2 \setminus \{0\}$ and its transpose $w^\top$. Then the single-layer potential with vector density ψ on Γ_ℓ is defined by

$$(U_\ell \psi)(x) := \int_{\Gamma_\ell} \Phi(x,y)\psi(y)\, ds(y), \quad x \in D, \quad \ell = 1, 2.$$

We search the solution of the boundary value problem (1)–(3) in the form

$$u(x) = (U_1\psi_1)(x) + (U_2\psi_2)(x), \quad x \in D. \tag{5}$$

From the boundary behavior properties of the single-layer elasticity potential [21], we obtain

$$u(x) = (S_{\ell 1}\psi_1)(x) + (S_{\ell 2}\psi_2)(x), \quad x \in \Gamma_\ell, \quad \ell = 1, 2 \tag{6}$$

and

$$(Tu)(x) = \frac{1}{2}\psi_2(x) + (D_{21}\psi_1)(x) + (D_{22}\psi_2)(x), \qquad x \in \Gamma_2. \tag{7}$$

Here, the boundary integral operators $S_{\ell k}$ and $D_{\ell k}$ are defined by

$$(S_{\ell k}\varphi)(x) = \int_{\Gamma_\ell} \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma_k\ ,$$

$$(D_{\ell k}\varphi)(x) = \int_{\Gamma_\ell} T_x\Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma_k\ .$$

Taking into account the boundary conditions (2) and (3) we receive from (6) a system of integral equations

$$\begin{cases} S_{11}\psi_1 + S_{12}\psi_2 = 0 & \text{on} \quad \Gamma_1, \\ \frac{1}{2}\psi_2 + D_{21}\psi_1 + D_{22}\psi_2 = g & \text{on} \quad \Gamma_2 \end{cases} \tag{8}$$

and the condition (4) leads to the integral equation

$$S_{21}\psi_1 + S_{22}\psi_2 = f \quad \text{on} \quad \Gamma_2. \tag{9}$$

Theorem 2. *The inverse boundary value problem* (1)–(4) *is equivalent to the system of integral equations* (8)–(9).

We will call the equations (8) as the "field" equations and the equation (9) as the "data" equation.

In general, there exist three different iterative methods to solve the system (8)–(9) by linearization:

A. Given initial guess for the boundary Γ_1 and the densities ψ_1 and ψ_2, we linearize all three equations in order to update all the unknowns.
B. Given initial guess for the boundary Γ_1, we solve the subsystem (8) to obtain the densities. Then, keeping the densities fixed we solve the linearized "data" equation (9) to obtain the update for the boundary.

C. Given initial guess for the densities, we solve the linearized "field" equations (8) to obtain Γ_1 and then we solve the linearized "data" equation (9) to obtain the new densities.

The linearization, using Fréchet derivatives of the operators, and the regularization of the ill-posed equations are needed in all methods. However, the iterative method A requires the calculation of the Fréchet derivatives of the operators with respect to all the unknowns and the selection of three regularization parameters at every step. Thus, we prefer to use one of the so-called two-step methods B or C. Between the two methods, the method B is preferable since we solve first a well-posed linear system and then we linearize the "data" equation.

3. Implementation of the two-step method B

3.1. Numerical solution of the "field" integral equations. Assume that boundary curves Γ_1 and Γ_2 have parametric representation

$$\Gamma_\ell = \{x_\ell(t) = (x_{\ell 1}(t), x_{\ell 2}(t))| \quad t \in [0, 2\pi]\}, \quad \ell = 1, 2,$$

where $x_{\ell 1}$, $x_{\ell 2}$ are 2π–periodic and twice continuously differentiable functions.

It gives us the following parametric form for the operator $S_{\ell k}$

$$(S_{\ell k}\psi_k)(x_\ell(t)) = \frac{1}{\pi}\int_0^{2\pi} K_{\ell k}(t,\tau)\psi_k(\tau)d\tau, \quad \ell, k = 1, 2,$$

where $K_{\ell k}(t,\tau) = \pi\Phi(x_\ell(t), x_k(\tau))$ and $\psi_k(t) = \psi(x_k(t))|x_k'(t)|$. Elementary calculations yield the representation of the matrix $K_{\ell\ell}$

$$K_{\ell\ell}(t,\tau) = -\frac{c_1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)I + \tilde{K}_{\ell\ell}(t,\tau), \quad t \neq \tau,$$

where

$$\tilde{K}_{\ell\ell}(t,\tau) = K_{\ell\ell}(t,\tau) + \frac{c_1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)I, \quad t \neq \tau$$

with the diagonal term

$$K_{\ell\ell}(t,t) = \frac{c_1}{2}\ln\left(\frac{1}{e|x_\ell(t)|^2}\right)I + c_2\frac{x_\ell'(t)\cdot x_\ell'(t)^\top}{|x_\ell'(t)|^2}.$$

Parametrization of integral operators $D_{\ell k}$ reads as following

$$(D_{\ell k}\psi_k)(x_\ell(t)) = \frac{1}{\pi}\int_0^{2\pi} L_{\ell k}(t,\tau)\psi_k(\tau)d\tau$$

with the matrices

$$L_{\ell k}(t,\tau) = c_3\frac{(x_\ell(t) - x_k(\tau))\cdot x_\ell'(t)}{|x_\ell'(t)||x_\ell(t) - x_k(\tau)|^2}Q -$$

$$-\frac{(x_\ell(t) - x_k(\tau))\cdot Qx_\ell'(t)}{|x_\ell'(t)||x_\ell(t) - x_k(\tau)|^2}\left\{c_3 I + c_4 J(x_\ell(t) - x_k(\tau))\right\}.$$

Here $c_3 = \dfrac{\mu}{2(\lambda+2\mu)}$ and $c_4 = \dfrac{\lambda+\mu}{\lambda+2\mu}$. The kernels $L_{\ell\ell}$ contain the singularity. The straightforward calculations lead to the following expression

$$L_{\ell\ell}(t,\tau) = \frac{c_3}{2|x'_\ell(t)|} \cot\frac{t-\tau}{2} Q + \tilde{L}_{\ell\ell}(t,\tau),$$

where

$$\tilde{L}_{\ell\ell}(t,\tau) = L_{\ell\ell}(t,\tau) - \frac{c_3}{2|x'_\ell(t)|}\cot\frac{t-\tau}{2}Q$$

with the diagonal term

$$\tilde{L}_{\ell\ell}(t,t) = \frac{c_3 x''_\ell(t)\cdot x'_\ell(t)}{2|x'_\ell(t)|^{3/2}}Q + \frac{x''_\ell(t)\cdot Qx'_\ell(t)}{2|x'_\ell(t)|^{3/2}}\left[c_3 I + c_4\frac{x'_\ell(t)\cdot x'_\ell(t)^\top}{|x'_\ell(t)|^2}\right].$$

Thus we obtain a system of parametrized integral equations

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\left\{\left[-\frac{c_1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)I + \tilde{K}_{11}(t,\tau)\right]\psi_1(\tau) + \right. \\ \qquad\qquad \left. + K_{12}(t,\tau)\psi_2(\tau)\right\}d\tau = 0, \\ \dfrac{\psi_2(t)}{2|x'_2(t)|} + \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\left\{L_{21}(t,\tau)\psi_1(\tau) + \right. \\ \qquad\qquad \left. + \left[\dfrac{c_3}{2|x'_2(t)|}\cot\dfrac{t-\tau}{2}Q + \tilde{L}_{11}(t,\tau)\right]\psi_2(\tau)\right\}d\tau = g(t). \end{cases} \tag{10}$$

For the numerical solution of integral equations (10) we combine a quadrature method and a collocation method based on trigonometric interpolation [3, 17]. For this we choose an equidistant mesh by setting $t_j = jh$, $h = \frac{\pi}{n}$, $j = 0, \ldots, 2n-1$ and use the following three quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\,d\tau \approx \frac{1}{2n}\sum_{k=0}^{2n-1} g(t_k), \tag{11}$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\ln\left(\frac{4}{e}\sin^2 t_j - \frac{\tau}{2}\right)d\tau \approx \sum_{k=0}^{2n-1} R_{|j-k|}\,g(t_k) \tag{12}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} g(\tau)\cot\tau - \frac{t_j}{2}d\tau \approx \sum_{k=0}^{2n-1} F_{j-k}\,g(t_k), \tag{13}$$

with the weights

$$R_j = -\frac{1}{2n}\left\{1 + 2\sum_{m=1}^{n-1}\frac{1}{m}\cos mjh + \frac{(-1)^j}{n}\right\}, \quad F_j = \frac{1}{n}\sum_{m=1}^{n-1}\sin mjh.$$

These interpolation quadrature formulas are obtained by replacing g by its trigonometric interpolation polynomial from the $2n$-dimensional space T_n and then integrating.

Thus we use quadrature rules (11) and (12) to approximate two types of integrals in the integral equations (10) and collocate the approximate equations to obtain the linear system

$$\begin{cases} \displaystyle\sum_{k=0}^{2n-1} \left\{ \left[-c_1 R_{|j-k|} I + \frac{1}{n}\, \tilde{K}_{11}(t_j, t_k) \right] \psi_{1n}(t_k) + \right. \\ \qquad\qquad \left. + \dfrac{1}{n}\, K_{12}(t_j, t_k) \psi_{2n}(t_k) \right\} = 0, \\ \dfrac{\psi_{2n}(t_j)}{2|x_2'(t_j)|} + \displaystyle\sum_{k=0}^{2n-1} \left\{ \frac{1}{n} L_{21}(t_j, t_k) \psi_{1n}(t_k) + \right. \\ \qquad\qquad \left. + \left[\dfrac{c_3}{|x_2'(t_k)|} F_{j-k} Q + \dfrac{1}{n} \tilde{L}_{22}(t_j, t_k) \right] \psi_{2n}(t_k) \right\} = g(t_j) \end{cases} \tag{14}$$

for $j = 0, 1, \ldots, 2n-1$, which we solve for the nodal values $\psi_{\ell n}(t_k)$, $\ell = 1, 2$ of $\psi_{\ell n} \in T_n$.

The convergence and error analysis for this quadrature method can be established on the basis of the collectively compact operators theory (see [8]) or on the basis of some estimate of trigonometric interpolation in Hölder spaces (see [19]).

Theorem 3. *For $f \in C^{p+1,\beta}[0, 2\pi]$ and a sufficiently large n the system (14) has an unique solution with $\psi_{\ell n} \in T_n$ and for the exact solutions ψ_ℓ of (10) we have the error estimates*

$$\|\psi_\ell - \psi_{\ell n}\|_{m,\alpha} \le C \frac{\ln n}{n^{p-m+\beta-\alpha}} \|\psi_\ell\|_{p,\beta}, \quad \ell = 1, 2$$

for $0 \le m \le p$, $0 < \alpha \le \beta < 1$ and some constant $C > 0$ depending only on α, β, m, p.

3.2. **Numerical solution of "data" integral equation equation.** According to our algorithm we need to find the correction for Γ_1 from the "data" equation (9), where the densities ψ_ℓ, $\ell = 1, 2$ are known. For simplicity we consider only star-like interior curves, i.e., we choose a parametrization in polar coordinates of the form

$$x_1(t) = \{r(t)c(t) : t \in [0, 2\pi]\}, \tag{15}$$

where $c(t) = (\cos t, \sin t)$ and $r : \mathbb{R} \to (0, \infty)$ is a 2π periodic function representing the radial distance from the origin. Also we use the following notation $S_r\psi = S_{21}\psi$. However, we wish to emphasize that the concepts described below, in principle, are not confined to star-like boundaries only.

For the given r and ψ_ℓ, $\ell = 1, 2$ we solve the linearized ill-posed integral equation

$$(S'[r, \psi_1]q)(t) = f(t) - (S_r\psi_1)(t) - (S_{22}\psi_2)(t) \tag{16}$$

with respect to the function q. Here the Fréchet derivative of the operator S_r has the following representation

$$(S'[r,\psi]q)(t) = \frac{1}{\pi}\int_0^{2\pi} q(\tau)N_r(t,\tau)\psi(\tau)d\tau,$$

where

$$N_r(t,\tau) = -c_1\, c(\tau)\cdot\nabla_{x_1(\tau)} \ln|x_2(t)-x_1(\tau)|I+ \\ + c_2\,(c(\tau),\partial_{x_1(\tau)})J(x_2(t)-x_1(\tau)).$$

Here $(c(\tau),\partial_{x_1(\tau)})J(x_2(t)-x_1(\tau))$ is the tensor obtained by applying $(c(\tau), \partial_{x_1(\tau)})$ to each column of $J(x_2(t)-x_1(\tau))$.

Theorem 4. *The Fréchet derivative operator $S'[r,\widetilde{\psi}_1]$ is injective at the exact solution.*

Proof. Assume $S'[r,\widetilde{\psi}_1]q = 0$. We introduce a function

$$V(x) = \int_{\Gamma_1}(\zeta(y),\partial_y)\Phi(x,y)\psi_1(y)\,ds(y), \qquad x\in\mathbb{R}^2\setminus\Gamma_1,$$

where $\zeta(x_1(t)) = q(t)c(t)$, $t\in[0,2\pi]$.

Clearly the function V satisfies the Navier equation

$$\Delta^* V = 0 \quad \text{in } \mathbb{R}^2\setminus\Gamma_1$$

and by the assumption

$$V^+|_{\Gamma_1} = 0.$$

It is known, [13], that for sufficiently small q, the perturbed interior curve as given in polar coordinates by

$$\Gamma_{1,r+q} = \{(r(t)+q(t))c(t) : t\in[0,2\pi]\}$$

can be represented in terms of the outward unit normal vector ν to $\Gamma_{1,r}$ as follows

$$\Gamma_{1,r+q} = \{r(t)c(t) + \widetilde{q}(t)\nu(t) : t\in[0,2\pi]\}.$$

Hence, the function V can be rewritten in the form

$$V(x) = \int_0^{2\pi}(\nu(\tau),\partial_{x_1(\tau)})\Phi(x,x_1(\tau))\,\widetilde{q}(\tau)\widetilde{\psi}_1(\tau)\,|x_1'(\tau)|\,d\tau, \qquad x\in\mathbb{R}^2\setminus\Gamma_1.$$

Recalling

$$\Phi(x,y) = \frac{c_1}{\pi}\,\ln\frac{1}{|x-y|}\,I + \frac{c_2}{\pi}\,\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\,\vec{e}_i\otimes\vec{e}_j,$$

and having introduced ε_{ij} the two-dimensional Ricci tensor

$$\tau_i = \varepsilon_{ji}\nu_j, \quad (\varepsilon_{ij}) = Q, \quad \nu = -Q\tau,$$

we rewrite the $(\nu(y),\partial_y)\Phi(x,y)$ in terms of the tangential derivative as follows

$$(\nu(y),\partial_y)\Phi(x,y) = \frac{c_1}{\pi}\,\frac{\partial}{\partial\nu(y)}\ln\frac{1}{|x-y|}\,I- \\ -\frac{c_2}{\pi}\,\varepsilon_{ik}\frac{\partial}{\partial\tau(y)}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\,\vec{e}_k\otimes\vec{e}_j$$

By [2, Theorem 4.5] we obtain that the function V can be continuously extended to the boundary Γ_1, i.e.,

$$V(x_1(t))^{\pm} = \mp c_1 \widetilde{\psi}_1(t)\widetilde{q}(t) + \\ + \int_0^{2\pi} (\nu(\tau), \partial_{x_1(\tau)})\Phi(x_1(t), x_1(\tau))\, \widetilde{q}(\tau)\widetilde{\psi}_1(\tau)\, |x_1'(\tau)|\, d\tau.$$

The function V behaves as $o(1)$ at infinity. By the uniqueness of the exterior and interior Dirichlet problem [21, p.55] we have

$$c_1\widetilde{\psi}_1(t)\widetilde{q}(t) = 0, \quad t \in [0, 2\pi].$$

The function u given by (5) solves the Dirichlet problem in the interior of Γ_1. By uniqueness of the solution to the Dirichlet problem for the Navier equation u has to vanish in the interior of Γ_1 and hence $Tu^- = 0$ on Γ_1.

The jump relations imply $Tu^+ = \psi_1$. Employing Holmgren's uniqueness theorem similar to the case for the Helmholtz equation [1, Theorem 2.3.] one can show that the Cauchy data (u^+, Tu^+) cannot be identically zero on an open subset and hence $\widetilde{\psi}_1$ cannot vanish on an open subset of $[0, 2\pi]$. □

For the numerical solution of (16) we apply tha collocation method with the approximation of q in the form

$$q_m = \sum_{i=0}^{2m} q_{mi} l_i, \quad m \in \mathbb{N}, n > m,$$

where $l_i(t) = \cos it$ for $i = 0, \dots, m$ and $l_i(t) = \sin(m-i)t$ for $i = m+1, \dots 2m$. Then the following linear system needs to be solved

$$\sum_{j=0}^{2m} q_{mj} A_{ij} = b_i, \quad i = 0, \dots, 2n-1 \tag{17}$$

with

$$A_{ij} = \frac{1}{n} \sum_{k=0}^{2n-1} l_j(t_k) N_r(t_i, t_k)\psi_{1n}(t_k)$$

and

$$b_i = f(t_i) - \sum_{k=0}^{2n-1} \left\{ \frac{1}{n} K_{21}(t_i, t_k)\psi_{1n}(t_k) + \right. \\ \left. + \left[-c_1 R_{|i-k|} I + \frac{1}{n}\, K_{22}(t_i, t_k) \right] \psi_{2n}(t_k) \right\}.$$

Due to ill-possedness of (17) and its over-determination we apply the least-squares method and the Tikhonov regularization with the regularization parameter $\alpha > 0$.

3.3. **Algorithm for the two-step method B.** Now we summarize the algorithm.

1. Choose some starting value r.
2. Solve the system of well-posed integral equations (8) (see subsec. 3.1).
3. For the given r, ψ_1 and ψ_2 solve the linearized ill-posed integral equation (9) with respect to function q (see subsec. 3.2).
4. Calculate an approximation for the radial function $r = r + \beta q$, where β is a relaxation parameter for the Newton method.
5. Repeat steps 2-4 until a stopping criterion is satisfied.

4. Numerical examples

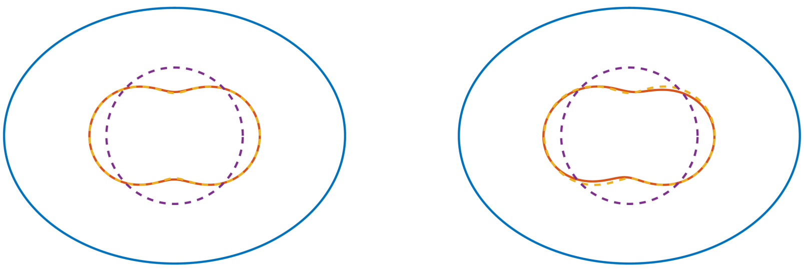
The Cauchy data on Γ_2 were generated by solving the direct problem (1)-(3) for $g = (1,1)^\top$ on Γ_2 and calculating $f = (f_1, f_2)^\top$ as the restriction of the solution on Γ_2. Note that when generating the "exact" Cauchy data we used a finer mesh in order to avoid the "inverse crime". The noisy data were formed as

$$f_\ell^\delta = f_\ell + \delta(2\eta - 1)\|f_\ell\|_{L_2(\Gamma_2)}, \quad \ell = 1,2$$

with the noise level δ and the uniformly distributed random variable η in $(0,1)$. The stopping rule was chosen as

$$\frac{\|q\|_{L_2(\Gamma_1)}}{\|r\|_{L_2(\Gamma_1)}} < \epsilon.$$

We demonstrate the feasibility of the proposed methods for the inverse problem (1)-(3) with $\mu = \lambda = 1$ and with following boundaries



a). Reconstruction for exact data after 21 iterations ($\alpha = 1E-10$)

b). Reconstruction for 5% nosy in the data after 16 iterations ($\alpha = 1E-2$)

Fig. 1. Reconstruction of the boundary Γ_1 for Ex. 1

Example 1: The exterior boundary curve Γ_2 is a elipse $\Gamma_2 = \{x_2(t) = (2\cos t, 1.5\sin t), t \in [0, 2\pi]\}$ and the interior boundary curve Γ_1 (to be reconstructed) is peanut-shaped with radial function

$$r(t) = \sqrt{\cos^2 t + 0.25\sin^2 t}.$$

Example 2: The exterior boundary curve Γ_2 is a rounded rectangle with radial function

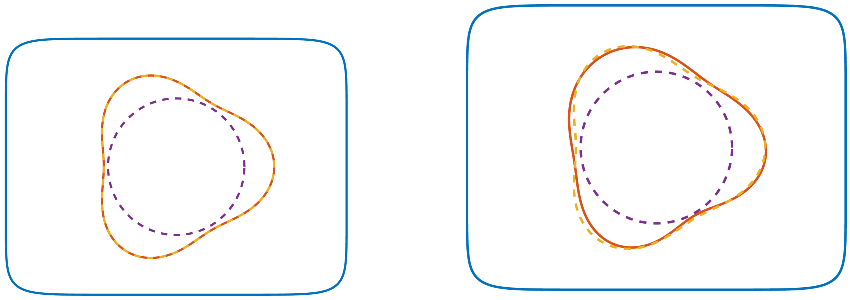
$$r_2(t) = ((1/2\cos t)^{10} + (2/3\sin t)^{10})^{-0.1}$$

and Γ_1 is a boundary with radial function

$$r_1(t) = 1 + 0.15\cos 3t.$$

The results of the numerical experiments for exact and noisy data with $\delta = 5\%$ are reflected on Fig. 1 and Fig. 2. Here we used the following discretization parameters $n = 32$, $m = 4$, $\epsilon = 0.0001$ and $\beta = 0.2$.

Thus, as we see from this preliminary study the non-linear integral equation approach provides accurate reconstruction for exact and noisy data.



a). Reconstruction for exact data after 21 iterations ($\alpha = 1E-10$)

b). Reconstruction for 5% nosy in the data after 20 iterations ($\alpha = 1E-2$)

FIG. 2. Reconstruction of the boundary Γ_1 for Ex. 2

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Roman Chapko, Vasyl Vavrychuk,
Faculty of Applied Mathematics and Informatics,
Ivan Franko National University of Lviv,
1, Universytets'ka Str., Lviv, 79000, Ukraine;

Olha Ivanyshyn Yaman,
Department of Mathematics,
Izmir Institute of Technology,
Gulbahce, Urla, Izmir, 35430, Turkey.

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