Журнал обчислювальної та прикладної математики | 2019 № 1 (130) | Journal of Numerical & Applied Mathematics

UDC 519.6

# THE SYSTEM OF POTAPOV'S FUNDAMENTAL MATRIX INEQUALITIES ASSOCIATED WITH A MATRICIAL STIELTJES TYPE POWER MOMENT PROBLEM

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Резюме. В статті показано, що множина розв'язків матричної проблеми силових моментів типу Стільєса співпадає з множиною розв'язків системи фундаментальної матриці нерівностей Потапова.

Abstract. The paper shows that the solution set of a matricial Stieltjes-type truncated power moment problem coincides with the solution set of the corresponding system of Potapov's fundamental matrix inequalities.

## 1. Introduction and preliminaries

The starting point of studying power moment problems on semi-infinite intervals was the famous two part memoir of T. J. Stieltjes [52, 53]. A complete theory of the treatment of power moment problems on semi-infinite intervals in the scalar case was developed by M. G. Krein in collaboration with A. A. Nudelman (see [45, Section 10], [46], [47, Chapter V]). What concerns an operator-theoretic treatment of the power moment problems named after Hamburger and Stieltjes and its interrelations, we refer the reader to Simon [51].

In the 1970's, V. P. Potapov developed a special approach to discuss matrix versions of classical interpolation and moment problems. The main idea of his method is based on transforming such problems into equivalent matrix inequalities with respect to the Löwner semi-ordering. Using this strategy, several matricial interpolation and moment problems could successfully be handled (see, e.g. [6, 7, 13–16, 18, 20–22, 32, 33, 37–44, 48, 54]). L. A. Sakhnovich enriched Potapov's method by unifying the particular instances of Potapov's procedure under the framework of one type of operator identities [9, 35, 50]. Matrix versions of the classical Stieltjes moment problem were studied by Adamyan/Tkachenko [1, 2], Andô [4], Bolotnikov [5, 6, 8], Bolotnikov/Sakhnovich [9], Chen/Hu [11], Chen/Li [12], Dyukarev [17, 18], Dyukarev/Katsnelson [21, 22], and Hu/Chen [34]. The considerations of this paper deal with the more general case of an arbitrary semi-infinite interval $[\alpha, \infty)$, where $\alpha$ is an arbitrarily given real number. This problem has already been treated by other methods in [27, 28].

In order to formulate the concrete moment problem, we are going to study, we first review some notation. Throughout this paper, let $p$ and $q$ be positive

*Key words.* Stieltjes moment problem; Potapov's fundamental matrix inequalities; Herglotz–Nevanlinna functions; Stieltjes functions.

integers. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. For every choice of $\upsilon, \omega \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\upsilon,\omega}$ be the set of all integers $k$ for which $\upsilon \leq k \leq \omega$ holds. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p\times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is short for $\mathcal{X}^{p\times 1}$. If $(\Omega, \mathfrak{A})$ is a measurable space, then each countably additive mapping whose domain is $\mathfrak{A}$ and whose values belong to the set $\mathbb{C}_{\geq}^{q\times q}$ of all non-negative Hermitian complex $q \times q$ matrices is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. By $\mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ we denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$. For the integration theory for non-negative Hermitian measures, we refer to [36, 49]. If $\mu = [\mu_{jk}]_{j,k=0}^{q}$ is a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$ and if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then we use $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$ to denote the set of all Borel-measurable functions $f\colon \Omega \to \mathbb{K}$ for which the integral exists, i. e., that $\int_\Omega |f| \mathrm{d}\tilde{\mu}_{jk} < \infty$ for every choice of $j$ and $k$ in $\mathbb{Z}_{1,q}$, where $\tilde{\mu}_{jk}$ is the variation of the complex measure $\mu_{jk}$. If $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$, then let $\int_A f \mathrm{d}\mu := [\int_\Omega 1_A f \mathrm{d}\mu_{jk}]_{j,k=1}^{q}$ for all $A \in \mathfrak{A}$ and we will also write $\int_A f(\omega)\mu(\mathrm{d}\omega)$ for this integral.

Let $\mathfrak{B}_\mathbb{R}$ (resp. $\mathfrak{B}_\mathbb{C}$) be the $\sigma$-algebra of all Borel subsets of $\mathbb{R}$ (resp. $\mathbb{C}$). For all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the $\sigma$-algebra of all Borel subsets of $\Omega$, let $\mathcal{M}_{\geq}^{q}(\Omega) := \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{B}_\Omega)$ and, for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}_{\geq,\kappa}^{q}(\Omega)$ be the set of all $\sigma \in \mathcal{M}_{\geq}^{q}(\Omega)$ such that for all $j \in \mathbb{Z}_{0,\kappa}$ the function $f_j\colon \Omega \to \mathbb{C}$ defined by $f_j(t) := t^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and if $\sigma \in \mathcal{M}_{\geq,\kappa}^{q}(\Omega)$, then we set

$$s_j^{[\sigma]} := \int_\Omega t^j \sigma(\mathrm{d}t) \qquad \text{for each } j \in \mathbb{Z}_{0,\kappa}. \tag{1}$$

The following matricial power moment problem lies in the background of our considerations:

**Problem** $\mathsf{MP}[\Omega; (s_j)_{j=0}^{m}, \leq]$**:** Let $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_{\geq}^{q}[\Omega; (s_j)_{j=0}^{m}, \leq]$ of all $\sigma \in \mathcal{M}_{\geq,m}^{q}(\Omega)$ for which the matrix $s_m - s_m^{[\sigma]}$ is non-negative Hermitian and for which, in the case $m > 0$, moreover $s_j^{[\sigma]} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,m-1}$.

The considerations of this paper are mostly concentrated on the case that the set $\Omega$ is a one-sided bounded and closed infinite interval of the real axis. Such moment problems are called to be of Stieltjes type. We are going to follow Potapov's strategy to solve the moment problem $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$, where $\alpha$ is an arbitrarily given real number. After the reformulation of the moment problem in the language of the members of a class of distinguished matrix-valued functions, a first step consists of finding a convenient system of matrix inequalities such that the solution set of the moment problem coincides with the solution set of the system of matrix inequalities. In a second step, one proves a parametrization of the solution set of the system of matrix inequalities, where the case that $m$ is an even integer and the case that $m$ is an odd integer are treated seperately. This paper is aimed at doing the fist step. We are going to construct the system of matrix inequalities in question. It will turn out that the solution set of the moment problem (obtained via Stieltjes transformation) coincides with the solution set of a certain system of Potapov's fundamental matrix inequalities. Further considerations to solve these inequalities will be stated in a subsequent paper.

In Section 2, we recall necessary and sufficient conditions of solvability of the moment problems in question. In Section 3, we give a reformulation of the moment problem, using certain matrix-valued functions. Section 4 is aimed at showing that every solution of the moment problem fulfills necessarily the corresponding system of Potapov's fundamental matrix inequalities. Some integral estimates for the scalar case are given in Section 5. In Section 6, we will prove that each solution of the system of Potapov's fundamental matrix inequalities is a solution of the moment problem as well.

At the end of this section, let us now introduce some further notations, which are useful for our considerations. We will write $I_q$ for the identity matrix in $\mathbb{C}^{q\times q}$, whereas $0_{p\times q}$ is the null matrix belonging to $\mathbb{C}^{p\times q}$. If the size of the identity matrix or the null matrix is obvious, then we will also omit the indexes. The notations $\mathbb{C}^{q\times q}_{\mathrm{H}}$ and $\mathbb{C}^{q\times q}_{\geq}$ stand for the set of all Hermitian complex $q\times q$ matrices and the set of all non-negative Hermitian complex matrices, respectively. If $A$ and $B$ are complex $q\times q$ matrices, then we will write $A\leq B$ or $B\geq A$ to indicate that $A$ and $B$ are Hermitian matrices such that the matrix $B-A$ is non-negative Hermitian. For each $A\in\mathbb{C}^{p\times q}$, let $\mathcal{N}(A)$ be the null space of $A$ and let $\mathcal{R}(A)$ be the column space of $A$. For each $A\in\mathbb{C}^{q\times q}$, we will use $\Re A$ and $\Im A$ to denote the real part of $A$ and the imaginary part of $A$, respectively: $\Re A := \frac{1}{2}(A+A^*)$ and $\Im A := \frac{1}{2\mathrm{i}}(A-A^*)$. Furthermore, for each $A\in\mathbb{C}^{p\times q}$, let $\|A\|_{\mathrm{F}}$ be the Frobenius norm of $A$ and let $\|A\|_{\mathrm{S}}$ be the operator norm of $A$. For each $x\in\mathbb{C}^q$, we write $\|x\|_{\mathrm{E}}$ for the Euclidean norm of $x$. If $n\in\mathbb{N}$, if $(p_j)_{j=1}^n$ is a sequence of positive integers, and if $x_j\in\mathbb{C}^{p_j\times q}$ for each $j\in\mathbb{Z}_{1,n}$, then let $\operatorname{col}(x_j)_{j=1}^n := \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$. If $n\in\mathbb{N}$, if $(q_k)_{k=1}^n$ is a sequence of positive integers, and if $y_k\in\mathbb{C}^{p\times q_k}$ for each $k\in\mathbb{Z}_{1,n}$, then let $\operatorname{row}(y_k)_{k=1}^n := [y_1, y_2, \dots, y_n]$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are non-empty sets with $\mathcal{Z}\subseteq\mathcal{X}$ and if $f\colon\mathcal{X}\to\mathcal{Y}$ is a mapping, then $\operatorname{Rstr}_{\mathcal{Z}} f$ stands for the restriction of $f$ onto $\mathcal{Z}$. Furthermore, let $\Pi_+ := \{z\in\mathbb{C}\colon \Im z\in(0,\infty)\}$ and let $\Pi_- := \{z\in\mathbb{C}\colon \Im z\in(-\infty,0)\}$.

## 2. On the solvability of matricial power moment problems

In this section, we recall a necessary and sufficient condition for the solvability of the Stieltjes moment problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$, where $\alpha$ is an arbitrarily given real number and where $m$ is an arbitrarily given non-negative integer. First we introduce certain sets of sequences of complex $q\times q$ matrices, which are determined by the properties of particular block Hankel matrices built of them. For each $n\in\mathbb{N}_0$, let $\mathcal{H}^{\geq}_{q,2n}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q\times q$ matrices such that the block Hankel matrix $H_n := [s_{j+k}]_{j,k=0}^n$ is non-negative Hermitian. Furthermore, let $\mathcal{H}^{\geq}_{q,\infty}$ be the set of all sequences

$(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that, for all $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$. The elements of the set $\mathcal{H}_{q,2\kappa}^{\geq}$, where $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, are called *Hankel non-negative definite* sequences. For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there are matrices $s_{2n+1} \in \mathbb{C}^{q\times q}$ and $s_{2n+2} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ to denote the set of sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there is some $s_{2n+2} \in \mathbb{C}^{q\times q}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. For all $m \in \mathbb{N}_0$, the elements of the set $\mathcal{H}_{q,m}^{\geq,\mathrm{e}}$ are called *Hankel non-negative definite extendable* sequences. For technical reasons, we set $\mathcal{H}_{q,\infty}^{\geq,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\geq}$. Observe that the solvability of the matricial Hamburger moment problems can be characterized by the introduced classes of sequences of complex $q \times q$ matrices:

**Theorem 2.1** (see, e. g. [10, Theorem 3.2] or [20, Theorem 4.16]). *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then*

$$\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \neq \emptyset$$

*if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$.*

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then let the sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ be defined by

$$s_{\alpha\triangleright j} := -\alpha s_j + s_{j+1} \qquad \text{for all } j \in \mathbb{Z}_{0,\kappa-1}. \tag{2}$$

The sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ is called the *sequence generated from $(s_j)_{j=0}^{\kappa}$ by right-sided $\alpha$-shifting.* (An analogous left-sided version is discussed in [25, Definition 2.1].) The sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ is used to define further sets of sequences of complex matrices, which are useful to discuss the Stieltjes moment problems we consider. Let $\mathcal{K}_{q,0,\alpha}^{\geq} := \mathcal{H}_{q,0}^{\geq}$. For every choice of $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^{\geq} := \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \colon (s_{\alpha\triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}\}$. For all $m \in \mathbb{N}_0$, by $\mathfrak{S}_m(\mathbb{C}^{q\times q})$ we denote the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices. Then we set $\mathcal{K}_{q,2n+1,\alpha}^{\geq} := \{(s_j)_{j=0}^{2n+1} \in \mathfrak{S}_{2n+1}(\mathbb{C}^{q\times q}) \colon \{(s_j)_{j=0}^{2n}, (s_{\alpha\triangleright j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\geq}\}$. For all $m \in \mathbb{N}_0$, let $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix $s_{m+1}$ such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq}$. We have $\mathcal{K}_{q,2n,\alpha}^{\geq,\mathrm{e}} = \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \colon (s_{\alpha\triangleright j})_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}\}$ for all $n \in \mathbb{N}$ and $\mathcal{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}} = \{(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}} \colon (s_{\alpha\triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}\}$ for all $n \in \mathbb{N}_0$. Obviously, $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}} \subseteq \mathcal{K}_{q,m,\alpha}^{\geq}$. Furthermore, if $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$), then we easily see that $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq}$ (resp. $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq,\mathrm{e}}$) holds true for all $\ell \in \mathbb{Z}_{0,m}$. Thus, for all $\alpha \in \mathbb{R}$, let $\mathcal{K}_{q,\infty,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{m}$ belongs to $\mathcal{K}_{q,m,\alpha}^{\geq}$ for all $m \in \mathbb{N}_0$, and let $\mathcal{K}_{q,\infty,\alpha}^{\geq,\mathrm{e}} := \mathcal{K}_{q,\infty,\alpha}^{\geq}$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we call a sequence $(s_j)_{j=0}^{\kappa}$ $[\alpha,\infty)$-*Stieltjes right-sided non-negative definite* (resp. $[\alpha,\infty)$-*Stieltjes* *right-sided non-negative definite expendable*) if it belongs to $\mathcal{K}^{\geq}_{q,\kappa,\alpha}$ (resp. to $\mathcal{K}^{\geq,\mathrm{e}}_{q,\kappa,\alpha}$). Note that left versions of these notions are used in [25, Definition 1.3].

Using the introduced sets of sequences of complex $q \times q$ matrices, we are able to recall solvability criterions of the problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$:

**Theorem 2.2** ( [19, Theorem 1.4]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}^q_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$.*

For the description of the solution set $\mathcal{M}^q_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$ of Problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$, it is essential that one can suppose extendable data without loss of generality:

**Theorem 2.3** ( [19, Theorem 5.2]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}^{\geq}_{q,m,\alpha}$. Then there is a unique sequence $(\tilde{s}_j)_{j=0}^{m} \in \mathcal{K}^{\geq,\mathrm{e}}_{q,m,\alpha}$ such that the sets $\mathcal{M}^q_{\geq}[[\alpha,\infty);(\tilde{s}_j)_{j=0}^{m},\leq]$ and $\mathcal{M}^q_{\geq}[[\alpha,\infty);(s_j)_{j=0}^{m},\leq]$ coincide.*

## 3. Some classes of holomorphic matrix-valued functions

The class $\mathcal{R}_q(\Pi_+)$ of all $q \times q$ Herglotz–Nevanlinna functions in the upper half-plane $\Pi_+$ consists of all matrix-valued functions $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ which are holomorphic in $\Pi_+$ and which satisfy $\Im[F(\Pi_+)] \subseteq \mathbb{C}^{q\times q}_{\geq}$. Detailed considerations of matrix-valued Herglotz–Nevanlinna functions can be found in [26, 31]. In particular, the functions belonging to $\mathcal{R}_q(\Pi_+)$ admit a well-known integral representation:

**Theorem 3.1.** (a) *For each $F \in \mathcal{R}_q(\Pi_+)$, there exist unique matrices $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $B \in \mathbb{C}^{q\times q}_{\geq}$ and a unique non-negative Hermitian measure $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that*

$$F(z) = A + zB + \int_{\mathbb{R}} \frac{1+tz}{t-z}\nu(\mathrm{d}t) \qquad \text{for each } z \in \Pi_+. \tag{3}$$

(b) *If $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$, if $B \in \mathbb{C}^{q\times q}_{\geq}$, and if $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$, then $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (3) belongs to $\mathcal{R}_q(\Pi_+)$.*

For each $F \in \mathcal{R}_q(\Pi_+)$, the unique triple $(A,B,\nu) \in \mathbb{C}^{q\times q}_{\mathrm{H}} \times \mathbb{C}^{q\times q}_{\geq} \times \mathcal{M}^q_{\geq}(\mathbb{R})$ for which the representation (3) holds true is called the *Nevanlinna parametrization of $F$* and we will also write $(A_F, B_F, \nu_F)$ for $(A,B,\nu)$. In particular, $\nu_F$ is said to be the *Nevanlinna measure of $F$*. If $F$ belongs to $\mathcal{R}_1(\Pi_+)$, then $\mu_F\colon \mathfrak{B}_{\mathbb{R}} \to [0,\infty]$ defined by

$$\mu_F(B) := \int_B (1+t^2)\nu_F(\mathrm{d}t) \qquad \text{for all } B \in \mathfrak{B}_{\mathbb{R}} \tag{4}$$

is a measure, which is called the *spectral measure of $F$*. By $\mathcal{R}'_q(\Pi_+)$ we denote the set of all $F \in \mathcal{R}_q(\Pi_+)$ for which $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(t) := 1+t^2$ belongs to $\mathcal{L}^1(\mathbb{R},\mathfrak{B}_{\mathbb{R}},\nu_F;\mathbb{R})$. Obviously, $\mathcal{R}'_q(\Pi_+) = \{F \in \mathcal{R}_q(\Pi_+)\colon \nu_F \in \mathcal{M}^q_{\geq,2}(\mathbb{R})\}$. If $F$ belongs to $\mathcal{R}'_q(\Pi_+)$, then $\mu_F\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}_{\geq}$ given by (4) is a well-defined non-negative Hermitian $q \times q$ measure belonging to $\mathcal{M}^q_{\geq}(\mathbb{R})$, which is said to be the *matricial spectral measure of* $F$. Obviously, for functions which belong to $\mathcal{R}'_1(\Pi_+)$, the notions spectral measure and matricial spectral measure coincide. For our considerations, the class $\mathcal{R}'_{0,q}(\Pi_+)$ of all $F \in \mathcal{R}_q(\Pi_+)$ for which

$$\sup_{y\in[1,\infty)} y\|F(\mathrm{i}y)\|_{\mathrm{S}} < \infty \tag{5}$$

holds true plays an essential role. The class $\mathcal{R}'_{0,q}(\Pi_+)$ is a subclass of $\mathcal{R}'_q(\Pi_+)$ (see, e. g. [26, Lemma 6.1]). The functions belonging to $\mathcal{R}'_{0,q}(\Pi_+)$ admit a particular integral representation:

**Theorem 3.2.** (a) *For each $F \in \mathcal{R}'_{0,q}(\Pi_+)$, there is a unique $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that*

$$F(z) = \int_{\mathbb{R}} \frac{1}{t-z}\mu(\mathrm{d}t) \qquad \text{for each } z \in \Pi_+, \tag{6}$$

*namely the matricial spectral measure of $F$, and*

$$\mu(\mathbb{R}) = \lim_{y\to\infty}(y\Im[F(\mathrm{i}y)]) = -\mathrm{i}\lim_{y\to\infty}[yF(\mathrm{i}y)] = \mathrm{i}\lim_{y\to\infty}[yF^*(\mathrm{i}y)].$$

(b) *If $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ is a matrix-valued function for which there exists a non-negative Hermitian measure $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that* (6) *holds true, then $F$ belongs to $\mathcal{R}'_{0,q}(\Pi_+)$.*

A proof of Theorem 3.2 is given, e. g., in [14, Theorem 8.7]. If $F \in \mathcal{R}'_{0,q}(\Pi_+)$, then the unique $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ for which (6) holds true is also called the *Stieltjes measure of* $F$. If a non-negative Hermitian $q \times q$ measure $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ is given, then $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (6) is said to be the *Stieltjes transform of* $\mu$.

**Lemma 3.3.** *Let $M \in \mathbb{C}^{q\times q}$ and let $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ be a matrix-valued function which is holomorphic in $\Pi_+$ and which satisfies the inequality*

$$\begin{bmatrix} M & F(z) \\ F^*(z) & \frac{F(z)-F^*(z)}{z-\overline{z}} \end{bmatrix} \geq 0$$

*for each $z \in \Pi_+$. Then $F$ belongs to $\mathcal{R}'_{0,q}(\Pi_+)$ and the inequality*

$$\sup_{y\in(0,\infty)} y\|F(\mathrm{i}y)\|_{\mathrm{S}} \leq \|M\|_{\mathrm{S}}$$

*holds true. Furthermore, the Stieltjes measure $\mu$ of $F$ fulfills $\mu(\mathbb{R}) \leq M$.*

A proof of Lemma 3.3 is given, e. g., in [14, Lemma 8.9].

In view of the Stieltjes moment problem, a further class of matrix-valued functions plays a key role: For each $\alpha \in \mathbb{R}$, let $\mathcal{S}_{q;[\alpha,\infty)}$ be the set of all matrix-valued functions $S\colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ which are holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$ and which satisfy $\Im[S(\Pi_+)] \subseteq \mathbb{C}^{q\times q}_{\geq}$ as well as $S((-\infty,\alpha)) \subseteq \mathbb{C}^{q\times q}_{\geq}$. In [29, Theorems 3.1 and 3.6, Proposition 2.16], integral representations of functions belonging to $\mathcal{S}_{q;[\alpha,\infty)}$ are proved. Furthermore, several characterizations of the class $\mathcal{S}_{q;[\alpha,\infty)}$ are given in [29, Section 4]. For each $\alpha \in \mathbb{R}$, let $\mathcal{S}_{0,q;[\alpha,\infty)}$ be the class of all $F \in \mathcal{S}_{q;[\alpha,\infty)}$ which satisfy (5). The functions belonging to $\mathcal{S}_{0,q;[\alpha,\infty)}$ admit a particular integral representation. Before we state this, let us note the following:

**Remark 3.4.** *For every choice of $\alpha \in \mathbb{R}$ and $z \in \mathbb{C} \setminus [\alpha, \infty)$, the function $b_{\alpha,z}\colon [\alpha, \infty) \to \mathbb{C}$ given by $b_{\alpha,z}(t) := 1/(t-z)$ is a bounded and continuous function which, in particular, belongs to $\mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ for all $\sigma \in \mathcal{M}^q_{\geq}([\alpha,\infty))$.*

**Theorem 3.5** ( [29, Theorem 5.1])**.** *Let $\alpha \in \mathbb{R}$.*

(a) *If $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$, then there is a unique $\sigma \in \mathcal{M}^q_{\geq}([\alpha,\infty))$ such that*

$$S(z) = \int_{[\alpha,\infty)} \frac{1}{t-z} \sigma(\mathrm{d}t) \qquad \text{for each } z \in \mathbb{C} \setminus [\alpha,\infty). \tag{7}$$

(b) *If $\sigma \in \mathcal{M}^q_{\geq}([\alpha,\infty))$ is such that $S\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ can be represented via (7), then $S$ belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$.*

If $F \in \mathcal{S}_{0,q;[\alpha,\infty)}$ is given, then the unique $\sigma \in \mathcal{M}^q_{\geq}([\alpha,\infty))$ which fulfills the representation (7) of $F$ is called the *$[\alpha,\infty)$-Stieltjes measure of $F$*. If $\sigma \in \mathcal{M}^q_{\geq}([\alpha,\infty))$ is given, then $F\colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ defined by (7) is said to be the *$[\alpha,\infty)$-Stieltjes transform of $\sigma$*. In view of Theorem 3.5, the moment problem $\mathsf{MP}[[\alpha,\infty); (s_j)_{j=0}^m, \leq]$ admits a reformulation in the language of $[\alpha,\infty)$-Stieltjes transforms:

**Problem $\mathsf{S}[[\alpha,\infty); (s_j)_{j=0}^m, \leq]$:** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^m, \leq]$ of all $F \in \mathcal{S}_{0,q;[\alpha,\infty)}$ the $[\alpha,\infty)$-Stieltjes measure of which belongs to

$$\mathcal{M}^q_{\geq}[[\alpha,\infty); (s_j)_{j=0}^m, \leq].$$

**Remark 3.6.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{0,q;[\alpha,\infty)}$. Then $F_\square := \operatorname{Rstr}_{\Pi_+} F$ belongs to $\mathcal{R}'_{0,q}(\Pi_+)$, the matricial spectral measure $\mu_\square$ of $F_\square$ fulfills $\mu_\square((-\infty,\alpha)) = 0$, and $\sigma := \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \mu_\square$ is exactly the $[\alpha,\infty)$-Stieltjes measure of $F$ (see [29, Proposition 2.16]).*

## 4. From the Stieltjes moment problem to the system of Potapov's fundamental inequalities

In this section, we introduce the system of Potapov's fundamental matrices corresponding to the matricial Stieltjes moment problem $\mathsf{MP}[[\alpha,\infty); (s_j)_{j=0}^m, \leq]$. We will see that each solution of this moment problem fulfills necessarily the system of Potapov's fundamental matrix inequalities. First it seems to be useful to introduce further notations and, in particular, several block Hankel matrices which will play a key role in our considerations. For technical reason, let $s_{-1} := 0_{p\times q}$.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $H_n := [s_{j+k}]_{j,k=0}^n$, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let $K_n := [s_{j+k+1}]_{j,k=0}^n$, and, for each $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$, let $G_n := [s_{j+k+2}]_{j,k=0}^n$. If $m$ and $n$ are integers such that $-1 \leq m \leq n \leq \kappa$, then we set $y_{m,n} := \operatorname{col}(s_j)_{j=m}^n$ and $z_{m,n} := \operatorname{row}(s_k)_{k=m}^n$. Let $u_0 := 0_{p\times q}$, $\mathfrak{u}_0 := 0_{p\times q}$, $w_0 := 0_{p\times q}$, and $\mathfrak{w}_0 := 0_{p\times q}$. For all $n \in \mathbb{N}$ with $n \leq \kappa+1$, let $u_n := -y_{-1,n-1}$, and $w_n := z_{-1,n-1}$. Further, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $\mathfrak{u}_n := \left[\begin{smallmatrix} -y_{n+1,2n} \\ 0_{p\times q} \end{smallmatrix}\right]$

and $\mathfrak{w}_n := [z_{n+1,2n}, 0_{p\times q}]$. If a real number $\alpha$ is additionally given, then we continue to use the notation given by (2), and we set $H_{\alpha\triangleright n} := [s_{\alpha\triangleright j+k}]_{j,k=0}^{n}$ for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.

For each $n \in \mathbb{N}_0$, we set

$$T_{q,n} := [\delta_{j,k+1} I_q]_{j,k=0}^{n}, \quad v_{q,n} := \operatorname{col}(\delta_{j,0} I_q)_{j=0}^{n}, \quad \text{and} \quad \mathfrak{v}_{q,n} := \operatorname{col}(\delta_{n-j,0} I_q)_{j=0}^{n},$$

where $\delta_{j,k}$ is the Kronecker delta: $\delta_{j,k} := 1$ if $j = k$ and $\delta_{j,k} := 0$ if $j \neq k$. Obviously, $T_{q,n}^* = [\delta_{j+1,k} I_q]_{j,k=0}^{n}$ for each $n \in \mathbb{N}_0$.

It seems to be useful to recall well-known Lyapunov identities for block Hankel matrices. (These equations can be also easily proved by straightforward calculation.)

**Remark 4.1.** *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices.*

(a) *For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then $H_n T_{q,n}^* - T_{p,n} H_n = u_n v_{q,n}^* - v_{p,n} w_n$ and $H_n T_{q,n} - T_{p,n}^* H_n = \mathfrak{u}_n \mathfrak{v}_{q,n}^* - \mathfrak{v}_{p,n} \mathfrak{w}_n$. In particular, if $p = q$ and if $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$, then $H_n T_{q,n}^* - T_{q,n} H_n = u_n v_{q,n}^* - v_{q,n} u_n^*$ and $H_n T_{q,n} - T_{q,n}^* H_n = \mathfrak{u}_n \mathfrak{v}_{q,n}^* - \mathfrak{v}_{q,n} \mathfrak{u}_n^*$ for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$.*

(b) *For each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, we have $H_{\alpha\triangleright n} = -\alpha H_n + K_n$, $v_{p,n} v_{p,n}^* H_n = \left[R_{T_{p,n}}(\alpha)\right]^{-1} H_n - T_{p,n} H_{\alpha\triangleright n}$, and, in the case that $p = q$ and $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$ hold true, moreover $H_{\alpha\triangleright n} T_{q,n}^* - T_{q,n} H_{\alpha\triangleright n} = (-\alpha u_n - y_{0,n}) v_{q,n}^* - v_{q,n}(-\alpha u_n - y_{0,n})^*$ for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.*

**Remark 4.2.** *For each $n \in \mathbb{N}_0$, the matrix-valued functions $R_{T_{q,n}} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ and $R_{T_{q,n}^*} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by $R_{T_{q,n}}(z) := (I_{(n+1)q} - zT_{q,n})^{-1}$ and $R_{T_{q,n}^*}(z) := (I_{(n+1)q} - zT_{q,n}^*)^{-1}$ are well-defined matrix polynomials of degree $n$, which can be represented, for each $z \in \mathbb{C}$, via $R_{T_{q,n}}(z) = \sum_{j=0}^{n} z^j T_{q,n}^j$ and $R_{T_{q,n}^*}(z) = \sum_{j=0}^{n} z^j (T_{q,n}^*)^j$, respectively. In particular, $R_{T_{q,n}^*}(z) = [R_{T_{q,n}}(\overline{z})]^*$ for all $z \in \mathbb{C}$.*

For each $n \in \mathbb{N}_0$, let $E_{q,n} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ and $F_{q,n} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ be defined by

$$E_{q,n}(z) := \operatorname{col}(z^j I_q)_{j=0}^{n} \qquad \text{and} \qquad F_{q,n}(z) := zE_{q,n}(z), \tag{8}$$

respectively. Obviously, for each $n \in \mathbb{N}_0$ and each $z \in \mathbb{C}$, we have $R_{T_{q,n}}(z) v_{q,n} = E_{q,n}(z)$.

**Notation 4.3.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Further, let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G} \setminus \mathbb{R} \neq \emptyset$ and let $f \colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $P_{2n}^{[f]} \colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q\times(n+2)q}$ be defined by*

$$P_{2n}^{[f]}(z) := \begin{bmatrix} H_n & R_{T_{q,n}}(z)[v_{q,n} f(z) - u_n] \\ (R_{T_{q,n}}(z)[v_{q,n} f(z) - u_n])^* & \frac{f(z) - f^*(z)}{z - \overline{z}} \end{bmatrix}. \tag{9}$$

*If $\kappa \geq 1$, then, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let $P^{[f]}_{2n+1}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(n+2)q\times(n+2)q}$ be given by*

$$P^{[f]}_{2n+1}(z) := \left[\begin{array}{c|c} H_{\alpha\triangleright n} & \begin{matrix} R_{T_{q,n}}(z)(v_{q,n}[(z-\alpha)f(z)] \\ -(-\alpha u_n - y_{0,n})) \end{matrix} \\ \hline \begin{matrix} [R_{T_{q,n}}(z)(v_{q,n}[(z-\alpha)f(z)] \\ -(-\alpha u_n - y_{0,n}))]^* \end{matrix} & \frac{(z-\alpha)f(z)-[(z-\alpha)f(z)]^*}{z-\overline{z}} \end{array}\right]. \tag{10}$$

*Furthermore, let $P^{[f]}_{-1}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by*

$$P^{[f]}_{-1}(z) := \frac{(z-\alpha)f(z) - [(z-\alpha)f(z)]^*}{z-\overline{z}}.$$

With respect to the Stieltjes moment problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ if, $\mathcal{G} = \mathbb{C}$, then the functions (9) and (10) are called the Potapov fundamental matrix-valued functions connected to the Stieltjes moment problem (generated by $f$). If these matrices are both non-negative Hermitian, then one says that the Potapov's fundamental matrix inequalities for the function $f$ are fulfilled.

**Remark 4.4.** *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q\times q$ matrices, let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G}\setminus\mathbb{R} \neq \emptyset$, and let $S\colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Straightforward calculations show then that the following statements hold true:*

(a) *For every choice of $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $z \in \mathcal{G}\setminus\mathbb{R}$, we have*

$$\begin{bmatrix} s_0 & S(z) \\ S^*(z) & \frac{S(z)-S^*(z)}{z-\overline{z}} \end{bmatrix} = [v_{q,n+1}, \mathfrak{v}_{q,n+1}]^* P^{[S]}_{2n}(z)[v_{q,n+1}, \mathfrak{v}_{q,n+1}]. \tag{11}$$

(b) *If $\kappa \geq 1$, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and each $z \in \mathcal{G}\setminus\mathbb{R}$, then*

$$\begin{aligned} &\begin{bmatrix} -\alpha s_0 + s_1 & (z-\alpha)S(z) + s_0 \\ [(z-\alpha)S(z)+s_0]^* & \frac{(z-\alpha)S(z)-[(z-\alpha)S(z)]^*}{z-\overline{z}} \end{bmatrix} = \\ &= [v_{q,n+1}, \mathfrak{v}_{q,n+1}]^* P^{[S]}_{2n+1}(z)[v_{q,n+1}, \mathfrak{v}_{q,n+1}]. \end{aligned} \tag{12}$$

**Notation 4.5.** *For each $n \in \mathbb{N}_0$, let $\tilde{A}_{2n}(z) := \operatorname{diag}([R_{T_{q,n}}(\overline{z})]^{-1}, I_q)$, let*

$$\tilde{B}_{2n}(z) := \begin{bmatrix} I_{(n+1)q} & (z-\overline{z})v_{q,n} \\ 0_{q\times(n+1)q} & I_q \end{bmatrix},$$

*let $\tilde{C}_{2n}(z) := \operatorname{diag}(R_{T_{q,n}}(z), I_q)$, let*

$$\tilde{A}_{2n+1}(z) := \tilde{A}_{2n}(z),$$

*let $\tilde{B}_{2n+1}(z) := \tilde{B}_{2n}(z)$, and let $\tilde{C}_{2n+1}(z) := \tilde{C}_{2n}(z)$.*

**Lemma 4.6.** *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of Hermitian complex $q\times q$ matrices. Let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G}\setminus\mathbb{R}\neq\emptyset$. Further, let $f\colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function, let $\mathcal{G}^\vee := \{z \in \mathbb{C}\colon \overline{z} \in \mathcal{G}\}$, and let $f^\vee\colon \mathcal{G}^\vee \to \mathbb{C}^{q\times q}$ be defined by $f^\vee(z) := f^*(\overline{z})$. For each $k \in \mathbb{Z}_{-1,\kappa}$ and each $z \in \mathcal{G}^\vee\setminus\mathbb{R}$, then $P^{[f^\vee]}_k(z) = X_k(z)P^{[f]}_k(\overline{z})X^*_k(z)$, where $X_k(z) := \tilde{C}_k(z)\tilde{B}_k(z)\tilde{A}_k(z)$.*

Taking into account Remark 4.1, Lemma 4.6 can be proved by straightforward calculations (, for details, see e. g. [30, Lemma 4.8]).

In the following, we will write $\mathfrak{B}_{p\times q}$ for the $\sigma$-algebra of all Borel subsets of $\mathbb{C}^{p\times q}$. Let $(\Omega, \mathfrak{A})$ be a measurable space and let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$. Then $\mu$ is absolutely continuous with respect to its trace measure $\tau := \operatorname{tr}\mu$. Let $\mu_\tau'$ be a version of the Radon–Nikodym derivative of $\mu$ with respect to $\tau$. A pair $[\Phi, \Psi]$ of an $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable mapping $\Phi\colon \Omega \to \mathbb{C}^{p\times q}$ and an $\mathfrak{A}$-$\mathfrak{B}_{r\times q}$-measurable mapping $\Psi\colon \Omega \to \mathbb{C}^{r\times q}$ is called left-integrable with respect to $\mu$ if $\Phi\mu_\tau'\Psi^*$ belongs to $[\mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})]^{p\times r}$. In this case, the corresponding integral is defined by $\int_\Omega \Phi \mathrm{d}\mu\Psi^* := \int_\Omega \Phi\mu_\tau'\Psi^*\mathrm{d}\tau$ and we also use the notation $\int_\Omega \Phi(\omega)\mu(\mathrm{d}\omega)\Psi^*(\omega)$ for it. In the following, when we write such an integral $\int_\Omega \Phi\mathrm{d}\mu\Psi^*$, then we also mean that the pair $[\Phi, \Psi]$ is left-integrable with respect to $\mu$. By $p \times q$-$\mathcal{L}^2(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ we denote the set of all $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable mappings for which the pair $[\Phi, \Phi]$ is left-integrable which respect to $\mu$. Furthermore, for each subset $A$ of $\Omega$, we will use $1_A$ to denote the indicator function of the set $A$ (defined on $\Omega$).

**Remark 4.7.** *Let $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $m \in \mathbb{N}_0$, and let $\sigma \in \mathcal{M}_{\geq}^{q}(\Omega)$. In view of Lemma 7.2, it is readily checked that $\sigma$ belongs to $\mathcal{M}_{\geq,2m}^{q}(\Omega)$ if and only if $\operatorname{Rstr}_\Omega E_{q,m}$ belongs to $(m+1)q \times q$-$\mathcal{L}^2(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$, where $E_{q,m}$ is given by (8). If $\sigma \in \mathcal{M}_{\geq,2m}^{q}(\Omega)$, then Lemma 7.2 also shows that, for each $n \in \mathbb{N}_0$ with $n \leq m$, the block Hankel matrix $H_n^{[\sigma]} := [s_{j+k}^{[\sigma]}]_{j,k=0}^{n}$ admits the integral representation*

$$H_n^{[\sigma]} = \int_\Omega E_{q,n}(t)\sigma(\mathrm{d}t)E_{q,n}^*(t). \tag{13}$$

If $\alpha \in \mathbb{R}$, if $\kappa \in \mathbb{N} \cup \{\infty\}$, and if $\sigma \in \mathcal{M}_{\geq,\kappa}^{q}([\alpha, \infty))$, then let $H_{\alpha\triangleright n}^{[\sigma]} := [s_{\alpha\triangleright j+k}^{[\sigma]}]_{j,k=0}^{n}$ for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.

**Remark 4.8.** *Let $\alpha \in \mathbb{R}$ and let $\sigma \in \mathcal{M}_{\geq,1}^{q}([\alpha, \infty))$. Using Proposition 7.4 and Remark 7.3, it is readily checked that the following statements hold true:*

(a) *The function $\phi\colon [\alpha, \infty) \to \mathbb{C}^{q\times q}$ defined by $\phi(t) := \sqrt{t-\alpha}I_q$ belongs to $q \times q$-$\mathcal{L}^2([\alpha, \infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ and $\sigma^\#\colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ given by*

$$\sigma^\#(B) := \int_B (\sqrt{t-\alpha}I_q)\sigma(\mathrm{d}t)(\sqrt{t-\alpha}I_q)^* \tag{14}$$

*belongs to $\mathcal{M}_{\geq}^{q}([\alpha, \infty))$.*

(b) *If $n \in \mathbb{N}_0$ and if $\sigma \in \mathcal{M}_{\geq,2n+1}^{q}([\alpha, \infty))$, then*

$$H_{\alpha\triangleright n}^{[\sigma]} = \int_{[\alpha,\infty)} \big[\sqrt{t-\alpha}E_{q,n}(t)\big]\sigma(\mathrm{d}t)\big[\sqrt{t-\alpha}E_{q,n}(t)\big]^*. \tag{15}$$

(c) *If $n \in \mathbb{N}_0$ and if $\sigma^\# \in \mathcal{M}_{\geq,2n}^{q}([\alpha, \infty))$, then $\sigma$ belongs to $\mathcal{M}_{\geq,2n+1}^{q}([\alpha, \infty))$ and furthermore $s_j^{[\sigma^\#]} = s_{j+1}^{[\sigma]} - \alpha s_j^{[\sigma]}$ for all $j \in \mathbb{Z}_{0,2n}$ and $H_n^{[\sigma^\#]} = H_{\alpha\triangleright n}^{[\sigma]}$.*

The next proposition shows that each solution of problem $\mathsf{MP}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ fulfills necessarily the system of the corresponding Potapov's fundamental matrix inequalities.

**Proposition 4.9.** *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q\times q$ matrices such that $\mathcal{M}_{\geq}^q[[\alpha,\infty);(s_j)_{j=0}^m,\leq] \neq \emptyset$. Let $\sigma \in \mathcal{M}_{\geq}^q[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ and let $S$ be the $[\alpha,\infty)$-Stieltjes transform of $\sigma$. For each $j \in \mathbb{Z}_{0,m}$, let $s_j^{[\sigma]}$ be given by (1). Then*

$$P_{2n}^{[S]}(z) = \int_{[\alpha,\infty)} \begin{bmatrix} E_{q,n}(t) \\ \frac{1}{t-z} I_q \end{bmatrix} \sigma(\mathrm{d}t) \begin{bmatrix} E_{q,n}(t) \\ \frac{1}{t-z} I_q \end{bmatrix}^* + \begin{bmatrix} \mathfrak{v}_{q,n} \\ 0_{q\times q} \end{bmatrix} (s_{2n} - s_{2n}^{[\sigma]}) \begin{bmatrix} \mathfrak{v}_{q,n} \\ 0_{q\times q} \end{bmatrix}^*$$

*for each $n \in \mathbb{N}_0$ with $2n \leq m$ and all $z \in \mathbb{C}\setminus\mathbb{R}$, where $E_{q,n}$ is given by (8), and*

$$\begin{aligned} P_{2n+1}^{[S]}(z) &= \\ &= \int_{[\alpha,\infty)} \left( \sqrt{t-\alpha} \begin{bmatrix} E_{q,n}(t) \\ \frac{1}{t-z} I_q \end{bmatrix} \right) \sigma(\mathrm{d}t) \left( \sqrt{t-\alpha} \begin{bmatrix} E_{q,n}(t) \\ \frac{1}{t-z} I_q \end{bmatrix} \right)^* + \\ &\quad + \begin{bmatrix} \mathfrak{v}_{q,n} \\ 0_{q\times q} \end{bmatrix} (s_{2n+1} - s_{2n+1}^{[\sigma]}) \begin{bmatrix} \mathfrak{v}_{q,n} \\ 0_{q\times q} \end{bmatrix}^* \end{aligned}$$

*for each $n \in \mathbb{N}_0$ with $2n+1 \leq m$ and all $z \in \mathbb{C}\setminus\mathbb{R}$. In particular, for every choice of $k \in \mathbb{Z}_{0,m}$ and $z \in \mathbb{C}\setminus\mathbb{R}$, the matrix $P_k^{[S]}(z)$ is non-negative Hermitian.*

Proposition 4.9 can be proved using standard arguments of integration theory of non-negative Hermitian measures (Lemma 7.2 and Remark 7.3). We omit the details.

## 5. Some integral estimates for the scalar case

In this section, we state some integral representations and estimates in the scalar case $q = 1$.

**Lemma 5.1.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ with Nevanlinna parametrization $(A, B, \nu)$ and spectral measure $\mu$. Then:*

(a) *For each $w \in \Pi_+$, the integral $\int_{\mathbb{R}} |t-w|^{-2}\mu(\mathrm{d}t)$ is finite and*

$$\Im F(w) = (\Im w)\left[B + \int_{\mathbb{R}} \frac{1}{|t-w|^2}\mu(\mathrm{d}t)\right]. \tag{16}$$

(b) *For each $w \in \Pi_+$, the integral $\int_{\mathbb{R}} |t[|t-w|^{-2} - (1+t^2)^{-1}] - \alpha|t-w|^{-2}|\mu(\mathrm{d}t)$ is finite and $F^\#\colon \Pi_+ \to \mathbb{C}$ defined by*

$$F^\#(w) := (w-\alpha)F(w) \tag{17}$$

*satisfies, for each $w \in \Pi_+$, the equation*

$$\begin{aligned}\Im F^{\#}(w) &= \\ &= (\Im w)\Bigg(A + B(2\Re w - \alpha) + \\ &+ \int_{\mathbb{R}}\left[t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}\right) - \frac{\alpha}{|t-w|^2}\right]\mu(\mathrm{d}t)\Bigg).\end{aligned} \tag{18}$$

*Proof.* In view of

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\mu(\mathrm{d}t) = \int_{\mathbb{R}} \frac{1}{1+t^2}(1+t^2)\nu(\mathrm{d}t) = \nu(\mathbb{R}) < \infty,$$

we see that, for each $w \in \Pi_+$, the function $\psi_w\colon \mathbb{R} \to \mathbb{C}$ given by the equation $\psi_w(t) := (t-w)^{-1} - t(1+t^2)^{-1}$ belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu; \mathbb{C})$. By virtue of a result due to R. Nevanlinna (see, e. g. [47, Theorem A.2]), for each $w \in \Pi_+$, we have

$$F(w) = A + Bw + \int_{\mathbb{R}}\left(\frac{1}{t-w} - \frac{t}{1+t^2}\right)\mu(\mathrm{d}t). \tag{19}$$

(a) Let $w \in \Pi_+$. For each $t \in \mathbb{R}$, then $\Im\psi_w(t) = (\Im w)|t-w|^{-2}$. Thus,

$$\int_{\mathbb{R}}\left|\frac{1}{|t-w|^2}\right|\mu(\mathrm{d}t) = \frac{1}{\Im w}\int_{\mathbb{R}} \Im\psi_w(t)\mu(\mathrm{d}t) \le \frac{1}{\Im w}\int_{\mathbb{R}} |\psi_w(t)|\mu(\mathrm{d}t) < \infty$$

and

$$\Im\left[\int_{\mathbb{R}} \psi_w(t)\mu(\mathrm{d}t)\right] = \int_{\mathbb{R}} \Im\psi_w(t)\mu(\mathrm{d}t) = (\Im w)\int_{\mathbb{R}} \frac{1}{|t-w|^2}\mu(\mathrm{d}t). \tag{20}$$

Because of $A \in \mathbb{R}$ and $B \in [0, \infty)$, we have $\Im A = 0$ and $\Im(wB) = (\Im w)B$. Consequently, from (19), and (20) we get then (16).

(b) Let $w \in \Pi_+$. In view of (17) and (19), we obtain

$$F^{\#}(w) = A(w-\alpha) + Bw(w-\alpha) + \int_{\mathbb{R}}\left[\frac{w-\alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2}\right]\mu(\mathrm{d}t). \tag{21}$$

For each $t \in \mathbb{R}$, we see that $(w-\alpha)\psi_w(t) = (w-\alpha)/(t-w) - t(w-\alpha)/(1+t^2)$ holds true. Hence, $(w-\alpha)\psi_w \in \mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu; \mathbb{C})$ and, for each $t \in \mathbb{R}$, we have furthermore $\Im[(w-\alpha)\psi_w(t)] = 2\mathrm{i}(\Im w)[t(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}) - \frac{\alpha}{|t-w|^2}]$. This implies

$$\int_{\mathbb{R}}\left|t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}\right) - \frac{\alpha}{|t-w|^2}\right|\mu(\mathrm{d}t) \le \frac{1}{\Im w}\int_{\mathbb{R}} |(w-\alpha)\psi_w(t)|\mu(\mathrm{d}t) < \infty$$

and

$$\begin{aligned}&\Im\left[\int_{\mathbb{R}}(w-\alpha)\psi_w(t)\mu(\mathrm{d}t)\right] = \\ &= (\Im w)\int_{\mathbb{R}}\left[t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}\right) - \frac{\alpha}{|t-w|^2}\right]\mu(\mathrm{d}t).\end{aligned} \tag{22}$$

Obviously, $\Im(w^2) = 2(\Re w)(\Im w)$. Hence, $\Im[w(w-\alpha)] = \Im(w^2) - \Im(w\alpha) = (\Im w)(2\Re w - \alpha)$. Thus, $\Im[Bw(w-\alpha)] = B(\Im w)(2\Re w - \alpha)$. Then, by virtue of (21), and (22), we get (18) from

$$\begin{aligned}\Im F^{\#}(w) &= \Im\left(A(w-\alpha) + Bw(w-\alpha) + \int_{\mathbb{R}}\left[\frac{w-\alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2}\right]\mu(\mathrm{d}t)\right)\\ &= \Im[A(w-\alpha)] + \Im[Bw(w-\alpha)] + \Im\left(\int_{\mathbb{R}}\left[\frac{w-\alpha}{t-w} - \frac{t(w-\alpha)}{1+t^2}\right]\mu(\mathrm{d}t)\right)\\ &= A\Im w + B(\Im w)(2\Re w - \alpha) +\\ &+ (\Im w)\int_{\mathbb{R}}\left[t\left(\frac{1}{|t-w|^2} - \frac{1}{1+t^2}\right) - \frac{\alpha}{|t-w|^2}\right]\mu(\mathrm{d}t).\end{aligned}$$

□

**Remark 5.2.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ with spectral measure $\mu$. Further, let $\ell_1, \ell_2 \in \mathbb{R}$ be such that $\ell_1 < \ell_2 < \alpha$. Then it is readily checked that for every choice of $a \in (-\infty, \ell_1)$ and $b \in (\ell_2, \infty)$, there exists a $K_{a,b} \in \mathbb{R}$ such that, for each $x \in [\ell_1, \ell_2]$, the inequality $\int_{\mathbb{R}\setminus(a,b)}(t-x)^{-2}\mu(\mathrm{d}t) < K_{a,b}$ holds true.*

**Remark 5.3.** *Let $r, s \in \mathbb{R}$. Then it is readily checked that the following statements hold true:*

(a) *If $r < s$ and $s \neq 0$, then there exists a number $a \in (-\infty, r) \cap (-\infty, 0)$ such that*

$$\left|t\left[\frac{1}{(t-x)^2+y^2} - \frac{1}{1+t^2}\right]\right| < \left(2 + \left|\frac{r}{s}\right|\right) \cdot \left|t\left[\frac{1}{(t-s)^2+1} - \frac{1}{1+t^2}\right]\right| \tag{23}$$

*is valid for every choice of $x \in [r, s]$ and $y \in (0, 1)$ and $t \in (-\infty, a]$.*

(b) *If $s < r$ and $r \neq 0$, then there exists a number $b \in (r, \infty) \cap (0, \infty)$ such that, for every choice of $x \in [s, r]$ and $y \in (0, 1)$ and $t \in [b, \infty)$, inequality (23) holds true.*

**Lemma 5.4.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ with spectral measure $\mu$. Further, let $\ell_1$ and $\ell_2$ be real numbers with $\ell_1 < \ell_2 < \alpha$. Then there are real numbers $a$, $b$, and $C$ with $a < \ell_1$ and $\ell_2 < b < \alpha$ such that $\int_{\mathbb{R}\setminus(a,b)}|t[\frac{1}{(t-x)^2+y^2} - \frac{1}{1+t^2}] - \frac{\alpha}{(t-x)^2+y^2}|\mu(\mathrm{d}t) < C$ holds true for every choice of $x \in [\ell_1, \ell_2]$ and $y \in (0, 1)$.*

Using Lemma 5.1 and Remarks 5.2 and 5.3, Lemma 5.4 can be proved analogous to the well-known special case $\alpha = 0$. However, in the general case of on arbitrary real number $\alpha$, these straightforward calculations are very lengthy. We omit the details.

**Lemma 5.5.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ be such that $F^{\#}\colon \Pi_+ \to \mathbb{C}$ defined by (17) belongs to $\mathcal{R}_1(\Pi_+)$. Further, let $\mu$ be the spectral measure of $F$ and let $\ell_1$ and $\ell_2$ be real numbers with $\ell_1 < \ell_2 < \alpha$. Then there are real numbers $a$, $b$, and $C$ with $a < \ell_1$ and $\ell_2 < b < \alpha$ such that $\int_{(a,b)}|\frac{t-\alpha}{(t-x)^2+y^2}|\sigma(\mathrm{d}t) < C$ and $\int_{(a,b)}|\frac{1}{(t-x)^2}|\sigma(\mathrm{d}t) < C$ hold true for every choice of $x \in [\ell_1, \ell_2]$ and $y \in [0, \infty)$.*

Lemma 5.5 can be proved, using Lemmata 5.1 and 5.4 and Beppo Levi's Theorem of monotone convergence. We omit the details.

**Remark 5.6.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ be such that $F^\#\colon \Pi_+ \to \mathbb{C}$ defined by* (17) *belongs to $\mathcal{R}_1(\Pi_+)$. Let $\mu$ be the spectral measure of $F$ and let $\ell_1$ and $\ell_2$ be real numbers with $\ell_1 < \ell_2 < \alpha$. Then one can easily see from Remark 5.2 and Lemma 5.5 that there is a real number $C$ such that $\int_{\mathbb{R}}(t-x)^{-2}\mu(\mathrm{d}t) < C$ for all $x \in [\ell_1, \ell_2]$.*

**Lemma 5.7.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_1(\Pi_+)$ be such that $F^\#\colon \Pi_+ \to \mathbb{C}$ defined by* (17) *belongs to $\mathcal{R}_1(\Pi_+)$. Then the Nevanlinna measure $\nu$ of $F$ and the spectral measure $\mu$ of $F$ fulfill $\nu((-\infty,\alpha)) = 0$ and $\mu((-\infty,\alpha)) = 0$.*

*Proof.* (I) In the first step of the proof, we consider arbitrary real numbers $\ell_1$ and $\ell_2$ with $\ell_1 < \ell_2 < \alpha$. Let $(A, B, \nu)$ be the Nevanlinna parametrization of $F$. Because of Remark 5.6, there is a $C \in \mathbb{R}$ such that $\int_{\mathbb{R}}(t-x)^{-2}\mu(\mathrm{d}t) < C$ is true for all $x \in [\ell_1, \ell_2]$. Since $F$ belongs to $\mathcal{R}_1(\Pi_+)$, for each $x \in [\ell_1, \ell_2]$ and each $\epsilon \in (0,\infty)$, from Lemma 5.1 we get then $0 \le \Im F(x+\mathrm{i}\epsilon) = \epsilon(B + \int_{\mathbb{R}}[(t-x)^2+\epsilon^2]^{-1}\mu(\mathrm{d}t)) < \epsilon(B+C)$ and, consequently,

$$0 \le \int_{[\ell_1,\ell_2]} \Im F(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x) \le \epsilon(B+C)(\ell_2-\ell_1), \tag{24}$$

where $\lambda^{(1)}$ is the Lebesgue measure defined on $\mathfrak{B}_{\mathbb{R}}$. In view of $F \in \mathcal{R}_1(\Pi_+)$, the inversion formula of Stieltjes–Perron (see, e. g. [47, Appendix, p. 390]) yields

$$\frac{1}{2}[\sigma(\{\ell_1\}) + \sigma(\{\ell_2\})] + \sigma((\ell_1,\ell_2)) = \frac{1}{\pi}\lim_{\epsilon\to 0+0}\int_{[\ell_1,\ell_2]} \Im F(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x). \tag{25}$$

Combining (25) and (24), we obtain $\sigma((\ell_1,\ell_2)) = 0$, from

$$\begin{aligned}0 &\le \sigma((\ell_1,\ell_2)) \le \frac{1}{2}[\sigma(\{\ell_1\}) + \sigma(\{\ell_2\})] + \sigma((\ell_1,\ell_2))\\ &= \frac{1}{\pi}\lim_{\epsilon\to 0+0}\int_{[\ell_1,\ell_2]} \Im F(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x) \le \frac{1}{\pi}\lim_{\epsilon\to 0+0}[\epsilon(B+C)(\ell_2-\ell_1)] = 0.\end{aligned}$$

(II) For each $n \in \mathbb{N}$, the real numbers $a_n := \alpha - (1+n)$ and $b_n := \alpha - \frac{1}{n}$ fulfill $a_n < b_n < \alpha$. Thus, part (I) of the proof provides us $\mu((a_n, b_n)) = 0$. Obviously, $(a_n, b_n) \subseteq (a_{n+1}, b_{n+1})$ for each $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty}(a_n, b_n) = (-\infty, \alpha)$. Hence, $\mu((-\infty,\alpha)) = \lim_{n\to\infty}\mu((a_n,b_n)) = 0$. Thus, $\nu((-\infty,\alpha)) = 0$ follows from

$$\begin{aligned}0 \le \nu((-\infty,\alpha)) &= \int_{(-\infty,\alpha)} 1\nu(\mathrm{d}t) \le\\ &\le \int_{(-\infty,\alpha)} (1+t^2)\nu(\mathrm{d}t) = \mu((-\infty,\alpha)) = 0.\end{aligned}$$

□

## 6. From the system of Potapov's fundamental matrix inequalities to the moment problem

Proposition 4.9 showed that the Stieltjes transform of an arbitrary solution of problem $\mathsf{MP}[[\alpha,\infty); (s_j)_{j=0}^m, \le]$ fulfills necessarily the system of corresponding Potapov's fundamental matrix inequalities. In this section, we are going to prove that the validity of the system of Potapov's fundamental matrix inequalities for a holomorphic $q \times q$ matrix-valued function defined on $\mathbb{C} \setminus [\alpha, \infty)$ is also sufficient to be the Stieltjes transform of some solution of this matricial Stieltjes-type moment problem. For the convenience of the reader, first we state two well-known facts.

**Remark 6.1.** *Let $\mathcal{D}$ be a discrete subset of $\Pi_+$ and let $F\colon \Pi_+ \setminus \mathcal{D} \to \mathbb{C}^{q\times q}$ be a matrix-valued function which is holomorphic in $\Pi_+ \setminus \mathcal{D}$ and which fulfills $\Im F(z) \in \mathbb{C}^{q\times q}_{\geq}$ for all $z \in \Pi_+ \setminus \mathcal{D}$. Then one can easily see from [16, Lemma 2.1.9] that there is a function $F^{\triangle} \in \mathcal{R}_q(\Pi_+)$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} F^{\triangle} = F$.*

**Remark 6.2.** *Let $A, B \in \mathbb{C}^{q\times q}$, let $M$ be an open subset of $\mathbb{R}$, and let $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R} \setminus M)$. In view of a well-known result on integrals which depend on a complex parameter (see, e. g. [24, Satz 5.8]), it is readily checked that $\phi\colon \Pi_+ \cup M \cup \Pi_- \to \mathbb{C}^{q\times q}$ given by*

$$\phi(z) := A + Bz + \int_{\mathbb{R}\setminus M} \frac{1+tz}{t-z}\nu(\mathrm{d}t)$$

*is holomorphic in $\Pi_+ \cup M \cup \Pi_-$.*

In the following, for all $\alpha \in \mathbb{R}$, let $\mathbb{C}_{\alpha,-} := \{z \in \mathbb{C}\colon \Re z \in (-\infty, \alpha)\}$.

**Lemma 6.3.** *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{R}_q(\Pi_+)$ be such that $F^{\#}\colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by $F^{\#}(w) := (w - \alpha)F(w)$ belongs to $\mathcal{R}_q(\Pi_+)$. Further, let $\nu$ be the Nevanlinna measure of $F$. Then $\nu((-\infty, \alpha)) = 0$ and the following two statements hold true:*

(a) *There is a function $F_\alpha\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ such that $\operatorname{Rstr}_{\Pi_+} F_\alpha = F$ and $F_\alpha((-\infty, \alpha)) \subseteq \mathbb{C}^{q\times q}_{\mathrm{H}}$ are fulfilled.*

(b) *There exists a unique function $S \in \mathcal{S}_{q;[\alpha,\infty)}$ with $\operatorname{Rstr}_{\Pi_+} S = F$.*

*Proof.* Since $F$ and $F^{\#}$ belong to $\mathcal{R}_q(\Pi_+)$, for all $u \in \mathbb{C}^q$, we see that $\{u^*Fu, u^*F^{\#}u\} \subseteq \mathcal{R}_1(\Pi_+)$ and that $u^*\nu u$ is the Nevanlinna measure of $u^*Fu$. Because of Lemma 5.7, for all $u \in \mathbb{C}^q$, we have $u^*\nu((-\infty, \alpha))u = (u^*\nu u)((-\infty, \alpha)) = 0 = u^*0_{q\times q}u$. Hence, $\nu((-\infty, \alpha)) = 0_{q\times q}$.

(a) Obviously, $\tilde{\nu} := \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \nu$ belongs to $\mathcal{M}^q_{\geq}([\alpha, \infty))$. By virtue of $F \in \mathcal{R}_q(\Pi_+)$ and Theorem 3.1, there are matrices $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $B \in \mathbb{C}^{q\times q}_{\geq}$ such that (3) holds true for each $z \in \Pi_+$. Remark 6.2 shows that $F_\alpha\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ given by

$$F_\alpha(z) := A + Bz + \int_{[\alpha,\infty)} \frac{1+tz}{t-z}\tilde{\nu}(\mathrm{d}t) \tag{26}$$

is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Comparing (3) and (26), we get $F_\alpha(z) = F(z)$ for each $z \in \Pi_+$. For every choice of $x \in \mathbb{R}$, we have

$$\left[\int_{[\alpha,\infty)} \frac{1+tx}{t-x}\tilde{\nu}(\mathrm{d}t)\right]^* = \int_{[\alpha,\infty)} \overline{\left(\frac{1+tx}{t-x}\right)}\tilde{\nu}(\mathrm{d}t) = \int_{[\alpha,\infty)} \frac{1+tx}{t-x}\tilde{\nu}(\mathrm{d}t).$$

In view of (26), $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, and $B \in \mathbb{C}_{\geq}^{q\times q}$, then $[F_\alpha(x)]^* = F_\alpha(x)$ follows for each $x \in (-\infty, \alpha)$.

(b) Because of part (a), there is a holomorphic function $S\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ such that

$$\operatorname{Rstr}_{\Pi_+} S = F \qquad \text{and} \qquad S((-\infty,\alpha)) \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q} \tag{27}$$

hold true. According to $\{F, F^\#\} \subseteq \mathcal{R}_q(\Pi_+)$ and (27), for all $z \in \Pi_+$, then

$$\Im S(z) = \Im F(z) \in \mathbb{C}_{\geq}^{q\times q} \quad \text{and} \quad \Im[(z-\alpha)S(z)] = \Im F^\#(z) \in \mathbb{C}_{\geq}^{q\times q}. \tag{28}$$

For all $z \in \mathbb{C}_{\alpha,-} \cap \Pi_+$, we have $\Im[(z-\alpha)S(z)] = [\Re(z-\alpha)]\Im S(z) + (\Im z)\Re S(z)$ and, by virtue of (28), consequently,

$$\Re S(z) = \frac{\Im[(z-\alpha)S(z)]}{\Im z} + [-\Re(z-\alpha)]\frac{\Im S(z)}{\Im z} \in \mathbb{C}_{\geq}^{q\times q}. \tag{29}$$

Now we consider an arbitrary monotonically nondecreasing sequence $(y_n)_{n=1}^\infty$ of positive real numbers with $\lim_{n\to\infty} y_n = 0$. Since the function $S$ is holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$, the functions $\Re S$ and $\Im S$ are continuous in $\mathbb{C}\setminus[\alpha,\infty)$. Thus, for each $x \in (-\infty,\alpha)$, we have $x + \mathrm{i}y_n \in \mathbb{C}_{\alpha,-} \cap \Pi_+$ for all $n \in \mathbb{N}$ and, hence, because of (29), and (28), then

$$\begin{aligned} \Re S(x) &= \lim_{n\to\infty} \Re S(x+\mathrm{i}y_n) \in \mathbb{C}_{\geq}^{q\times q} \quad \text{and} \\ \Im S(x) &= \lim_{n\to\infty} \Im S(x+\mathrm{i}y_n) \in \mathbb{C}_{\geq}^{q\times q}. \end{aligned} \tag{30}$$

Combining (27) and (30), for each $x \in (-\infty,\alpha)$, we get $\Re S(x) + \mathrm{i}\Im S(x) = S(x) = [S(x)]^* = \Re S(x) - \mathrm{i}\Im S(x)$ and, hence, $\Im S(x) = 0$. From (30) then $S(x) \in \mathbb{C}_{\geq}^{q\times q}$ follows for each $x \in (-\infty,\alpha)$. Consequently, $S \in \mathcal{S}_{q;[\alpha,\infty)}$. Now we consider an arbitrary $S^\square \in \mathcal{S}_{q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+} S^\square = F$. From (27) we get then $S^\square(z) = F(z) = S(z)$ for each $z \in \Pi_+$. Thus, the identity theorem for holomorphic functions provides us $S^\square = S$. □

**Proposition 6.4.** *Let $\alpha \in \mathbb{R}$ and let $\mathcal{D}$ be a discrete subset of $\Pi_+$. Let $F\colon \Pi_+\setminus\mathcal{D} \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function and let $F^\#\colon \Pi_+ \to \mathbb{C}^{q\times q}$ be defined by $F^\#(w) := (w-\alpha)F(w)$. Suppose $\{\Im F(w), \Im F^\#(w)\} \subseteq \mathbb{C}_{\geq}^{q\times q}$ for all $w \in \Pi_+\setminus\mathcal{D}$. Then there is a unique $S \in \mathcal{S}_{q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} S = F$.*

Proposition 6.4 can be easily proved using Remark 6.1, Lemma 6.3, and the identity theorem for holomorphic functions. We omit the details.

**Theorem 6.5.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q\times q$ matrices, and let $m \in \mathbb{Z}_{0,\kappa}$. Further, let $\mathcal{D}$ be a discrete subset of $\Pi_+$ and let $F\colon \Pi_+\setminus\mathcal{D} \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function such that*

$$P_m^{[F]}(z) \geq 0 \qquad \text{and} \qquad P_{m-1}^{[F]}(z) \geq 0 \qquad \text{for each } z \in \Pi_+\setminus\mathcal{D}. \tag{31}$$

*Then there exists a unique $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} S = F$. Moreover, the inequality $P_k^{[S]}(z) \geq 0$ holds true for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \mathbb{C}\setminus\mathbb{R}$.*

*Proof.* From (31) and Notation 4.3 we see that $H_n \geq 0$ for each $n \in \mathbb{N}_0$ with $2n \leq m$, that $H_{\alpha \triangleright n} \geq 0$ for each $n \in \mathbb{N}$ with $2n+1 \leq m$, that in particular $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,m}$, and that $\Im F(z) = (\Im z)\frac{F(z)-F^*(z)}{z-\overline{z}} \geq 0$ and $\Im[(z-\alpha)F(z)] = (\Im z)\frac{(z-\alpha)F(z)-[(z-\alpha)F(z)]^*}{z-\overline{z}} \geq 0$ hold true for each $z \in \Pi_+ \setminus \mathcal{D}$. Thus, because of Proposition 6.4, there exists a unique $S \in \mathcal{S}_{q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+ \setminus \mathcal{D}} S = F$. By continuity arguments, from (31) we get then $\{P_m^{[S]}(z), P_{m-1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{q\times q}$ for each $z \in \Pi_+$ and, consequently,

$$P_k^{[S]}(z) \geq 0 \qquad \text{for each } k \in \mathbb{Z}_{-1,m} \text{ and each } z \in \Pi_+. \tag{32}$$

In particular, $\tilde{S} := \operatorname{Rstr}_{\Pi_+} S$ fulfills

$$\begin{bmatrix} s_0 & \tilde{S}(z) \\ \tilde{S}^*(z) & \frac{\tilde{S}(z)-\tilde{S}^*(z)}{z-\overline{z}} \end{bmatrix} = P_0^{[S]}(z) \geq 0 \qquad \text{for each } z \in \Pi_+.$$

Consequently, Lemma 3.3 provides us $\tilde{S} \in \mathcal{R}'_{0,q}(\Pi_+)$ and $\sup_{y\in[1,\infty)} y\|S(\mathrm{i}y)\|_{\mathrm{S}} < \infty$. Hence, $S$ belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$. Then Theorem 3.5 shows that there is a $\sigma \in \mathcal{M}_{\geq}^q([\alpha,\infty))$ such that (7) holds true. Let $\tilde{S}^{\vee}\colon \Pi_- \to \mathbb{C}^{q\times q}$ be defined by $\tilde{S}^{\vee}(z) := S^*(\overline{z})$. Thus, from (7) we get $\tilde{S}^{\vee}(z) = [\int_{[\alpha,\infty)} \frac{1}{t-\overline{z}}\sigma(\mathrm{d}t)]^* = \int_{[\alpha,\infty)} \frac{1}{t-z}\sigma(\mathrm{d}t) = S(z)$ for each $z \in \Pi_-$. From (32) and Lemma 4.6 we see then that, for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \Pi_-$, there exists a matrix $X_k(z)$ such that $P_k^{[S]}(z) = P_k^{[\tilde{S}^{\vee}]}(z) = X_k(z)P_k^{[S]}(\overline{z})X_k^*(z)$ is fulfilled for all $k \in \mathbb{Z}_{-1,m}$ and all $z \in \Pi_-$. In view of (32), this implies $P_k^{[S]}(z) \geq 0$ for each $k \in \mathbb{Z}_{-1,m}$ and each $z \in \Pi_-$. Because of $\mathbb{C} \setminus \mathbb{R} = \Pi_+ \cup \Pi_-$, the proof is complete. ☐

**Remark 6.6.** *For each $n \in \mathbb{N}_0$ and every choice of $w$ and $z$ in $\mathbb{C}$, it is readily checked that*

$$(z-\overline{w})\left[R_{T_{q,n}^*}(w)\right]^* T_{q,n} R_{T_{q,n}}(z) = R_{T_{q,n}}(z) - \left[R_{T_{q,n}^*}(w)\right]^*.$$

**Lemma 6.7.** *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. Then*

$$\begin{aligned} &H_n T_{q,n}^* R_{T_{q,n}^*}(z) - \left[R_{T_{q,n}^*}(w)\right]^* T_{q,n} H_n + \\ &+ \left[R_{T_{q,n}^*}(w)\right]^* (v_{q,n} u_n^* - u_n v_{q,n}^*) R_{T_{q,n}^*}(z) = \\ &= (z-\overline{w})\left[R_{T_{q,n}^*}(w)\right]^* T_{q,n} H_n T_{q,n}^* R_{T_{q,n}^*}(z) \end{aligned} \tag{33}$$

*for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and every choice of $w$ and $z$ in $\mathbb{C}$. Furthermore,*

$$\begin{aligned} &H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) - \left[R_{T_{q,n}^*}(w)\right]^* T_{q,n} H_{\alpha\triangleright n} + \\ &+ \left[R_{T_{q,n}^*}(w)\right]^* \left[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n}) v_{q,n}^*\right] R_{T_{q,n}^*}(z) = \\ &= (z-\overline{w})\left[R_{T_{q,n}^*}(w)\right]^* T_{q,n} H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) \end{aligned} \tag{34}$$

*for all $\alpha \in \mathbb{R}$, all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, and every choice of $w$ and $z$ in $\mathbb{C}$.*

*Proof.* By virtue of Remark 4.1(a), we have

$$\begin{aligned}
&H_n T_{q,n}^* R_{T_{q,n}^*}(z) - \Big[R_{T_{q,n}^*}(w)\Big]^* T_{q,n} H_n + \\
&\quad + \Big[R_{T_{q,n}^*}(w)\Big]^* (v_{q,n} u_n^* - u_n v_{q,n}^*) R_{T_{q,n}^*}(z) = \\
&= \Big[R_{T_{q,n}^*}(w)\Big]^* \Bigg( \Big[R_{T_{q,n}^*}(w)\Big]^{-*} H_n T_{q,n}^* - T_{q,n} H_n R_{T_{q,n}^*}^{-1}(z) + \\
&\quad + (v_{q,n} u_n^* - u_n v_{q,n}^*) \Bigg) R_{T_{q,n}^*}(z) = \Big[R_{T_{q,n}^*}(w)\Big]^* \big[(I_{(n+1)q} - \overline{w} T_{q,n}) H_n T_{q,n}^* - \\
&\quad - T_{q,n} H_n (I_{(n+1)q} - z T_{q,n}^*) + (v_{q,n} u_n^* - u_n v_{q,n}^*) bigr] R_{T_{q,n}^*}(z) = \\
&= \Big[R_{T_{q,n}^*}(w)\Big]^* \Big[(z - \overline{w}) T_{q,n} H_n T_{q,n}^* - (T_{q,n} H_n - H_n T_{q,n}^*) + \\
&\quad + (v_{q,n} u_n^* - u_n v_{q,n}^*)\Big] R_{T_{q,n}^*}(z) = \Big[R_{T_{q,n}^*}(w)\Big]^* \big[(z - \overline{w}) T_{q,n} H_n T_{q,n}^*\big] R_{T_{q,n}^*}(z) = \\
&= (z - \overline{w}) \Big[R_{T_{q,n}^*}(w)\Big]^* T_{q,n} H_n T_{q,n}^* R_{T_{q,n}^*}(z).
\end{aligned}$$

Using Remark 4.1(b), equation (34) can be proved analogous to (33). □

**Notation 6.8.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Let $\mathcal{G}$ be a subset of $\mathbb{C}$ with $\mathcal{G} \setminus \mathbb{R} \neq \emptyset$ and let $f\colon \mathcal{G} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. For each $n \in \mathbb{N}_0$ with $2n \le \kappa$, let $F_{2n}\colon \mathcal{G} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by*

$$F_{2n}(z) := H_n T_{q,n}^* R_{T_{q,n}^*}(z) + R_{T_{q,n}}(z)[v_{q,n} f(z) - u_n] v_{q,n}^* R_{T_{q,n}^*}(z) \tag{35}$$

*and let $Q_{2n}^{[f]}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(2n+2)q\times(2n+2)q}$ be defined by*

$$Q_{2n}^{[f]}(z) := \begin{bmatrix} H_n & F_{2n}(z) \\ F_{2n}^*(z) & \frac{F_{2n}(z) - F_{2n}^*(z)}{z - \overline{z}} \end{bmatrix}. \tag{36}$$

*If $\kappa \ge 1$, then, for all $n \in \mathbb{N}_0$ with $2n+1 \le \kappa$, let $F_{2n+1}\colon \mathcal{G} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by*

$$\begin{aligned} F_{2n+1}(z) := H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) + R_{T_{q,n}}(z)\big[v_{q,n}(z - \alpha) f(z) - \\ - (-\alpha u_n - y_{0,n})\big] v_{q,n}^* R_{T_{q,n}^*}(z) \end{aligned} \tag{37}$$

*and let $Q_{2n+1}^{[f]}\colon \mathcal{G} \setminus \mathbb{R} \to \mathbb{C}^{(2n+2)q\times(2n+2)q}$ be defined by*

$$Q_{2n+1}^{[f]}(z) := \begin{bmatrix} H_{\alpha\triangleright n} & F_{2n+1}(z) \\ F_{2n+1}^*(z) & \frac{F_{2n+1}(z) - F_{2n+1}^*(z)}{z - \overline{z}} \end{bmatrix}. \tag{38}$$

Further, for each $k \in \mathbb{N}_0$, let $m_{2k} := k$ and $m_{2k+1} := k$.

**Proposition 6.9.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. Let $f\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Further, for each $k \in \mathbb{N}_0$, let $F_k\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{(m_k+1)q\times(m_k+1)q}$ be defined by Notation 6.8. For all $k \in \mathbb{Z}_{0,\kappa}$, then there are functions $\Gamma_k\colon \mathbb{C} \setminus \mathbb{R} \to$*

$\mathbb{C}^{(m_k+2)q\times(2m_k+2)q}$ *and* $\Delta_k\colon \mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{(2m_k+2)q\times(m_k+2)q}$ *such that* $P_k^{[f]}(z) = \Gamma_k(z)Q_k^{[f]}(z)\Gamma_k^*(z)$ *and* $Q_k^{[f]}(z) = \Delta_k(z)P_k^{[f]}(z)\Delta_k^*(z)$ *hold true for each* $z\in\mathbb{C}\setminus\mathbb{R}$.

*Proof.* (I) In the trivial case $k=0$, choose $\Gamma_0(z) := I_{2q}$ and $\Delta_0(z) := I_{2q}$ for all $z\in\mathbb{C}\setminus\mathbb{R}$.

(II) Now we consider the case that $\kappa\geq 1$ and that $n\in\mathbb{N}_0$ is such that $2n+1\leq\kappa$. Let

$$\begin{aligned}\Delta_{2n+1}(z) &:= \begin{bmatrix} I_{(n+1)q} & 0\\ [R_{T_{q,n}^*}(z)]^*T_{q,n} & [R_{T_{q,n}^*}(z)]^*v_{q,n}\end{bmatrix} \quad\text{and}\\ \Gamma_{2n+1}(z) &:= \begin{bmatrix} I_{(n+1)q} & 0\\ -v_{q,n}^*[R_{T_{q,n}^*}(z)]^*T_{q,n} & v_{q,n}^*\end{bmatrix}\end{aligned}\tag{39}$$

for all $z\in\mathbb{C}\setminus\mathbb{R}$. Since $s_j^* = s_j$ holds true for each $j\in\mathbb{Z}_{0,\kappa}$, we have $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$. We consider an arbitrary $z\in\mathbb{C}\setminus\mathbb{R}$. Let

$$B_{2n+1}(z) := R_{T_{q,n}}(z)[v_{q,n}(z-\alpha)f(z) - (-\alpha u_n - y_{0,n})],\tag{40}$$

let

$$C_{2n+1}(z) := \frac{(z-\alpha)f(z) - [(z-\alpha)f(z)]^*}{z-\overline{z}},\tag{41}$$

and let

$$\Delta_{2n+1}(z)P_{2n+1}^{[f]}(z)\Delta_{2n+1}^*(z) = \begin{bmatrix} X_{2n+1}(z) & Y_{2n+1}(z)\\ Z_{2n+1}(z) & W_{2n+1}(z)\end{bmatrix}\tag{42}$$

be the $(n+1)q\times(n+1)q$ block representation of $\Delta_{2n+1}(z)P_{2n+1}^{[f]}(z)\Delta_{2n+1}^*(z)$. Then

$$P_{2n+1}^{[f]}(z) = \begin{bmatrix} H_{\alpha\triangleright n} & B_{2n+1}(z)\\ B_{2n+1}^*(z) & C_{2n+1}(z)\end{bmatrix}.$$

Consequently, using (42) and (39), straightforward calculations show that

$$\begin{aligned} X_{2n+1}(z) &= H_{\alpha\triangleright n},\\ Y_{2n+1}(z) &= H_{\alpha\triangleright n}T_{q,n}^*R_{T_{q,n}^*}(z) + B_{2n+1}(z)v_{q,n}^*R_{T_{q,n}^*}(z),\end{aligned}\tag{43}$$

$$Z_{2n+1}(z) = \left[R_{T_{q,n}^*}(z)\right]^*T_{q,n}H_{\alpha\triangleright n} + \left[R_{T_{q,n}^*}(z)\right]^*v_{q,n}B_{2n+1}^*(z),\tag{44}$$

and

$$\begin{aligned} W_{2n+1}(z) = {}&\left[R_{T_{q,n}^*}(z)\right]^*T_{q,n}H_{\alpha\triangleright n}T_{q,n}^*R_{T_{q,n}^*}(z) +\\ &+\left[R_{T_{q,n}^*}(z)\right]^*T_{q,n}B_{2n+1}(z)v_{q,n}^*R_{T_{q,n}^*}(z) +\\ &+\left[R_{T_{q,n}^*}(z)\right]^*v_{q,n}B_{2n+1}^*(z)T_{q,n}^*R_{T_{q,n}^*}(z) +\\ &+\left[R_{T_{q,n}^*}(z)\right]^*v_{q,n}C_{2n+1}(z)v_{q,n}^*R_{T_{q,n}^*}(z)\end{aligned}\tag{45}$$

hold true. Because of (44), (40), and (37), we see that

$$\begin{aligned} Y_{2n+1}(z) &= H_{\alpha\triangleright n}T_{q,n}^*R_{T_{q,n}^*}(z) + R_{T_{q,n}}(z)\big[v_{q,n}(z-\alpha)f(z) -\\ &\quad - (-\alpha u_n - y_{0,n})\big]v_{q,n}^*R_{T_{q,n}^*}(z) =\\ &= F_{2n+1}(z)\end{aligned}\tag{46}$$

is valid. From (44), $H^*_{\alpha\triangleright n} = H_{\alpha\triangleright n}$, (43), and (46) we obtain then

$$Z_{2n+1}(z) = Y^*_{2n+1}(z) = F^*_{2n+1}(z). \tag{47}$$

Using (45), it follows

$$\begin{aligned} W_{2n+1}(z) = & \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) + \\ & + \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z) + \\ & + \Big(\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z)\Big)^* + \\ & + \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} C_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z). \end{aligned} \tag{48}$$

In view of Lemma 6.7, we have

$$\begin{aligned} & (z-\overline{z})\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_{\alpha\triangleright n} + \\ & \quad + \Big[R_{T^*_{q,n}}(z)\Big]^* [v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n}) v^*_{q,n}] R_{T^*_{q,n}}(z). \end{aligned} \tag{49}$$

By virtue of (40), Remark 6.6, and (37), we conclude

$$\begin{aligned} & (z-\overline{z})\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z) = (z-\overline{z})\Big[R_{T^*_{q,n}}(z)\Big]^* \times \\ & \quad \times T_{q,n} R_{T_{q,n}}(z)[v_{q,n}(z-\alpha) f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = \Big(R_{T_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^*\Big) \times \\ & \quad \times [v_{q,n}(z-\alpha) f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = R_{T_{q,n}}(z)[v_{q,n}(z-\alpha) f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) - \\ & \quad - \Big[R_{T^*_{q,n}}(z)\Big]^* [v_{q,n}(z-\alpha) f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = F_{2n+1}(z) - H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) - \\ & \quad - \Big[R_{T^*_{q,n}}(z)\Big]^* [v_{q,n}(z-\alpha) f(z) - (-\alpha u_n - y_{0,n})] v^*_{q,n} R_{T^*_{q,n}}(z) = \\ & = F_{2n+1}(z) - H_{\alpha\triangleright n} T^*_{q,n} R_{T^*_{q,n}}(z) - \\ & \quad - \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n}(z-\alpha) f(z) v^*_{q,n} R_{T^*_{q,n}}(z) + \\ & \quad + \Big[R_{T^*_{q,n}}(z)\Big]^* (-\alpha u_n - y_{0,n}) v^*_{q,n} R_{T^*_{q,n}}(z), \end{aligned} \tag{50}$$

which implies

$$\begin{aligned} & (z-\overline{z})\Big(\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} B_{2n+1}(z) v^*_{q,n} R_{T^*_{q,n}}(z)\Big)^* = \\ & = -F^*_{2n+1}(z) + \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H^*_{\alpha\triangleright n} + \end{aligned} \tag{51}$$

$$+ \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v_{q,n}^* R_{T_{q,n}^*}(z) -$$
$$- \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}(-\alpha u_n - y_{0,n})^* R_{T_{q,n}^*}(z).$$

Taking into account (41) we get

$$\begin{aligned}(z-\overline{z})&\left[R_{T_{q,n}^*}(z)\right]^* v_{q,n} C_{2n+1}(z) v_{q,n}^* R_{T_{q,n}^*}(z) = \\ &= \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}(z-\alpha) f(z) v_{q,n}^* R_{T_{q,n}^*}(z) - \\ &- \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v_{q,n}^* R_{T_{q,n}^*}(z).\end{aligned} \tag{52}$$

In view of (48), we obtain

$$\begin{aligned}(z-\overline{z})W_{2n+1}(z) = {} & (z-\overline{z})\left[R_{T_{q,n}^*}(z)\right]^* T_{q,n} H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) + \\ & + (z-\overline{z})\left[R_{T_{q,n}^*}(z)\right]^* T_{q,n} B_{2n+1}(z) v_{q,n}^* R_{T_{q,n}^*}(z) + \\ & + (z-\overline{z})\left(\left[R_{T_{q,n}^*}(z)\right]^* T_{q,n} B_{2n+1}(z) v_{q,n}^* R_{T_{q,n}^*}(z)\right)^* + \\ & + (z-\overline{z})\left[R_{T_{q,n}^*}(z)\right]^* v_{q,n} C_{2n+1}(z) v_{q,n}^* R_{T_{q,n}^*}(z)\end{aligned}$$

and, using (49), (50), (51), (52), and $H_{\alpha\triangleright n}^* = H_{\alpha\triangleright n}$, consequently,

$$\begin{aligned}(z-\overline{z})W_{2n+1}(z) = {} & H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) - \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n} H_{\alpha\triangleright n} + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* \left[v_{q,n}(-\alpha u_n - y_{0,n})^* - (-\alpha u_n - y_{0,n}) v_{q,n}^*\right] R_{T_{q,n}^*}(z) + \\ & + F_{2n+1}(z) - H_{\alpha\triangleright n} T_{q,n}^* R_{T_{q,n}^*}(z) - \\ & - \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}(z-\alpha) f(z) v_{q,n}^* R_{T_{q,n}^*}(z) + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* (-\alpha u_n - y_{0,n}) v_{q,n}^* R_{T_{q,n}^*}(z) - F_{2n+1}^*(z) + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* T_{q,n} H_{\alpha\triangleright n}^* + \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v_{q,n}^* R_{T_{q,n}^*}(z) + \\ & - \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}(-\alpha u_n - y_{0,n})^* R_{T_{q,n}^*}(z) + \\ & + \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}(z-\alpha) f(z) v_{q,n}^* R_{T_{q,n}^*}(z) - \\ & - \left[R_{T_{q,n}^*}(z)\right]^* v_{q,n}[(z-\alpha)f(z)]^* v_{q,n}^* R_{T_{q,n}^*}(z) = F_{2n+1}(z) - F_{2n+1}^*(z).\end{aligned} \tag{53}$$

From (42), (43), (46), (47), (53), and (38) we infer

$$\Delta_{2n+1}(z) P_{2n+1}^{[f]}(z) \Delta_{2n+1}^*(z) = Q_{2n+1}^{[f]}(z). \tag{54}$$

In view of $v_{q,n}^*[R_{T_{q,n}^*}^*(z)]v_{q,n} = I_q$, we easily see that the matrices $\Gamma_{2n+1}(z)$ and $\Delta_{2n+1}(z)$ given by (39) obviously fulfill

$$\Gamma_{2n+1}(z)\Delta_{2n+1}(z) = I_{(n+2)q}. \tag{55}$$

Thus, because of (54), we obtain

$$\begin{aligned}P^{[f]}_{2n+1}(z) &= I_{(n+2)q}P^{[f]}_{2n+1}(z)I^*_{(n+2)q} = \\ &= \Gamma_{2n+1}(z)\Delta_{2n+1}(z)P^{[f]}_{2n+1}(z)\Gamma^*_{2n+1}(z)\Delta^*_{2n+1}(z) = \\ &= \Gamma_{2n+1}(z)Q^{[f]}_{2n+1}(z)\Gamma^*_{2n+1}(z).\end{aligned}$$

In this case $k = 2n+1$ with some $n \in \mathbb{N}_0$, the proof is complete.

(III) Now we consider the case that $\kappa \geq 2$ and that there is an $n \in \mathbb{N}$ such that $k = 2n$. Let $\Gamma_{2n} := \Gamma_{2n+1}$ and let $\Delta_{2n} := \Delta_{2n+1}$. We consider again an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$. Let

$$\Delta_{2n}(z)P^{[f]}_{2n}(z)\Delta^*_{2n}(z) = \begin{bmatrix} X_{2n}(z) & Y_{2n}(z) \\ Z_{2n}(z) & W_{2n}(z)\end{bmatrix} \tag{56}$$

be the $(n+1)q \times (n+1)q$ block representation of $\Delta_{2n}(z)P^{[f]}_{2n}(z)\Delta^*_{2n}(z)$. Setting

$$B_{2n}(z) := R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n] \quad \text{and} \quad C_{2n}(z) := \frac{f(z) - f^*(z)}{z - \overline{z}}, \tag{57}$$

we have $P^{[f]}_{2n}(z) = \begin{bmatrix} H_n & B_{2n}(z) \\ B^*_{2n}(z) & C_{2n}(z)\end{bmatrix}$. Consequently, from (56) we easily see then that

$$X_{2n}(z) = H_n, \qquad Y_{2n}(z) = H_nT^*_{q,n}R_{T^*_{q,n}}(z) + B_{2n}(z)v^*_{q,n}R_{T^*_{q,n}}(z), \tag{58}$$

$$Z_{2n}(z) = \left[R_{T^*_{q,n}}(z)\right]^*T_{q,n}H_n + \left[R_{T^*_{q,n}}(z)\right]^*v_{q,n}B^*_{2n}(z), \tag{59}$$

and

$$\begin{aligned}W_{2n}(z) = &\left[R_{T^*_{q,n}}(z)\right]^*T_{q,n}H_nT^*_{q,n}R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^*v_{q,n}B^*_{2n}(z)T^*_{q,n}R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^*T_{q,n}B_{2n}(z)v^*_{q,n}R_{T^*_{q,n}}(z) + \\ &+ \left[R_{T^*_{q,n}}(z)\right]^*v_{q,n}C_{2n}(z)v^*_{q,n}R_{T^*_{q,n}}(z)\end{aligned} \tag{60}$$

hold true. Because of (58), (57), and (35), we obtain

$$Y_{2n}(z) = H_nT^*_{q,n}R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z)[v_{q,n}f(z) - u_n]v^*_{q,n}R_{T^*_{q,n}}(z) = F_{2n}(z). \tag{61}$$

Since $s^*_j = s_j$ is supposed for each $j \in \mathbb{Z}_{0,\kappa}$, we get $H^*_n = H_n$. Consequently, in view of (59), (58), and (61), then

$$Z_{2n}(z) = Y^*_{2n}(z) = F^*_{2n}(z) \tag{62}$$

follows. By virtue of (60) and (57), we see that

$$\begin{aligned} W_{2n}(z) = &\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_n T^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} [f^*(z) v^*_{q,n} - u^*_n] [R_{T_{q,n}}(z)]^* T^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} \left[\frac{f(z) - f^*(z)}{z - \overline{z}}\right] v^*_{q,n} R_{T^*_{q,n}}(z) \end{aligned} \tag{63}$$

holds true. Taking into account (63) and Remark 6.6, we conclude

$$\begin{aligned} W_{2n}(z) = &\Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_n T^*_{q,n} R_{T^*_{q,n}}(z) + \\ + &\Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} [f^*(z) v^*_{q,n} - u^*_n] \left[\frac{1}{z - \overline{z}}\Big(R_{T^*_{q,n}}(z) - [R_{T_{q,n}}(z)]^*\Big)\right] + \\ + &\left[\frac{1}{z - \overline{z}}\Big(R_{T_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^*\Big)\right] [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) + \\ + &\Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} \left[\frac{f(z) - f^*(z)}{z - \overline{z}}\right] v^*_{q,n} R_{T^*_{q,n}}(z). \end{aligned} \tag{64}$$

Using Lemma 6.7, the equation $H^*_n = H_n$, (35), and (64), we infer

$$\begin{aligned} W_{2n}(z) = &\frac{1}{z - \overline{z}} \Big\{ H_n T^*_{q,n} R_{T^*_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^* T_{q,n} H_n + \\ &+ \Big[R_{T^*_{q,n}}(z)\Big]^* (v_{q,n} u^*_n - u_n v^*_{q,n}) R_{T^*_{q,n}}(z) + \\ &+ \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} [f^*(z) v^*_{q,n} - u^*_n] \Big(R_{T^*_{q,n}}(z) - [R_{T_{q,n}}(z)]^*\Big) + \\ &+ \Big(R_{T_{q,n}}(z) - \Big[R_{T^*_{q,n}}(z)\Big]^*\Big) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) + \\ &+ \Big[R_{T^*_{q,n}}(z)\Big]^* v_{q,n} [f(z) - f^*(z)] v^*_{q,n} R_{T^*_{q,n}}(z) \Big\} = \\ &= \frac{1}{z - \overline{z}} \Big\{ H_n T^*_{q,n} R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z) - \\ &- \Big(H_n T^*_{q,n} R_{T^*_{q,n}}(z) + R_{T_{q,n}}(z) [v_{q,n} f(z) - u_n] v^*_{q,n} R_{T^*_{q,n}}(z)\Big)^* \Big\} = \\ &= \frac{1}{z - \overline{z}} [F_{2n}(z) - F^*_{2n}(z)]. \end{aligned}$$

Thus, (56), the first equation in (58), (61), (62), and (36) show that

$$\Delta_{2n}(z) P^{[f]}_{2n}(z) \Delta^*_{2n}(z) = \begin{bmatrix} H_n & F_{2n}(z) \\ F^*_{2n}(z) & \frac{F_{2n}(z) - F^*_{2n}(z)}{z - \overline{z}} \end{bmatrix} = Q^{[f]}_{2n}(z) \tag{65}$$

is valid. Because of $\Gamma_{2n} = \Gamma_{2n+1}$ and $\Delta_{2n} = \Delta_{2n+1}$, equation (55) implies $\Gamma_{2n}(z)\Delta_{2n}(z) = I_{(n+2)q}$. Consequently, from (65) we get

$$P^{[f]}_{2n}(z) = \Gamma_{2n}(z) \Delta_{2n}(z) P^{[f]}_{2n}(z) \Delta^*_{2n}(z) \Gamma^*_{2n}(z) = \Gamma_{2n}(z) Q^{[f]}_{2n}(z) \Gamma^*_{2n}(z). \qquad \square$$

**Remark 6.10.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Let $f\colon \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function. In view of Proposition 6.9 and Lemma 3.3, it is readily checked that the following statements hold true:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. If $P_{2n}^{[f]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for each $z \in \mathbb{C}\setminus\mathbb{R}$, then $F_{2n}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by (35) belongs to $\mathcal{R}'_{0,(n+1)q}(\Pi_+)$ and the matricial spectral measure $\mu_{2n}$ of $F_{2n}$ fulfills $\mu_{2n}(\mathbb{R}) \leq H_n$.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. If $P_{2n+1}^{[f]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for each $z \in \mathbb{C} \setminus \mathbb{R}$, then $F_{2n+1}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ defined by (37) belongs to $\mathcal{R}'_{0,(n+1)q}(\Pi_+)$ and the matricial spectral measure $\mu_{2n+1}$ of $F_{2n+1}$ fulfills $\mu_{2n+1}(\mathbb{R}) \leq H_{\alpha\rhd n}$.*

**Lemma 6.11.** *Let $\alpha \in \mathbb{R}$, let $f\colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$, let $F_{2n}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be defined by (35), and let $\Psi_{2n}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by*

$$\Psi_{2n}(z) := R_{T_{q,n}}(z)(H_n T_{q,n}^* - u_n v_{q,n}^* - zT_{q,n}H_nT_{q,n}^*)R_{T_{q,n}^*}(z). \tag{66}$$

*Then $\Psi_{2n}$ is a continuous matrix-valued function such that $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. In view of (8), furthermore,*

$$F_{2n}(z) = \Psi_{2n}(z) + E_{q,n}(z)f(z)E_{q,n}^*(\overline{z}) \qquad \text{for each } z \in \Pi_+. \tag{67}$$

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $F_{2n+1}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be defined by (37). Then $\Psi_{2n+1}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by*

$$\begin{aligned}\Psi_{2n+1}(z) := R_{T_{q,n}}(z)\big[H_{\alpha\rhd n}T_{q,n}^* - (-\alpha u_n - y_{0,n})v_{q,n}^* -\\ - zT_{q,n}H_{\alpha\rhd n}T_{q,n}^*\big]R_{T_{q,n}^*}(z)\end{aligned} \tag{68}$$

*is continuous and fulfills $\Psi_{2n+1}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ as well as*

$$F_{2n+1}(z) = \Psi_{2n+1}(z) + E_{q,n}(z)[(z-\alpha)f(z)]E_{q,n}^*(\overline{z}) \qquad \text{for each } z \in \Pi_+.$$

*Proof.* (a) The case $n = 0$ is trivial. Suppose now $0 < 2n \leq \kappa$. Remark 4.2 shows that $\Psi_{2n}$ is continuous. For each $x \in \mathbb{R}$, we have $R_{T_{q,n}^*}(x) = [R_{T_{q,n}}(\overline{x})]^* = [R_{T_{q,n}}(x)]^*$ and, consequently,

$$[\Psi_{2n}(x)]^* = R_{T_{q,n}}(x)(T_{q,n}H_n - v_{q,n}u_n^* - xT_{q,n}H_n^*T_{q,n}^*)R_{T_{q,n}^*}(x),$$

which, in view of $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n}$, i. e., $H_n^* = H_n$, implies that

$$\begin{aligned}[\Psi_{2n}(x)]^* &= R_{T_{q,n}}(x)\big(-[H_nT_{q,n}^* - T_{q,n}H_n]+\\ &\quad + H_nT_{q,n}^* - v_{q,n}u_n^* - xT_{q,n}H_nT_{q,n}^*\big)R_{T_{q,n}^*}(x) =\\ &= R_{T_{q,n}}(x)\big(-[u_nv_{q,n}^* - v_{q,n}u_n^*] + H_nT_{q,n}^* - v_{q,n}u_n^* - xT_{q,n}H_nT_{q,n}^*\big)R_{T_{q,n}^*}(x) =\\ &= R_{T_{q,n}}(x)(H_nT_{q,n}^* - u_nv_{q,n}^* - xT_{q,n}H_nT_{q,n}^*)R_{T_{q,n}^*}(x) = \Psi_{2n}(x)\end{aligned}$$

holds true for all $x \in \mathbb{R}$. Hence, $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. Taking into account (35), Remark 4.2, and (66), for all $z \in \Pi_+$, we conclude

$$\begin{aligned} F_{2n}(z) &= R_{T_{q,n}}(z)\big[R_{T_{q,n}}(z)\big]^{-1}H_nT_{q,n}^*R_{T_{q,n}^*}(z)+\\ &\quad + R_{T_{q,n}}(z)v_{q,n}f(z)v_{q,n}^*R_{T_{q,n}^*}(z) - R_{T_{q,n}}(z)u_nv_{q,n}^*R_{T_{q,n}^*}(z) =\\ &= R_{T_{q,n}}(z)\big[(I_{(n+1)q} - zT_{q,n})H_nT_{q,n}^* - u_nv_{q,n}^*\big]R_{T_{q,n}^*}(z)+\\ &\quad + R_{T_{q,n}}(z)v_{q,n}f(z)v_{q,n}^*\big[R_{T_{q,n}}(\overline{z})\big]^* =\\ &= R_{T_{q,n}}(z)(H_nT_{q,n}^* - zT_{q,n}H_nT_{q,n}^* - u_nv_{q,n}^*)R_{T_{q,n}^*}(z)+\\ &\quad + R_{T_{q,n}}(z)v_{q,n}f(z)[R_{T_{q,n}}(\overline{z})v_{q,n}]^* =\\ &= \Psi_{2n}(z) + E_{q,n}(z)f(z)E_{q,n}^*(\overline{z}). \end{aligned}$$

(b) Part (b) can be proved analogously. We omit the details. □

**Lemma 6.12.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \le \kappa$. Further, let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that*

$$\begin{aligned} &P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \quad \text{and}\\ &P_{2n+1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \quad \text{for all} \quad z \in \Pi_+. \end{aligned} \tag{69}$$

*Then the $[\alpha,\infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $\mathcal{M}_{\geq,1}^{q}([\alpha,\infty))$.*

*Proof.* (I) For all $z \in \Pi_+$, from Remark 4.4 we see that (11) holds true and, in view of (69), hence, that the block matrix on the left-hand side of (11) is non-negative Hermitian. Consequently, since $S$ is holomorphic in $\mathbb{C} \setminus [\alpha,\infty)$, Lemma 3.3 yields that $F := \operatorname{Rstr}_{\Pi_+} S$ belongs to $\mathcal{R}_{0,q}'(\Pi_+)$ and that the matricial spectral measure $\mu$ of $F$ fulfills $\mu(\mathbb{R}) \le s_0$. Thus, Remark 3.6 provides us $\sigma([\alpha,\infty)) = \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \mu([\alpha,\infty)) = \mu([\alpha,\infty)) \le \mu(\mathbb{R}) \le s_0$. Because of (69) and (9), we have $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. In particular, $s_0 \in \mathbb{C}_{\geq}^{q\times q}$. Hence,

$$s_0^* = s_0 \qquad \text{and} \qquad \{u^*\sigma([\alpha,\infty))u, u^*s_0u\} \subseteq [0,\infty) \qquad \text{for all } u \in \mathbb{C}^q. \tag{70}$$

(II) In the second part of the proof, we consider an arbitrary $n \in \mathbb{N}$ and an arbitrary $u \in \mathbb{C}^q$. From Remark 3.4 we see then that

$$\int_{[\alpha,\infty)} \left|\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}\right|(u^*\sigma u)(\mathrm{d}t) = n\int_{[\alpha,\infty)} \left|\frac{1}{t-(\mathrm{i}n+\alpha)}\right|(u^*\sigma u)(\mathrm{d}t) < \infty. \tag{71}$$

In view of

$$\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)} = -\frac{n^2}{|t-\alpha-\mathrm{i}n|^2} + \mathrm{i}\frac{(t-\alpha)n}{|t-\alpha-\mathrm{i}n|^2} \tag{72}$$

and (71), we obtain

$$\begin{aligned} &\int_{[\alpha,\infty)} \left|-\frac{n^2}{|t-\alpha-\mathrm{i}n|^2}\right|(u^*\sigma u)(\mathrm{d}t) =\\ &= \int_{[\alpha,\infty)} \left|\Re\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}\right]\right|(u^*\sigma u)(\mathrm{d}t) < \infty \end{aligned}$$

and

$$\begin{aligned}\int_{[\alpha,\infty)} \left| \frac{(t-\alpha)n}{|t-\alpha-\mathrm{i}n|^2} \right| (u^*\sigma u)(\mathrm{d}t) &= \\ = \int_{[\alpha,\infty)} \left| \Im\left[ \frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)} \right] \right| (u^*\sigma u)(\mathrm{d}t) &< \infty.\end{aligned} \tag{73}$$

For each $t \in [\alpha, \infty)$, we have

$$n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right] = \frac{(t-\alpha)n}{t-\alpha-\mathrm{i}n} = \frac{(t-\alpha)^2 n}{(t-\alpha)^2+n^2} + \mathrm{i}\frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}. \tag{74}$$

Consequently, the function $g_n \colon [\alpha,\infty) \to \mathbb{C}$ given by $g_n(t) := n[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1]$ fulfills $|\Re[g_n(t)]| = n(t-\alpha)^2[(t-\alpha)^2+n^2]^{-1} \le n = n\cdot 1_{[\alpha,\infty)}(t)$ and

$$\begin{aligned}|\Im[g_n(t)]| = (t-\alpha)n^2\left[(t-\alpha)^2+n^2\right]^{-1} &\le \\ \le 2|t-\alpha|n^2\left[(t-\alpha)^2+n^2\right]^{-1} &\le n = n\cdot 1_{[\alpha,\infty)}(t)\end{aligned}$$

for each $t \in [\alpha,\infty)$. This implies $\int_{[\alpha,\infty)} |\Re[g_n(t)]|(u^*\sigma u)(\mathrm{d}t) \le nu^*\sigma([\alpha,\infty))u < \infty$ and $\int_{[\alpha,\infty)} |\Im[g_n(t)]|(u^*\sigma u)(\mathrm{d}t) \le nu^*\sigma([\alpha,\infty))u < \infty$. Thus,

$$g_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, u^*\sigma u; \mathbb{C}). \tag{75}$$

Using Theorem 3.5, Remark 7.1, (72), and (73), we conclude

$$\begin{aligned}u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u &= u^*\left(\mathrm{i}n\int_{[\alpha,\infty)}[t-(\mathrm{i}n+\alpha)]^{-1}\sigma(\mathrm{d}t)\right)u = \\ &= \int_{[\alpha,\infty)} \frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}(u^*\sigma u)(\mathrm{d}t) = \\ &= \int_{[\alpha,\infty)} \left[-\frac{n^2}{|t-\alpha-\mathrm{i}n|^2} + \mathrm{i}\frac{(t-\alpha)n}{|t-\alpha-\mathrm{i}n|^2}\right](u^*\sigma u)(\mathrm{d}t) = \\ &= -n^2\int_{[\alpha,\infty)} \frac{1}{|t-\alpha-\mathrm{i}n|^2}(u^*\sigma u)(\mathrm{d}t) + \mathrm{i}n\int_{[\alpha,\infty)} \frac{t-\alpha}{|t-\alpha-\mathrm{i}n|^2}(u^*\sigma u)(\mathrm{d}t)\end{aligned}$$

and, in particular,

$$\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) = -n^2\int_{[\alpha,\infty)} |t-\alpha-\mathrm{i}n|^{-2}(u^*\sigma u)(\mathrm{d}t). \tag{76}$$

Taking into account $\sigma([\alpha,\infty)) \le s_0$, (76), and that $1 - n^2|t-\alpha-\mathrm{i}n|^{-1} = (t-\alpha)^2[(t-\alpha)^2+n^2]^{-1}$ holds true, for each $t \in [\alpha,\infty)$, we get

$$\begin{aligned}&\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*s_0u \ge \Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*\sigma([\alpha,\infty))u \\ &= \int_{[\alpha,\infty)}\left(1-\frac{n^2}{|t-\alpha-\mathrm{i}n|^2}\right)(u^*\sigma u)(\mathrm{d}t) = \int_{[\alpha,\infty)} \frac{(t-\alpha)^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \ge 0\end{aligned}$$

and, consequently,

$$\begin{aligned}[\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) + u^*s_0u]^2 &\ge \\ \ge [\Re(u^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)]u) &+ u^*\sigma([\alpha,\infty))u]^2.\end{aligned} \tag{77}$$

Because of (70), (77), and again (70), it follows

$$\begin{aligned}
&|nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0]u|^2 = n^2|u^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)]u+u^*s_0u|^2 = \\
&= n^2\Big([\Re(u^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)]u)+u^*s_0u]^2+[\Im(u^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)]u)]^2\Big) \geq \\
&\geq n^2\Big([\Re(u^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)]u)+u^*\sigma([\alpha,\infty))u]^2+[\Im(u^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)]u)]^2\Big) = \\
&= n^2|u^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)]u+u^*\sigma([\alpha,\infty))u|^2 = \\
&= |nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+\sigma([\alpha,\infty))]u|^2
\end{aligned}$$

and, therefore,

$$|nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0]u| \geq |nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+\sigma([\alpha,\infty))]u|. \tag{78}$$

Since $S$ belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$, the function $G\colon \Pi_+ \to \mathbb{C}^{q\times q}$ given by $G(w) := wS(w+\alpha)+s_0$ is holomorphic in $\Pi_+$. From Remark 4.4 we know that, for all $z \in \mathbb{C}\setminus[\alpha,\infty)$, equation (12) is true. Hence, from (69) we see that the block matrix on the left-hand side of (12) is non-negative Hermitian. Consequently, we conclude

$$\begin{aligned}
&\begin{bmatrix} -\alpha s_0+s_1 & G(w) \\ G^*(w) & \frac{G(w)-G^*(w)}{w-\overline{w}} \end{bmatrix} = \begin{bmatrix} -\alpha s_0+s_1 & wS(w+\alpha)+s_0 \\ [wS(w+\alpha)+s_0]^* & \frac{[wS(w+\alpha)+s_0]-[wS(w+\alpha)+s_0]^*}{w-\overline{w}} \end{bmatrix} = \\
&= \begin{bmatrix} -\alpha s_0+s_1 & [(w+\alpha)-\alpha]S(w+\alpha)+s_0 \\ ([(w+\alpha)-\alpha]S(w+\alpha)+s_0)^* & \frac{[(w+\alpha)-\alpha]S(w+\alpha)-([(w+\alpha)-\alpha]S(w+\alpha))^*}{w-\overline{w}} \end{bmatrix} \in \mathbb{C}_{\geq}^{2q\times 2q}.
\end{aligned} \tag{79}$$

Since $G$ is holomorphic, from (79) and Lemma 3.3 then $\sup_{y\in(0,\infty)}(y\|G(\mathrm{i}y)\|_{\mathrm{S}}) \leq \|-\alpha s_0+s_1\|_{\mathrm{S}}$ and, hence, $\sup_{n\in\mathbb{N}}(n\|\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0\|_{\mathrm{S}}) \leq \|-\alpha s_0+s_1\|_{\mathrm{S}}$ follows. Thus, the Bunjakowski–Cauchy–Schwarz inequality provides us

$$\begin{aligned}
|u^*(n[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0])u| &\leq \|n[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0]u\|_{\mathrm{E}} \cdot \|u\|_{\mathrm{E}} \leq \\
&\leq n\|\mathrm{i}n \cdot S(\mathrm{i}n+\alpha)+s_0\|_{\mathrm{S}} \cdot \|u\|_{\mathrm{E}}^2 \leq \|-\alpha s_0+s_1\|_{\mathrm{S}} \cdot \|u\|_{\mathrm{E}}^2.
\end{aligned} \tag{80}$$

For each $t \in [\alpha,\infty)$, we have $|t-\alpha| = \liminf_{n\to\infty}(t-\alpha)n^2[(t-\alpha)^2+n^2]^{-1}$. Then

$$\begin{aligned}
\int_{[\alpha,\infty)} |t-\alpha|(u^*\sigma u)(\mathrm{d}t) &= \int_{[\alpha,\infty)} \liminf_{n\to\infty} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \leq \\
&\leq \liminf_{n\to\infty} \int_{[\alpha,\infty)} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t),
\end{aligned} \tag{81}$$

by virtue of Fatou's lemma. Obviously, from (75) and (74) we infer

$$\begin{aligned}
&\int_{[\alpha,\infty)} \Im\left(n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\right)(u^*\sigma u)(\mathrm{d}t) = \\
&= \int_{[\alpha,\infty)} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t)
\end{aligned} \tag{82}$$

and

$$\begin{aligned}
&\int_{[\alpha,\infty)} \Im\left(n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\right)(u^*\sigma u)(\mathrm{d}t) = \\
&= \Im\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right](u^*\sigma u)(\mathrm{d}t)\right) \leq \\
&\leq \left|\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right](u^*\sigma u)(\mathrm{d}t)\right|.
\end{aligned} \tag{83}$$

(III) Since (75) holds true for every choice of $u \in \mathbb{C}^q$ and $n \in \mathbb{N}$, Remark 7.1 yields $g_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$. Hence, Remark 7.1 shows that

$$\begin{aligned}
&\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right](u^*\sigma u)(\mathrm{d}t) = \\
&= u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u
\end{aligned} \tag{84}$$

is valid for each $u \in \mathbb{C}^q$ and each $n \in \mathbb{N}$. Combining (82), (83), and (84), we have

$$\begin{aligned}
0 &\leq \int_{[\alpha,\infty)} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \leq \\
&\leq \left|u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u\right|
\end{aligned} \tag{85}$$

for each $u \in \mathbb{C}^q$ and each $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and all $t \in [\alpha,\infty)$, we see that $g_n(t) - n \cdot 1_{[\alpha,\infty)}(t) = \tilde{g}_n(t)$ holds true, where $\tilde{g}_n\colon [\alpha,\infty) \to \mathbb{C}$ is given by $\tilde{g}_n(t) := \mathrm{i}n^2[t-(\mathrm{i}n+\alpha)]^{-1}$. Thus, for each $n \in \mathbb{N}$, we get $\tilde{g}_n = g_n - n \cdot 1_{[\alpha,\infty)}$, and, since $g_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$, then $\tilde{g}_n \in \mathcal{L}^1([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$ and

$$\int_{[\alpha,\infty)} \tilde{g}_n \mathrm{d}\sigma = \int_{[\alpha,\infty)} g_n \mathrm{d}\sigma - n\int_{[\alpha,\infty)} 1_{[\alpha,\infty)} \mathrm{d}\sigma = \int_{[\alpha,\infty)} g_n \mathrm{d}\sigma - n\sigma([\alpha,\infty))$$

hold true as well. Consequently, for each $n \in \mathbb{N}$, we conclude

$$\begin{aligned}
&\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t) = \int_{[\alpha,\infty)} \frac{\mathrm{i}n^2}{t-(\mathrm{i}n+\alpha)}\sigma(\mathrm{d}t) + n\sigma([\alpha,\infty)) = \\
&= \mathrm{i}n^2\int_{[\alpha,\infty)} \frac{1}{t-(\mathrm{i}n+\alpha)}\sigma(\mathrm{d}t) + n\sigma([\alpha,\infty)) = n[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha) + \sigma([\alpha,\infty))].
\end{aligned}$$

Thus, because of (78), for each $u \in \mathbb{C}^q$ and each $n \in \mathbb{N}$, we obtain

$$\left|u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u\right| \leq |nu^*[\mathrm{i}n \cdot S(\mathrm{i}n+\alpha) + s_0]u|. \tag{86}$$

Taking into account (81), (85), (86), and (80), for each $u \in \mathbb{C}^q$, we get

$$\begin{aligned}\int_{[\alpha,\infty)} |t-\alpha|(u^*\sigma u)(\mathrm{d}t) &\leq \liminf_{n\to\infty} \int_{[\alpha,\infty)} \frac{(t-\alpha)n^2}{(t-\alpha)^2+n^2}(u^*\sigma u)(\mathrm{d}t) \leq \\ &\leq \liminf_{n\to\infty}\left|u^*\left(\int_{[\alpha,\infty)} n\left[\frac{\mathrm{i}n}{t-(\mathrm{i}n+\alpha)}+1\right]\sigma(\mathrm{d}t)\right)u\right| \leq \\ &\leq \liminf_{n\to\infty}|nu^*[\mathrm{i}n\cdot S(\mathrm{i}n+\alpha)+s_0]u| \leq \\ &\leq \liminf_{n\to\infty}\|-\alpha s_0+s_1\|_{\mathrm{S}}\cdot\|u\|_{\mathrm{E}}^2 = \|-\alpha s_0+s_1\|_{\mathrm{S}}\cdot\|u\|_{\mathrm{E}}^2 < \infty.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\int_{[\alpha,\infty)} |t|(u^*\sigma u)(\mathrm{d}t) &\leq \int_{[\alpha,\infty)} (|t-\alpha|+|\alpha|)(u^*\sigma u)(\mathrm{d}t) = \\ &= \int_{[\alpha,\infty)} |t-\alpha|(u^*\sigma u)(\mathrm{d}t) + \int_{[\alpha,\infty)} |\alpha|(u^*\sigma u)(\mathrm{d}t) \leq \\ &\leq \|-\alpha s_0+s_1\|_{\mathrm{S}}\cdot\|u\|_{\mathrm{E}}^2 + |\alpha|(u^*\sigma u)([\alpha,\infty)) < \infty\end{aligned}$$

is true for all $u \in \mathbb{C}^q$. Thus, Remark 7.1 provides us $\sigma \in \mathcal{M}_{\geq,1}^q([\alpha,\infty))$. □

**Lemma 6.13.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$, and let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Further, let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ and $P_{2n+1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ hold true for all $z \in \Pi_+$. Then:*

(a) *The $[\alpha,\infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $\mathcal{M}_{\geq,1}^q([\alpha,\infty))$.*
(b) *The function $\phi\colon [\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by $\phi(t) := \sqrt{t-\alpha}I_q$ belongs to $q\times q$-$\mathcal{L}^2([\alpha,\infty),\mathfrak{B}_{[\alpha,\infty)},\sigma;\mathbb{C})$ and $\sigma^{\#}\colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ defined by (14) belongs to $\mathcal{M}_{\geq}^q([\alpha,\infty))$.*
(c) *The function $\tilde{S}\colon \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by $\tilde{S}(z) := (z-\alpha)S(z)$ and the $[\alpha,\infty)$-Stieltjes transform $S^{[\sigma^{\#}]}$ of $\sigma^{\#}$ fulfill $\tilde{S}(z) = S^{[\sigma^{\#}]}(z) - \sigma([\alpha,\infty))$ for each $z \in \mathbb{C}\setminus[\alpha,\infty)$.*
(d) *The function $(\tilde{S})_\square := \operatorname{Rstr}_{\Pi_+}\tilde{S}$ belongs to $\mathcal{R}_q'(\Pi_+)$ and $(\tilde{\sigma})_\square\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}$ given by $(\tilde{\sigma})_\square(B) := \sigma^{\#}(B\cap[\alpha,\infty))$ is exactly the matricial spectral measure of $(\tilde{S})_\square$.*

*Proof.* (a) Part (a) is proved in Lemma 6.12.

(b) In view of (a), part (b) follows immediately from Remark 4.8.

(c) Let $z \in \mathbb{C}\setminus[\alpha,\infty)$. According to Remark 3.4 and Theorem 3.5, the function $g_{\alpha,z}\colon [\alpha,\infty) \to \mathbb{C}$ given by $g_{\alpha,z}(t) := (z-\alpha)/(t-z)$ belongs to $\mathcal{L}^1([\alpha,\infty),\mathfrak{B}_{[\alpha,\infty)},\sigma;\mathbb{C})$ and

$$(z-\alpha)S(z) = \int_{[\alpha,\infty)} \frac{z-\alpha}{t-z}\sigma(\mathrm{d}t)$$

is true. Consequently, in view of Lemma 7.2, we get that the pair $[g_{\alpha,z}I_q, 1_{[\alpha,\infty)}I_q]$ is left-integrable with respect to $\sigma$ and that

$$\tilde{S}(z) = (z-\alpha)S(z) = \int_{[\alpha,\infty)} (\frac{z-\alpha}{t-z}I_q)\sigma(\mathrm{d}t)I_q^*.$$

Due to Remark 7.3, then the pair $[g_{\alpha,z}I_q + 1_{[\alpha,\infty)}I_q, I_q]$ is left-integrable with respect to $\sigma$ and

$$\int_{[\alpha,\infty)} \left[\left(\frac{z-\alpha}{t-z}+1\right)I_q\right]\sigma(\mathrm{d}t)I_q^* = \int_{[\alpha,\infty)} \left(\frac{z-\alpha}{t-z}I_q\right)\sigma(\mathrm{d}t)I_q^* + \int_{[\alpha,\infty)} I_q\sigma(\mathrm{d}t)I_q^*$$

is fulfilled. Taking into account

$$\sigma([\alpha,\infty)) = \int_{[\alpha,\infty)} 1_{[\alpha,\infty)}\mathrm{d}\sigma = \int_{[\alpha,\infty)} (1_{[\alpha,\infty)}I_q)\mathrm{d}\sigma(1_{[\alpha,\infty)}I_q)^* = \int_{[\alpha,\infty)} I_q\sigma(\mathrm{d}t)I_q^*$$

and that $(z-\alpha)/(t-z)+1 = (t-\alpha)/(t-z)$ holds true for each $t \in [\alpha,\infty)$, we get then

$$\tilde{S}(z) = \int_{[\alpha,\infty)} \left(\frac{t-\alpha}{t-z}I_q\right)\sigma(\mathrm{d}t)I_q^* - \sigma([\alpha,\infty)). \tag{87}$$

Because of Lemma 7.2, Proposition 7.4, and (14), we have

$$\begin{aligned} &\int_{[\alpha,\infty)} \left(\frac{t-\alpha}{t-z}I_q\right)\sigma(\mathrm{d}t)I_q^* = \\ &= \int_{[\alpha,\infty)} \left[\left(\frac{1}{t-z}I_q\right)(\sqrt{t-\alpha}I_q)\right]\sigma(\mathrm{d}t)\left[I_q(\sqrt{t-\alpha}I_q)\right]^* = \\ &= \int_{[\alpha,\infty)} \left(\frac{1}{t-z}I_q\right)\sigma^{\#}(\mathrm{d}t)I_q^* = \int_{[\alpha,\infty)} \frac{1}{t-z}\sigma^{\#}(\mathrm{d}t) = S^{[\sigma^{\#}]}(z). \end{aligned}$$

Thus, from (87) it follows $\tilde{S}(z) = S^{[\sigma^{\#}]}(z) - \sigma([\alpha,\infty))$ for each $z \in \mathbb{C}\setminus[\alpha,\infty)$.

(d) In view of Theorem 3.5, we have $S^{[\sigma^{\#}]} \in \mathcal{S}_{0,q;[\alpha,\infty)}$. Thus, Remark 3.6 shows that $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]} \in \mathcal{R}'_{0,q}(\Pi_+) \subseteq \mathcal{R}'_q(\Pi_+)$, that the matricial spectral measure $\mu^{\#}$ of $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]}$ fulfills $\sigma^{\#} = \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \mu^{\#}$, and that $\mu^{\#}(\mathbb{R}\setminus[\alpha,\infty)) = \mu^{\#}((-\infty,\alpha)) = 0_{q\times q}$. Consequently, $(\tilde{\sigma})_{\square}$ is the matricial spectral measure of $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]}$. From Theorem 3.1 one can see that $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ given by $F(z) := -\sigma([\alpha,\infty))$ belongs to $\mathcal{R}'_q(\Pi_+)$ and that the matricial spectral measure $\theta$ of $F$ fulfills $\theta(B) = 0_{q\times q}$ for all $B \in \mathfrak{B}_{\mathbb{R}}$ (see also [13, Beispiel 1.2.1]). Since $\tilde{S}(z) = S^{[\sigma^{\#}]}(z) - \sigma([\alpha,\infty))$ is valid for all $z \in \mathbb{C}\setminus[\alpha,\infty)$, we get $(\tilde{S})_{\square} = \mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]} + F$. Since $\mathrm{Rstr}_{\Pi_+} S^{[\sigma^{\#}]}$ and $F$ both belong to $\mathcal{R}'_q(\Pi_+)$, from [26, Remark 4.4] we see that $(\tilde{S})_{\square} \in \mathcal{R}'_q(\Pi_+)$ and that $(\tilde{\sigma})_{\square} + \theta$ is the matricial spectral measure of $(\tilde{S})_{\square}$. In view of $(\tilde{\sigma})_{\square} + \theta = (\tilde{\sigma})_{\square}$, the proof is complete. □

**Lemma 6.14.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that*

$$P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \qquad \text{for all } z \in \mathbb{C}\setminus\mathbb{R}. \tag{88}$$

*Then the $[\alpha,\infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $\mathcal{M}_{\geq,2n}^{q}([\alpha,\infty))$ and the inequality $H_n^{[\sigma]} \leq H_n$ holds true.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that*

$$\left\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\right\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q} \qquad \text{for all } z \in \mathbb{C}\setminus\mathbb{R}. \tag{89}$$

*Then the $[\alpha,\infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $\mathcal{M}_{\geq,2n+1}^{q}([\alpha,\infty))$ and the inequality $H_{\alpha\rhd n}^{[\sigma]} \leq H_{\alpha\rhd n}$ holds true.*

*Proof.* (a) Because of (88), we get $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q} \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$ and, in particular, $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2n}$. In view of $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$, we see that the function $S$ is holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$ and, using additionally [26, Propositions 8.9 and 8.8], we also obtain $\mathrm{Rstr}_{\Pi_+} S \in \mathcal{R}_{0,q}'(\Pi_+) \subseteq \mathcal{R}_q'(\Pi_+)$. Let $f := S$ and let $F_{2n}\colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by (35). Using Remark 6.10 and [26, Propositions 8.9 and 8.8], we conclude that $F_{2n} \in \mathcal{R}_{0,(n+1)q}'(\Pi_+) \subseteq \mathcal{R}_{(n+1)q}'(\Pi_+)$ and that the matricial spectral measure $\mu_{2n}$ of $F_{2n}$ fulfills $\mu_{2n}(\mathbb{R}) \leq H_n$. Let $\Psi_{2n}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given (66). Since $s_j^* = s_j$ holds true for each $j \in \mathbb{Z}_{0,2n}$, from Lemma 6.11 we see that $\Psi_{2n}$ is a continuous matrix-valued function with $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. Furthermore, Lemma 6.11 yields (67). According to Remark 3.6, the matricial spectral measure $\sigma_\square$ of $\mathrm{Rstr}_{\Pi_+} S$ fulfills $\sigma = \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}} \sigma_\square$ and $\sigma_\square(\mathbb{R}\setminus[\alpha,\infty)) = 0$. Standard arguments of measure theory show that we can choose sequences $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ of real numbers such that

$$\sigma_\square(\{a_k\}) = 0, \quad \sigma_\square(\{b_k\}) = 0, \quad \mu_{2n}(\{a_k\}) = 0, \quad \mu_{2n}(\{b_k\}) = 0, \tag{90}$$

$$a_k < b_k, \quad \text{and} \quad (a_k, b_k) \subseteq (a_{k+1}, b_{k+1}) \tag{91}$$

hold true for each $k \in \mathbb{N}$ and that $\bigcup_{k=1}^\infty (a_k, b_k) = \mathbb{R}$. In view of $F_{2n} \in \mathcal{R}_{(n+1)q}'(\Pi_+)$, a matricial version of Stieltjes' inversion formula (see [14, Theorem 8.6]), and (90) provide us

$$\begin{aligned}\mu_{2n}((a_k,b_k)) &= \frac{1}{2}[\mu_{2n}(\{a_k\}) + \mu_{2n}(\{b_k\})] + \mu_{2n}((a_k,b_k)) = \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0+0}\int_{[a_k,b_k]} \Im F_{2n}(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x)\end{aligned} \tag{92}$$

for all $k \in \mathbb{N}$, where $\lambda^{(1)}$ is the Lebesgue measure defined on $\mathfrak{B}_{\mathbb{R}}$. The function $E_{q,n}\colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ given by (8) is holomorphic in $\mathbb{C}$. Moreover, $\Psi_{2n}$ is continuous with $\Psi_{2n}(\mathbb{R}) \subseteq \mathbb{C}_{\mathrm{H}}^{(n+1)q\times(n+1)q}$. Thus, for all $k \in \mathbb{N}$, we get from (67), a matricial version of Stieltjes' inversion formula (see [14, Theorem 8.6]) and (90) that

$$\begin{aligned}
&\frac{1}{\pi}\lim_{\epsilon\to 0+0}\int_{[a_k,b_k]}\Im F_{2n}(x+\mathrm{i}\epsilon)\lambda^{(1)}(\mathrm{d}x)=\\
&=\frac{1}{2}(E_{q,n}(a_k)\sigma_\square(\{a_k\})[E_{q,n}(a_k)]^*+E_{q,n}(b_k)\sigma_\square(\{b_k\})[E_{q,n}(b_k)]^*)+\\
&\quad+\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t)=\\
&=\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t).
\end{aligned} \tag{93}$$

Combining (93) and (92), we obtain

$$\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t)=\mu_{2n}((a_k,b_k))\le\mu_{2n}(\mathbb{R})\quad\text{for all } k\in\mathbb{N} \tag{94}$$

and, consequently,

$$\operatorname{tr}\left[\int_{(a_k,b_k)}E_{q,n}(t)\sigma_\square(\mathrm{d}t)E^*_{q,n}(t)\right]\le\operatorname{tr}[\mu_{2n}(\mathbb{R})]<\infty\quad\text{for all } k\in\mathbb{N}. \tag{95}$$

The trace measure $\tau:=\operatorname{tr}\sigma_\square$ of $\sigma_\square$ is a finite measure and $\sigma_\square$ is absolutely continuous with respect to $\tau$. We can choose a version $(\sigma_\square)'_\tau$ of the matricial Radon–Nikodym derivative of $\sigma_\square$ with respect to $\tau$ such that $(\sigma_\square)'_\tau(t)\in\mathbb{C}^{q\times q}_{\ge}$ for all $t\in\mathbb{R}$. For all $k\in\mathbb{N}$, then

$$g_k:=\|1_{(a_k,b_k)}(\operatorname{Rstr}_\mathbb{R}E_{q,n})\sqrt{(\sigma_\square)'_\tau}\|^2_\mathrm{F}\in\mathcal{L}^1(\mathbb{R},\mathfrak{B}_\mathbb{R},\tau;\mathbb{C})$$

and

$$\operatorname{tr}[\int_\mathbb{R}(1_{(a_k,b_k)}\operatorname{Rstr}_\mathbb{R}E_{q,n})\mathrm{d}\sigma_\square(1_{(a_k,b_k)}\operatorname{Rstr}_\mathbb{R}E_{q,n})^*]=\int_\mathbb{R}g_k\mathrm{d}\tau.$$

Thus, by virtue of (95), then

$$\int_\mathbb{R}g_k\mathrm{d}\tau\le\operatorname{tr}[\mu_{2n}(\mathbb{R})]<\infty \tag{96}$$

follows for all $k\in\mathbb{N}$. Obviously, $g\colon\mathbb{R}\to\mathbb{C}$ defined by $g(t):=\|E_{q,n}(t)\sqrt{(\sigma_\square)'_\tau}\|^2_\mathrm{F}$ is an $\mathfrak{B}_\mathbb{R}$-$\mathfrak{B}_\mathbb{C}$-measurable function with $g(\mathbb{R})\subseteq[0,\infty)$. For all $t\in\mathbb{R}$, we see that

$$\begin{aligned}
g(t)&=\left\|\left[\lim_{k\to\infty}1_{(a_k,b_k)}(t)\right]\cdot\left[(\operatorname{Rstr}_\mathbb{R}E_{q,n})\sqrt{(\sigma_\square)'_\tau}\right](t)\right\|^2_\mathrm{F}=\\
&=\lim_{k\to\infty}g_k(t)=\liminf_{k\to\infty}g_k(t).
\end{aligned} \tag{97}$$

In view of (97) and (96), Fatou's lemma yields then

$$\int_\mathbb{R}|g(t)|\tau(\mathrm{d}t)=\int_\mathbb{R}\liminf_{k\to\infty}g_k(t)\tau(\mathrm{d}t)\le\liminf_{k\to\infty}\int_\mathbb{R}g_k(t)\tau(\mathrm{d}t)\le\operatorname{tr}[\mu_{2n}(\mathbb{R})]<\infty,$$

and, consequently, $g\in\mathcal{L}^1(\mathbb{R},\mathfrak{B}_\mathbb{R},\tau;\mathbb{C})$. Because of Lemma 7.2, we get then $\operatorname{Rstr}_\mathbb{R}E_{q,n}\in(n+1)q\times q$-$\mathcal{L}^2(\mathbb{R},\mathfrak{B}_\mathbb{R},\sigma_\square;\mathbb{C})$. Hence, from $\sigma=\operatorname{Rstr}_{\mathfrak{B}_{[\alpha,\infty)}}\sigma_\square$

we obtain that $\operatorname{Rstr}_{[\alpha,\infty)} E_{q,n}$ belongs to $(n+1)q \times q\text{-}\mathcal{L}^2([\alpha,\infty), \mathfrak{B}_{[\alpha,\infty)}, \sigma; \mathbb{C})$, that

$$\int_{\mathbb{R}} E_{q,n}(t)\sigma_{\Box}(\mathrm{d}t)E^*_{q,n}(t) = \int_{[\alpha,\infty)} E_{q,n}(t)\sigma(\mathrm{d}t)E^*_{q,n}(t), \tag{98}$$

and that $\Theta_n \colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ defined by

$$\Theta_n(B) := \int_B E_{q,n}(t)\sigma_{\Box}(\mathrm{d}t)E^*_{q,n}(t) \tag{99}$$

is a well-defined non-negative Hermitian $(n+1)q \times (n+1)q$ measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$. Furthermore, applying Remark 4.7, we get $\sigma \in \mathcal{M}^q_{\geq,2n}([\alpha,\infty))$ and (13). Using (99), $\bigcup_{k=1}^{\infty}(a_k, b_k) = \mathbb{R}$, (91), $\Theta_n \in \mathcal{M}^{(n+1)q}_{\geq}(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$, (94), and $\mu_{2n} \in \mathcal{M}^{(n+1)q}_{\geq}(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$, we conclude

$$\begin{aligned}\int_{\mathbb{R}} E_{q,n}(t)\sigma_{\Box}(\mathrm{d}t)E^*_{q,n}(t) &= \Theta_n(\mathbb{R}) = \lim_{k\to\infty} \Theta_n((a_k, b_k)) = \\ = \lim_{k\to\infty} \int_{(a_k,b_k)} E_{q,n}(t)\sigma_{\Box}(\mathrm{d}t)E^*_{q,n}(t) &= \lim_{k\to\infty} \mu_{2n}((a_k, b_k)) = \mu_{2n}(\mathbb{R}).\end{aligned} \tag{100}$$

The combination of (13), (98), (100), and $\mu_{2n}(\mathbb{R}) \leq H_n$ provides us then

$$\begin{aligned}H^{[\sigma]}_n &= \int_{[\alpha,\infty)} E_{q,n}(t)\sigma(\mathrm{d}t)E^*_{q,n}(t) = \\ &= \int_{\mathbb{R}} \operatorname{Rstr}_{\mathbb{R}} E_{q,n} \mathrm{d}\sigma_{\Box} (\operatorname{Rstr}_{\mathbb{R}} E_{q,n})^* = \mu_{2n}(\mathbb{R}) \leq H_n.\end{aligned}$$

(b) Because of (89), we have $\{H_n, H_{\alpha\triangleright n}\} \subseteq \mathbb{C}^{(n+1)q\times(n+1)q}_{\geq} \subseteq \mathbb{C}^{(n+1)q\times(n+1)q}_{\mathrm{H}}$ and, consequently, $s^*_j = s_j$ for each $j \in \mathbb{Z}_{0,2n+1}$. Since $S$ belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}$, from (89) and Lemma 6.13 we infer that $\sigma$ belongs to $\mathcal{M}^q_{\geq,1}([\alpha,\infty))$, that $\sigma^{\#} \colon \mathfrak{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ defined by (14) belongs to $\mathcal{M}^q_{\geq}([\alpha,\infty))$, that $\tilde{S} \colon \mathbb{C} \setminus [\alpha,\infty) \to \mathbb{C}^{q\times q}$ given by $\tilde{S}(z) := (z-\alpha)S(z)$ is a function with $\operatorname{Rstr}_{\Pi_+} \tilde{S} \in \mathcal{R}'_q(\Pi_+)$, and that $(\sigma^{\#})_{\Box} \colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}$ given by $(\tilde{\sigma})_{\Box}(B) := \sigma^{\#}(B \cap [\alpha,\infty))$ is the matricial spectral measure of $(\tilde{S})_{\Box} := \operatorname{Rstr}_{\Pi_+} \tilde{S}$. Observe that Remark 4.8(b) shows that (15) holds true. Now part (b) can be proved analogous to part (a), where $F_{2n+1} \colon \Pi_+ \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by (37) and $\Psi_{2n+1} \colon \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ defined by (68) play the roles of $F_{2n}$ and $\Psi_{2n}$, respectively (for details, see also [48, Lemma 7.9]). □

**Remark 6.15.** *It is readily checked that if $E$ is non-negative Hermitian, then $\|B\|^2_{\mathrm{S}} \leq \|A\|_{\mathrm{S}} \cdot \|D\|_{\mathrm{S}}$ (see, e. g. [16, proof of Lemma 1.1.10]).*

**Remark 6.16.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q\times q}$. Using Remark 6.15 and the definition of the class $\mathcal{S}_{0,q;[\alpha,\infty)}$, it is readily checked that the following statements hold true:*

(a) *If $n \in \mathbb{N}_0$ is such that $2n \leq \kappa$ and if $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ is such that $P_{2n}^{[S]}(\mathrm{i}y) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for all $y \in (0,\infty)$, then*

$$\lim_{y\to\infty} R_{T_{q,n}}(\mathrm{i}y)[v_{q,n}S(\mathrm{i}y) - u_n] = 0. \tag{101}$$

(b) *If $\kappa \geq 1$ and $n \in \mathbb{N}_0$ are such that $2n+1 \leq \kappa$ and if $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ is such that $P_{2n+1}^{[S]}(\mathrm{i}y) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for each $y \in (0,\infty)$, then*

$$\lim_{y\to\infty} R_{T_{q,n}}(\mathrm{i}y)[v_{q,n}(\mathrm{i}y-\alpha)S(\mathrm{i}y) - (-\alpha u_n - y_{0,n})] = 0.$$

**Remark 6.17.** *Let $n \in \mathbb{N}_0$ and let $y \in \mathbb{R}$. If $u \in \mathbb{C}^{(n+1)q\times p}$ is such that $\lim_{y\to\infty}[u^* R_{T_{q,n}}(\mathrm{i}y)u] = 0$, then from Remark 4.2 one can easily see that $u = 0$ holds true.*

**Remark 6.18.** *Let $n \in \mathbb{N}$ and let $(d_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. If $d_0 = 0_{q\times q}$ and if the block Hankel matrix $[d_{j+k}]_{j,k=0}^{n}$ is non-negative Hermitian, then a characterization of non-negative Hermitian block matrices by their blocks (see [3, 23], or [16, Lemmata 1.1.9 and 1.1.7]), it is readily proved by induction that $d_j = 0_{q\times q}$ for all $j \in \mathbb{Z}_{0,2n-1}$.*

**Lemma 6.19.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for all $z \in \mathbb{C}\setminus\mathbb{R}$. Then the $[\alpha,\infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $\mathcal{M}_{\geq,2n}^{q}([\alpha,\infty))$ and $S$ belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$ and let $S \in \mathcal{S}_{0,q;[\alpha,\infty)}$ be such that $\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ holds true for each $z \in \mathbb{C}\setminus\mathbb{R}$. Then the $[\alpha,\infty)$-Stieltjes measure $\sigma$ of $S$ belongs to $\mathcal{M}_{\geq,2n+1}^{q}([\alpha,\infty))$ and $S$ belongs to $\mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.*

*Proof.* (a) Lemma 6.14 yields $\sigma \in \mathcal{M}_{\geq,2n}^{q}([\alpha,\infty))$ and $H_n^{[\sigma]} \leq H_n$. If $n = 0$, then $\sigma \in \mathcal{M}_{\geq}^{q}[[\alpha,\infty); (s_j)_{j=0}^{2n}, \leq]$ follows. Suppose now $n \geq 1$. Remark 6.16 shows that (101) is valid. Obviously, $\sigma \in \mathcal{M}_{\geq}^{q}[[\alpha,\infty); (s_j^{[\sigma]})_{j=0}^{2n}, \leq]$, where $(s_j^{[\sigma]})_{j=0}^{2n}$ is defined by (1). Thus, Proposition 4.9 and Remark 6.16 provide us $\lim_{y\to\infty} R_{T_{q,n}}(\mathrm{i}y)[v_{q,n}S(\mathrm{i}y) - u_n^{[\sigma]}] = 0$ where $s_{-1}^{[\sigma]} := 0_{q\times q}$ and where $u_n^{[\sigma]} := -\operatorname{col}(s_{j-1}^{[\sigma]})_{j=0}^{n}$. Using additionally (101), we can conclude

$$\lim_{y\to\infty} (u_n^{[\sigma]} - u_n)^* R_{T_{q,n}}(\mathrm{i}y)(u_n^{[\sigma]} - u_n) = 0.$$

Consequently, Remark 6.17 yields $u_n^{[\sigma]} = u_n$. Let $d_j := s_j - s_j^{[\sigma]}$ for each $j \in \mathbb{Z}_{0,2n}$. Then $u_n^{[\sigma]} = u_n$ and $n \geq 1$ imply $d_0 = 0_{q\times q}$. Furthermore, the inequality $H_n^{[\sigma]} \leq H_n$ shows that the block Hankel matrix $[d_{j+k}]_{j,k=0}^{n}$ is non-negative Hermitian. Thus, $d_{2n} \in \mathbb{C}_{\geq}^{q\times q}$ and Remark 6.18 yield $d_j = 0_{q\times q}$ for each $j \in \mathbb{Z}_{0,2n-1}$. Hence, $\sigma$ belongs to $\mathcal{M}_{\geq}^{q}[[\alpha,\infty); (s_j)_{j=0}^{2n}, \leq]$.

(b) Part (b) can be proved analogous to part (a). We omit the details. □

Now we are able to prove that the solution set of the (reformulated) truncated Stieltjes-type moment problem and the solution set of the corresponding system of the fundamental Potapov's matrix inequalities coincide.

**Theorem 6.20.** *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Let $\mathcal{D}$ be a discrete subset of $\Pi_+$ and let $S\colon \mathbb{C}\setminus[\alpha, \infty) \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Then the following statements are equivalent:*
   (i) *$S \in \mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.*
   (ii) *$P_{2n-1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ and $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for all $z \in \Pi_+ \setminus \mathcal{D}$.*

(b) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Then the following statements are equivalent:*
   (iii) *$S \in \mathcal{S}_{0,q;[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$.*
   (iv) *$\{P_{2n}^{[S]}(z), P_{2n+1}^{[S]}(z)\} \subseteq \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ for all $z \in \Pi_+ \setminus \mathcal{D}$.*

*Proof.* (i)⇒(ii), (iii)⇒(iv): Use Proposition 4.9.

(ii)⇒(i): Let $m := 2n$. Observe that the function $F := \operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} S$ is holomorphic. Because of (ii), the inequalities $P_{m-1}^{[F]}(z) \geq 0$ and $P_m^{[F]}(z) \geq 0$ hold true for each $z \in \Pi_+ \setminus \mathcal{D}$. From Theorem 6.5 we get then that there is a unique function $\hat{S} \in \mathcal{S}_{0,q;[\alpha,\infty)}$ such that $\operatorname{Rstr}_{\Pi_+\setminus\mathcal{D}} \hat{S} = F$, namely $\hat{S} = S$, and that $P_k^{[\hat{S}]}(z) \geq 0$ are valid for all $k \in \mathbb{Z}_{-1,m}$ and all $z \in \mathbb{C} \setminus \mathbb{R}$. Applying Lemma 6.19, we get then (i).

(iv)⇒(iii): Let $m := 2n+1$ and use the same argumentation as in the proof of the implication "(ii)⇒(i)". □

## 7. Particular results on non-negative Hermitian measures

In this appendix, we summarize some facts of the integration theory of non-negative Hermitian measures. We consider a measurable space $(\Omega, \mathfrak{A})$ and use the notation $\mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ to denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$.

**Remark 7.1.** *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ and let $f\colon \Omega \to \mathbb{C}$ be a function. Then standard arguments of measure and integration theory show that the following statements are equivalent:*
   (i) *$f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$.*
   (ii) *$f \in \mathcal{L}^1(\Omega, \mathfrak{A}, B^*\mu B; \mathbb{C})$ for all $B \in \mathbb{C}^{q\times p}$.*
   (iii) *$f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})$ where $\tau := \operatorname{tr} \mu$ is the trace measure of $\mu$.*

*If (i) holds true, then*

$$\int_A f \mathrm{d}(B^*\mu B) = B^*(\int_A f \mathrm{d}\mu) B$$

*for all $A \in \mathfrak{A}$ and all $B \in \mathbb{C}^{q\times p}$.*

**Lemma 7.2.** *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ and let $\mu_{\tau}'$ be a version of the Radon–Nikodym derivative of $\mu$ with respect to the trace measure $\tau := \operatorname{tr}\mu$ of $\mu$. Let $f\colon \Omega \to \mathbb{C}$ and $g\colon \Omega \to \mathbb{C}$ be $\mathfrak{A}$-$\mathfrak{B}_{\mathbb{C}}$-measurable functions. Then the following statements are equivalent:*

(i) $f\overline{g} \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$.

(ii) *The pair $[fI_q, gI_q]$ is left-integrable with respect to $\mu$.*

*If (i) is fulfilled, then*

$$\int_{\Omega} f\overline{g}\mathrm{d}\mu = \int_{\Omega} (fI_q)\mathrm{d}\mu(gI_q)^*.$$

Lemma 7.2 can be proved by standard methods of measure and integration theory.

**Remark 7.3.** *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ and let $m, n \in \mathbb{N}$. For each $j \in \mathbb{Z}_{1,m}$, let $p_j \in \mathbb{N}$ and let $\Phi_j\colon \Omega \to \mathbb{C}^{p_j \times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{p_j \times q}$-measurable matrix-valued function. For each $k \in \mathbb{Z}_{1,n}$, let $r_k \in \mathbb{N}$ and let $\Psi_k\colon \Omega \to \mathbb{C}^{r_k \times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{r_k \times q}$-measurable matrix-valued function. Suppose that, for every choice of $j \in \mathbb{Z}_{1,m}$ and $k \in \mathbb{Z}_{1,n}$ the pair $[\Phi_j, \Psi_k]$ is left-integrable with respect to $\mu$. Let $s, t \in \mathbb{N}$. For each $j \in \mathbb{Z}_{1,m}$, let $A_j \in \mathbb{C}^{s \times p_j}$, and, for each $k \in \mathbb{Z}_{1,n}$, let $B_k \in \mathbb{C}^{t \times r_k}$. Then it is readily checked that the pair*

$$[\sum_{j=1}^{m} A_j\Phi_j, \sum_{k=1}^{n} B_k\Psi_k]$$

*is left-integrable with respect to $\mu$ and that*

$$\int_{\Omega} \left(\sum_{j=1}^{m} A_j\Phi_j\right)\mathrm{d}\mu\left(\sum_{k=1}^{m} B_k\Psi_k\right)^* = \sum_{j=1}^{m}\sum_{k=1}^{n} A_j\left(\int_{\Omega} \Phi\mathrm{d}\mu\Psi^*\right)B_k^*.$$

**Proposition 7.4.** *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$, let $\tau := \operatorname{tr}\mu$ be the trace measure of $\mu$, and let $\mu_{\tau}'$ be a version of the Radon–Nikodym derivative of $\mu$ with respect to $\tau$. Furthermore, let $\Theta \in p \times q$-$\mathcal{L}^2(\Omega, \mathfrak{A}, \mu; \mathbb{C})$. Then:*

(a) *$\mu_{\Theta}\colon \mathfrak{A} \to \mathbb{C}^{p \times p}$ defined by*

$$\mu_{\Theta}(A) := \int_{A} \Theta\mathrm{d}\mu\Theta^*$$

*belongs to $\mathcal{M}_{\geq}^{p}(\Omega, \mathfrak{A})$.*

(b) *The non-negative Hermitian measure $\mu_{\Theta}$ is absolutely continuous with respect to $\tau$ and $\Theta\mu_{\tau}'\Theta^*$ is a version of the Radon–Nikodym derivative of $\mu_{\Theta}$ with respect to $\tau$.*

(c) *Let $r, s \in \mathbb{N}$, let $\Phi\colon \Omega \to \mathbb{C}^{r \times p}$ be an $\mathfrak{A}$-$\mathfrak{B}_{r \times p}$-measurable function and let $\Psi\colon \Omega \to \mathbb{C}^{s \times p}$ be an $\mathfrak{A}$-$\mathfrak{B}_{s \times p}$-measurable function. Then the pair $[\Phi, \Psi]$ is left-integrable with respect to $\mu_{\Theta}$ if and only if the pair $[\Phi\Theta, \Psi\Theta]$ is left-integrable with respect to $\mu$. In this case,*

$$\int_{\Omega} \Phi\mathrm{d}\mu_{\Theta}\Psi^* = \int_{\Omega} (\Phi\Theta)\mathrm{d}\mu(\Psi\Theta)^*.$$

Proposition 7.4 can be proved by standard arguments of measure and integration theory.

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Received 10.03.2019.