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THE CLASSICAL ORTHOGONAL POLYNOMIALS IN RESONANT EQUATIONS

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Резюме. У статті запропоновано теорію та алгоритм для знаходження часткових розв'язків резонансних рівнянь, пов'язаних із класичними ортогональними поліномами. Це дає можливість отримати загальний розв'язок у явному вигляді. Алгоритм підходить зокрема для систем комп'ютерної алгебри, наприклад, Maple. Резонансні рівняння є невід'ємною частиною різних застосувань, наприклад, ефективного функціонально-дискретного методу (FD-метод) для розв'язування операторних рівнянь і проблеми власних значень на основі збурень і ідеї гомотопії. Ці рівняння виникають також і в контексті суперсиметричних операторів Казіміра для ді-спінової алгебри, а також рівнянь типу $A^2u = f$ з заданим оператором A в деякому банаховому просторі, наприклад, бігармонічного рівняння.

Abstract. In the present paper we propose a theory and an algorithm for particular solutions of resonant equations related to the classical orthogonal polynomials. This enable us to obtain the general solution in explicit form. The algorithm is particulary suitable for computer algebra tools like Maple. The resonant equations are an essential part of various applications e.g. of the efficient functional-discrete method (FD-method) for solving operator equations and of eigenvalue problems based on the perturbation and the homotopy ideas. These equations arose also in the context of supersymmetric Casimir operators for the di-spin algebra as well as of the equations of type $A^2u = f$ with a given operator A in some Banach space, for example, of the biharmonic equation.

1. Introduction

There are various definitions of resonant equations, see e.g. [1, 2], where a boundary value problem is called resonant, when the operator, defined by the differential equation and by the boundary conditions does not possess the inverse. In the present paper we follow the definition from [7, 16, 19] and call an equation of the form $Lf = g$ with $Lg = 0$ resonant. In other words, the right-hand side of the resonant equation belongs to the kernel $K(L)$ of the operator L. These equations are interesting both from theoretical point of view and from the practical side in various applications. For example, in [16] was proposed the so called functional-discrete method (FD-method) for solving of operator equations and of eigenvalue problems. The method is based on the

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ideas of perturbation of the operator involved and on the homotopy idea. This approach was applied to various problems in particulary to eigenvalue problems in [9–13] and has been proven to possess a super exponential convergence rate. An essential part of the algorithm are some inhomogeneous equations with a resonant component in the sense of the definition above. Resonant equations arise in the theory of supersymmetric Casimir operators and of di-spin algebra [7]. They can be used to study the equations of the type $A^2u = 0$ with some given operator A. Substituting $Au = v$ we reduce this equation to the pair $Av = 0, Au = v$ where the second equation is resonant.

Their importance for praxis can be explained by the following example. Let the mathematical model of some system be the operator equation

$$Au - \lambda u = f$$

in some Hilbert space H, where the system is characterized by the operator A and the parameter λ. The element f describes external perturbation. The operator A is completely defined by its eigenvalues $\lambda_1, \lambda_2, ...$ and by the corresponding eigenvectors $u_1, u_2, ...$ If the perturbation is of the kind $f = \alpha u_k$ for some fixed α, k, i.e, the equation is resonant, then the solution of the mathematical model is $u = \frac{\alpha}{\lambda_k - \lambda} u_k$. One can see that the norm $\|u\|$, which can be interpreted as “amplitude”, tends to infinity as the system parameter λ tends to the so called resonant frequency λ_k. This phenomenon is called resonance and can be observed in the nature and many technical applications, e.g. in magnetic resonance imaging or nuclear spin tomography etc.

The present article deals with the resonant equations associated with the ordinary differential operators of the hypergeometric or confluent hypergeometric type, defining the classical orthogonal polynomials, i.e.

$$\mathcal{A}_n = \sigma(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx} + \lambda_n \tag{1}$$

where $\sigma(x) = a_2x^2 + a_1x + a_0$ is a polynomial of the degree not greater then two, $\tau(x) = b_1x + b_0$ - a polynomial of the degree not greater then one and $\lambda_n = \lambda(n) = -nb_1 - n(n-1)a_2$ depends on the integer parameter $n \geq 0$ but not on the variable x. We consider the differential operators defining the classical orthogonal polynomials (as the first linear independent solution of the corresponding homogeneous differential equation) and the corresponding functions of the second kind (the second linear independent solution) and the resonant equations of the first and of the second kind with the corresponding right hand side. We propose a theory describing particular solutions of the inhomogeneous resonant equations. We propose a theory and an algorithm to compute such solutions, which is especially convenient for the computer algebra tools like Maple and prove that the functions generated by this algorithm satisfy the resonant differential equation. Incidently we prove a new differentiation formula which represents the derivative of a classical orthogonal polynomial through the linear combination of the same and of a neighboring polynomial and which is unified for all classical orthogonal polynomials. Its coefficients are expressed through the coefficients of $\sigma(x), \tau(x)$ and the coefficients of the recurrence relation. Such formulas are well known in the literature (see e.g. [5, 5, 24, 28]), but for each concrete orthogonal polynomial only.

2. Representation of particular solutions of resonant equations

A classical orthogonal polynomial (Jacobi, Laguerre or Hermite) $\hat{P}_n(x)$ (see e.g. [6, 24, 28]) satisfies the homogeneous differential equation

$$\mathcal{A}_n u(x) = 0 \tag{2}$$

and is called also the function of the first kind. Let $\hat{Q}_n(x)$ be the second linear independent solution of the homogeneous differential equation, which is called the function of the second kind. Then the general solution of the homogeneous differential equation (2) is given by

$$u(x) = c_1 \hat{P}_n(x) + c_2 \hat{Q}_n(x), \tag{3}$$

where c_1, c_2 are arbitrary constants.

Let us consider the resonant equations of the type

$$\mathcal{A}_n u_n(x) = R_n(x). \tag{4}$$

In the case when $R_n(x)$ is a classical orthogonal polynomial $\hat{P}_n(x)$ (the function of the first kind), the inhomogeneous differential equation (4) is called the resonant equation of the first kind. The inhomogeneous differential equation (4) with the right-hand side $\hat{Q}_n(x)$ instead of $R_n(x)$ is called the resonant differential equation of the second kind. Both functions $\hat{P}_n(x)$ and $\hat{Q}_n(x)$ satisfy the same homogeneous differential equation (2) and the same recurrence relation

$$R_{n+1}(x) = (\alpha(n)x + \beta(n))R_n(x) - \gamma(n)R_{n-1}(x), \ n = 1, 2, ... \tag{5}$$

with some coefficients $\alpha(n) = \alpha_n, \beta(n) = \beta_n, \gamma(n) = \gamma_n$ (see e.g. [6, 23, 24, 28]). If we change in the differential operator $\mathcal{A}_n$ the integer $n \geq 0$ to a real ν then the corresponding solutions $\hat{P}_\nu(x), \hat{Q}_\nu(x)$ become the hypergeometric or confluent hypergeometric functions [5,6]. Since $R_n(x)$ satisfies the homogeneous differential equation (2), then we can differentiate this equation by n in the following way: 1) switch from the integer $n \geq 0$ to a real ν, 2)differentiate by ν and 3)replace the real ν by the integer n. In regard of (1) we obtain $\mathcal{A}_n \frac{dR_n}{dn} = -\lambda'(n) R_n$ or $A_n \left(-\frac{1}{\lambda'(n)} \frac{dR_n}{dn}\right) = R_n$, which means that the function

$$u_n(x) = -\frac{1}{\lambda'(n)} \frac{dR_n}{dn} \tag{6}$$

is a particular solution of the resonant equation. Using this relation and differentiating (5) by n we obtain

$$\begin{aligned} u_{n+1}(x) = -\frac{1}{\lambda'(n+1)} & \left[-\lambda'(n)(\alpha(n)x + \beta(n))\, u_n(x) + \right. \\ & + \lambda'(n-1)\, \gamma(n) u_{n-1}(x) + \\ & \left. + \left(\alpha'(n)x + \beta'(n)\right) R_n(x) - \gamma'(n) R_{n-1}(x)\right], \ n = 1, 2, ... \end{aligned} \tag{7}$$

The general solution of the resonant equation (4) is given by

$$u(x) = c_1\hat{P}_n(x) + c_2\hat{Q}_n(x) + u_n^{(k)}(x), \tag{8}$$

where $u_n^{(k)}(x), k = 1, 2$ is a particular solution of the corresponding inhomogeneous resonant equation. Below we propose an algorithm to find the particular solutions, which is especially suitable for computer algebra tools like Maple etc. Since our algorithm below for particular solutions of the resonant differential equations of the first and of the second kind (3) is based on the same recurrence relation (5) it is valid for the resonant equations of both types and we use the notation $R_n(x)$ below for both $\hat{P}_n(x)$ and $\hat{Q}_n(x)$. The following general result on the particular solutions of the resonant equations has been proven in [19].

Theorem 1. *Let $A : X \to X$ be a linear operator acting in a Banach space X, the set $K(A) \subset X$ be the kernel of A and a connected set $\Sigma(A)$ in the complex plane be the spectral set of A. If $f(\lambda) \in K(A-\lambda E), \lambda \in \Sigma(A)$ is a differentiable function then the solution of the resonant equation*

$$(A - \lambda E)u = f(\lambda) \tag{9}$$

can be represented by

$$u(\lambda) = \frac{df(\lambda)}{d\lambda} \tag{10}$$

The proof of this theorem is based on the equivalent equation

$$(A - \lambda_0 I)\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f(\lambda)$$

with some fixed λ_0 and on passing to the limit $\lambda \to \lambda_0$.

3. An algorithm for computation of particular solutions. A general differentiation formula for classical orthogonal polynomials

Now we are at the position to formulate an algorithm for the particular solutions of the resonant equations associated with a differential operator of the hypergeometric type, defining classical orthogonal polynomials. This algorithm is especially suitable for computer algebra tools like Maple etc.

Algorithm 1. *Problem: Given a resonant equation of the first or of the second kind, return a given number N of particular solutions.*

Inputs: The number N and the right hand side $R_\nu(x)$ of the resonant equation.

Outputs: The particular solutions $u_0(x), u_1(x), ..., u_N(x)$.

1. Find

$$\chi_0(x) = -\frac{1}{\lambda'(\nu)}\left.\frac{d\,R_\nu(x)}{d\nu}\right|_{\nu=0}, \ \chi_1(x) = -\frac{1}{\lambda'(\nu)}\left.\frac{d\,R_\nu(x)}{d\nu}\right|_{\nu=1}. \tag{11}$$

Due to (6) these are particular solutions.

2. Compute $u_2(x)$ in accordance with (7) using the initial conditions

$$u_0(x) = \chi_0(x) + c_0P_0(x) + d_0Q_0(x),\ u_1(x) = \chi_1(x) + c_1P_1(x) + d_1Q_1(x) \tag{12}$$

with undefined coefficients c_0, c_1, d_0, d_1.

3. Find c_0, c_1, d_0, d_1 *from the condition that* $u_2(x)$ *satisfies the resonant differential equation (3).*

4. For $n = 2$ *step 1 until* $n = N$ *compute* $u_n(x)$ *by (7) and return* $u_n(x)$.

Using Theorem 1 we prove below that the sequence $u_n(x)$ generated by this algorithm satisfy the resonant equation for all $n = 0, 1, 2, \dots$.

Theorem 2. *All functions* $u_{n+1}(x)$ *generated by the recursion (7) with the initial conditions (12) satisfy the resonant differential equation (4).*

Proof. We use the mathematical induction and, first of all, note that the functions $u_p(x)$, $p = 0, 1, 2$ satisfy the resonant equation by construction and due to Theorem 1. Let us assume that all the functions $u_p(x), p = 0, 1, \dots, n$ satisfy the resonant differential equation (4) and prove that then the function $u_{n+1}(x)$ is its solution too.

First of all we notice that

$$\begin{aligned}
&\mathcal{A}_{n+1}u_n(x) = \sigma(x)\frac{d^2u_n}{dx^2} + \tau(x)\frac{du_n}{dx} + \lambda(n+1)u_n = \\
&\quad = \mathcal{A}_n u_n(x) + (\lambda(n+1) - \lambda(n))u_n = R_n(x) + (\lambda(n+1) - \lambda(n))u_n, \\
&\mathcal{A}_{n+1}u_{n-1}(x) = \mathcal{A}_{n-1}u_{n-1}(x) + (\lambda(n+1) - \lambda(n-1))u_{n-1} = \\
&\quad = R_{n-1}(x) + (\lambda(n+1) - \lambda(n-1))u_{n-1}, \\
&\frac{d}{dx}\left[(\alpha'(n)x + \beta'(n))R_n(x) - \gamma'(n)R_{n-1}(x)\right] = \\
&\quad = \alpha'(n)R_n(x) + (\alpha'(n)x + \beta'(n))\frac{dR_n(x)}{dx} - \gamma'(n)\frac{dR_{n-1}(x)}{dx},
\end{aligned} \tag{13}$$

Further we use the differentiation formula for the classical orthogonal polynomials (which is the same for the functions of the second kind too) and which represents the derivative of these functions through the same functions of index n and the function of the index $n-1$ with some coefficients independent of x (see, e.g. [23, §4, (12)] or [6, p.171,(15); p.189, (12); p.193, (14)] for concrete classical orthogonal polynomials):

$$\begin{aligned}
\sigma(x)\frac{dR_n}{dx} &= [q_1(n)x + q_2(n)]R_n(x) + s(n)R_{n-1}(x) = \\
&= [q_{1,n}x + q_{2,n}]R_n(x) + s_nR_{n-1}(x).
\end{aligned} \tag{14}$$

Substituting this expression as well as (13)into the formula for $\mathcal{A}_{n+1}u_{n+1}(x)$, we obtain

$$\begin{aligned}
&\mathcal{A}_{n+1}u_{n+1}(x) = \frac{\lambda'(n)}{\lambda'(n+1)}(\alpha(n)x + \beta(n))(\lambda(n+1) - \lambda(n))u_n(x) \\
&\frac{1}{\lambda'(n+1)}(\alpha'(n)x + \beta'(n))(\lambda(n+1) - \lambda(n))R_n(x) + \\
&\quad + \frac{\lambda'(n)}{\lambda'(n+1)}(\alpha(n)x + \beta(n))R_n(x) -
\end{aligned} \tag{15}$$

$$
\begin{aligned}
&- \frac{2\sigma(x)}{\lambda'(n+1)} \left[-\lambda'(n)\alpha(n)\frac{du_n(x)}{dx} + \alpha'(n)\frac{dR_n(x)}{dx} \right] - \\
&- \frac{\tau(x)}{\lambda'(n+1)} \left[-\lambda'(n)\alpha(n)u_n(x) - \alpha'(n)R_n(x) \right] - \\
&- \frac{1}{\lambda'(n+1)} (\lambda'(n-1)\gamma(n)(\lambda(n+1) - \lambda(n-1))u_{n-1}(x) + \\
&+ \lambda'(n-1)\gamma(n)R_{n-1}(x) - \\
&- \gamma'(n)(\lambda(n+1) - \lambda(n-1))R_{n-1}(x)) = \\
= &\frac{\lambda'(n)}{\lambda'(n+1)} \{ \ (\alpha(n)x + \beta(n))(\lambda(n+1) - \lambda(n)) + \\
&+ 2\alpha(n) \left[q_1(n)\, x + q_2(n) \right] + \lambda\tau(x) \ \} \, u_n(x) + \\
&+ \frac{\lambda'(n-1)}{\lambda'(n+1)} \{ \ \gamma(n)(\lambda(n+1) - \lambda(n)) + 2\alpha(n)s\,(n) \ \} \, u_{n-1}(x) + \mathcal{R}(x),
\end{aligned}
$$

where $\mathcal{R}(x)$ contains the functions $R_{n-1}(x), R_n(x)$ and its derivatives but not $u_{n-1}(x), u_n(x)$. Setting the coefficients in front of $u_{n-1}(x), u_n(x)$ equal to zero, we obtain

$$
\begin{aligned}
s(n) &= -\frac{\gamma(n)}{\alpha(n)} \left[b_1 + (2n-1)a_2 \right], \\
q_1(n) &= -\frac{1}{2} \left[b_1 + \lambda(n+1) - \lambda(n) \right] = n a_2, \\
q_2(n) &= -\frac{b_0}{2} - \frac{\beta(n)}{2\alpha(n)} \left[\lambda(n+1) - \lambda(n) \right] = -\frac{b_0}{2} + \frac{\beta(n)}{2\alpha(n)} \left[b_1 + 2n a_2 \right].
\end{aligned}
\tag{16}
$$

It is easy to check that the coefficients of the differentiation formulas for all classical orthogonal polynomials satisfy (16). For example, the Laguerre polynomials are defined by the confluent hypergeometric differential equation with $\sigma(x) = a_2x^2 + a_1x + a_0 = x, \tau(x) = b_1x + b_0 = \alpha + 1 - x, \lambda_n = \lambda(n) = -nb_1 - n(n-1)a_2 = n$; i.e. $a_2 = 0, a_1 = -1, a_0 = 0, b_1 = -1, b_0 = \alpha + 1$. Besides they satisfy the recurrence relation [6, §10.2]

$$
(n+1)L_{n+1}^{\alpha}(x) - (2n + \alpha + 1 - x)L_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x) = 0, \tag{17}
$$

i.e. $\alpha(n) = -\frac{1}{n+1}, \beta(n) = \frac{2n+\alpha+1}{n+1}, \gamma(n) = -\frac{n+\alpha}{n+1}$. Due to (16) we obtain $s(n) = n + \alpha, q_1(n) = 0, q_2(n) = -\frac{\alpha+1}{2} + \frac{2n+\alpha+1}{2} = n$ and (14) implies the well known differentiation formula (see e.g. [6, §10.2])

$$
x\frac{dL_n^{\alpha}(x)}{dx} = nL_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x). \tag{18}
$$

Now, using the recurrent relation (5) we obtain from (15) the equality

$$
\mathcal{A}_{n+1}u_{n+1}(x) = R_{n+1}(x), \tag{19}
$$

which proves the assertion. □

Remark 1. *At once with (16) we have obtained the coefficients of the general differentiation formula (14) which is valid for the general classical orthogonal* *polynomials and contains all particular cases of the polynomials by Jacobi, Laguerre, Hermite known from the literature [6, 24, 28]. This formula is much more convenient for use then the corresponding formula from [23, 24].*

4. Examples

Example 1. This example demonstrates the use of Algorithm 1 for the representation of the general solution of the following Laguerre resonant equation of the first kind

$$x\frac{d^2u(x)}{dx^2} + (1+\alpha-x)\frac{du(x)}{dx} + n\,u(x) = L_n^\alpha(x) \tag{20}$$

where

$$\begin{aligned} L_n^\alpha(x) &= \frac{(\alpha+1)_n}{n!}\Phi(-n,\alpha+1,x) = \\ &= \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\Phi(-n,\alpha+1,x) = \sum_{k=0}^{n}\left(\begin{array}{c} n+\alpha \\ n-k \end{array}\right)\frac{(-x)^k}{k!} \end{aligned} \tag{21}$$

is the Laguerre polynomial satisfying the corresponding homogeneous differential equation and $\Phi(-n,\alpha+1,x)$ is the confluent hypergeometric function satisfying a degenerate form of the hypergeometric differential equation when two of the three regular singularities merge into an irregular singularity [5, p. 189, formula (14)] and $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer-Symbol.

The second linear independent solution of the homogeneous differential equation is the Laguerre function of the second kind $l_n^\alpha(x)$ (see e.g. [25, pp.16,20]). The general solution of the homogeneous Laguerre differential equation is given by

$$u(x) = c_1L_n^\alpha(x) + c_2l_n^\alpha(x) \tag{22}$$

with arbitrary constants c_1, c_2. The general solution of the Laguerre resonant (inhomogeneous) equation is given by

$$u(x) = c_1L_n^\alpha(x) + c_2l_n^\alpha(x) + u_n(x) \tag{23}$$

where c_1, c_2 are arbitrary constants and $u_n(x)$ is a particular solution of the inhomogeneous (resonant) equation.

Solving the corresponding differential equation for the Laguerre function of the second kind [25, pp.16,20] by Maple we obtain the following representation of this function for non-integer α:

$$\begin{aligned} &l_n^\alpha(x) = \Gamma(1-\alpha)L_n^\alpha(x) - (-x)^{-\alpha}\,{}_1F_1(-n-\alpha,-\alpha+1;x) = \\ &= \Gamma(1-\alpha,-x)L_n^\alpha(x) - (-x)^{-\alpha}p_n^\alpha(x)\exp(x), \\ &p_{n+1}^\alpha(x) = \frac{1}{n+1}\left[(2n+\alpha+1-x)p_n^\alpha(x) - (n+\alpha)\,p_{n-1}^\alpha(x)\right], \\ &\quad n = 1,2,..., \\ &p_0^\alpha(x) = 1, p_1^\alpha(x) = 1-x. \end{aligned} \tag{24}$$

For non-negative natural $\alpha \in \mathbb{N}$ we have

$$\begin{aligned} l_n^\alpha(x) &= \mathrm{Ei}_1(-x) L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x)\, \exp(x), \\ p_{-1}^\alpha(x) &= (\alpha-1)!, \\ p_0^\alpha(x) &= x^{\alpha-1} + x^\alpha \left[U\,(2,2,-x) + (-1)^\alpha \alpha!\, U(1+\alpha, 1+\alpha, -x) \right], \end{aligned} \tag{25}$$

where

$$\mathrm{Ei}_1(x) = \int_z^\infty \frac{e^{-t}}{t}\, dt, \qquad |\mathrm{Arg}(z)| < \pi \tag{26}$$

is the exponential integral and $U(a,b,z)$ is the Kummer's function of the second kind. The last one is a solution of the Kummer's differential equation

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0. \tag{27}$$

The other linear independent solution of this differential equation is the Kummer's function of the first kind M defined e.g. by the hypergeometric series:

$$M(a,b,z) = \sum_{n=0}^\infty \frac{(a)_n z^n}{(b)_n n!} = {}_1F_1(a;b;z). \tag{28}$$

The Kummer's function of the second kind can be represented also as

$$\begin{aligned} U(a,b,z) &= \frac{\Gamma(1-b)}{\Gamma(a+1-b)} M(a,b,z) + \\ &+ \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a+1-b, 2-b, z). \end{aligned} \tag{29}$$

Note that the function at the second initial condition in (25) solves the following difference initial value problem

$$\begin{aligned} p_0^\alpha(x) &= x\, p_0^{\alpha-1}(x) + (\alpha-1)!, \quad \alpha = 1,2,..., \\ p_0^0(x) &= 0. \end{aligned} \tag{30}$$

Using Theorem 1 we can represent the particular solutions of the Laguerre resonant equation of the first kind also by

$$u_n(x) = \frac{\partial}{\partial \nu} \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)} \Phi(-n, \alpha+1, x) \bigg|_{n=\nu}, \; n = 0,1,... \tag{31}$$

From this expression we extract the following particular solutions containing the elementary functions only

$$\begin{aligned} \chi_0^\alpha(x) &= u_0(x) = -\ln(x) + \sum_{p=0}^{\alpha-1} \frac{(\alpha-p)_{p+1}}{(p+1)x^{p+1}}, \\ \chi_1^\alpha(x) &= u_1(x) = -L_1^\alpha(x)\ln(x) + \sum_{p=0}^{\alpha} \frac{k_p(\alpha)}{x^p}, \end{aligned} \tag{32}$$

where

$$k_{p+1}(\alpha) = p\sum_{i=1}^{\alpha-1} k_p(i), \quad p = 1, 2, ..., \alpha - 1,$$
$$k_1(\alpha) = \frac{\alpha(\alpha+1)}{2}, \; k_0(\alpha) = -\alpha - 2, \; \alpha = 2, 3, ..., \tag{33}$$

At the first step of our Algorithm 1 we use the ansates

$$u_0^\alpha(x) = \chi_0^\alpha(x) + c_0 L_0^\alpha(x) + d_0 l_0^\alpha(x),$$
$$u_1^\alpha(x) = \chi_1^\alpha(x) + c_1 L_1^\alpha(x) + d_1 l_1^\alpha(x) \tag{34}$$

with undefined coefficients c_0, d_0, c_1, d_1, obtain $u_2^\alpha(x)$ from the corresponding recurrence formula of our algorithm and choose c_0, d_0, c_1, d_1 so that $u_2^\alpha(x)$ satisfies the resonant differential equation. We get $d_0 = 0, d_1 = 0$ and $c_1 = 1 + c_0$. Now one can verify that

$$u_n^\alpha(x) = -L_n^\alpha(x)\ln(x) + \frac{p_n^\alpha(x)}{x^\alpha}, \tag{35}$$

where the polynomials $p_n^\alpha(x)$ satisfy the recurrence equation

$$p_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1} p_n^\alpha(x) - \frac{n+\alpha}{n+1} p_{n-1}^\alpha(x) +$$
$$+ \frac{(\alpha-1-x)}{(n+1)^2} L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2} L_{n-1}^\alpha(x), \; n = 1, 2, ... \tag{36}$$

with the initial conditions

$$p_0^\alpha(x) = \sum_{p=0}^{\alpha-1} \frac{x^{\alpha-p-1}(\alpha-p)_{p+1}}{p+1} + c_0 x^\alpha,$$
$$p_1^\alpha(x) = \sum_{p=0}^{\alpha} x^{\alpha-p} k_p(\alpha) + (1+c_0) x^\alpha L_1^\alpha(x). \tag{37}$$

Example 2. Now, let us consider the Laguerre resonant equation of the second kind

$$x\frac{d^2u(x)}{dx^2} + (1+\alpha-x)\frac{du(x)}{dx} + n\,u(x) = l_n^\alpha(x) \tag{38}$$

whith the Laguerre function of the second kind $l_n^\alpha(x)$. Due to Theorem 1 the formula

$$u_n(x) = -\frac{d}{d\nu} \;\; l_\nu^\alpha(x) \mid_{\nu=n} \tag{39}$$

defines a particular solution of (38), so that its general solution is given by

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) + u_n(x). \tag{40}$$

The use of formula (39) for arbitrary n is rather burdensome, therefore we use Algorithm 1, where we for the sake of simplicity set $\alpha = 0$. Solving differential equation (38) with Maple for $n = 0$, $n = 1$ we get

$$\begin{aligned}
\chi_0(x) &= -\int_1^x \frac{\exp(t)}{t} \int_1^t \mathrm{Ei}_1(-\xi)\exp(-\xi)d\xi dt,\\
\chi_1(x) &= [(1-x)\ \mathrm{Ei}_1(-x) - \exp(x)]\times\\
&\times \int_1^x [1+\mathrm{Ei}_1(-\xi)(-1+\xi)\exp(-\xi)](-1+\xi)d\xi +\\
&+ \int_1^x \exp(-\xi)[\mathrm{Ei}_1(-\xi)(-1+\xi)+\exp(-\xi)]^2 d\xi(-1+x).
\end{aligned} \tag{41}$$

As the ansatzes for initial values of our algorithm we use

$$u_0^0(x) = \chi_0(x) + c_0\mathrm{Ei}_1(-x) + d_0, \quad u_1^0(x) = \chi_1(x) + c_1 l_1^0(x) + d_1 L_1^0(x) \tag{42}$$

with undefined constants c_0, d_0, c_1, d_1. Differentiating the recurrence equation for the Laguerre functions of the second kind by n and in regard of (39) we obtain the following recurrence relation for particular solutions

$$\begin{aligned}
u_{n+1}^0(x) &= \frac{2n+1-x}{n+1}u_n^0(x) - \frac{n}{n+1}u_{n-1}^0(x) -\\
&- \frac{1+x}{(n+1)^2}l_n^0(x) + \frac{1}{(n+1)^2}l_{n-1}^0(x).
\end{aligned} \tag{43}$$

We substitute (42) into this equation with $n = 1$ and demand that the obtained function $u_2^0(x)$ satisfies the resonant differential equation (38) with $n = 2$, then we obtain

$$\begin{aligned}
c_0 &= -\mathrm{Ei}_1(-1)\exp(-1) - 1,\\
d_0 &= -[\mathrm{Ei}_1(-1)\exp(-1/2) + \exp(1/2)]^2,\\
c_1 &= 0, \quad d_1 = 0.
\end{aligned} \tag{44}$$

It can be verified by substitution into (43) that the following representation holds true

$$\begin{aligned}
u_n^0(x) &= p_n^0(x)\chi_1(x) + q_n^0(x)\chi_0(x) + v_n^0(x)\mathrm{Ei}_1(-x) +\\
&+ w_n^0(x)\exp(x) + q_n^0(x)d_0,
\end{aligned} \tag{45}$$

where the polynomials $p_n^0(x), q_n^0(x)$ satisfy the recurrence relation for the Laguerre polynomials with the initial conditions

$$p_0^0(x) = 0, \quad p_1^0(x) = 1, \quad q_0^0(x) = 1, \quad q_1^0(x) = 0.$$

The polynomials $w_n^0(x)$ satisfy the inhomogeneous recurrence relation for the Laguerre polynomials

$$\begin{aligned}
w_{n+1}^0(x) &= \frac{2n+1-x}{n+1}w_n^0(x) - \frac{n}{n+1}w_{n-1}^0(x) - \frac{1+x}{(n+1)^2}p_n^0(x) +\\
&+ \frac{1}{(n+1)^2}p_{n-1}^0(x), \quad n = 1, 2, \ldots
\end{aligned} \tag{46}$$

with the initial conditions

$$w_1^0(x) = 0, \quad w_2^0(x) = \frac{x+1}{4}.$$

The polynomials $v_n^0(x)$ solve the following discrete initial value problem

$$\begin{aligned} v_{n+1}^0(x) &= \frac{2n+1-x}{n+1} v_n^0(x) - \frac{n}{n+1} v_{n-1}^0(x) - \frac{1+x}{(n+1)^2} L_n^0(x) + \\ &+ \frac{1}{(n+1)^2} L_{n-1}^0(x), \quad n = 1, 2, ..., \\ v_1^0(x) &= 0, \quad v_2^0(x) = \frac{x^2 - 2c_0}{4}. \end{aligned} \tag{47}$$

Below we give some particular solutions of the Laguerre resonant equation of the second kind obtained by our algorithm:

$$\begin{aligned} u_0^0(x) &= \chi_0(x) + c_0 \mathrm{Ei}_1(-x) + d_0, \ u_1^0(x) = \chi_1(x), \\ u_2^0(x) &= -\frac{x-3}{2}\chi_1(x) - \frac{1}{2}\chi_0(x) + \\ &+ \frac{x^2 - 2c_0}{4} \mathrm{Ei}_1(-x) - \frac{x^2-1}{8} \exp(x) - \frac{1}{2} d_0, \\ u_3^0(x) &= \left(\frac{1}{6}x^2 - \frac{4}{3}x + \frac{11}{6}\right)\chi_1(x) + \left(\frac{1}{6}x - \frac{5}{6}\right)\chi_0(x) + \\ &+ \left(-\frac{5}{36}x^3 + \frac{7}{12}x^2 + \frac{c_0}{6}x - \frac{5c_0}{6}\right)\mathrm{Ei}_1(-x) + \\ &+ \left(\frac{1}{24}x^3 - \frac{11}{72}x^2 - \frac{23}{72}x - \frac{1}{72}\right)\exp(x) + \left(\frac{1}{6}x - \frac{5}{6}\right)d_0, \end{aligned} \tag{48}$$

where c_0, d_0 are given by (44) and $\chi_0(x), \chi_1(x)$ – by (41).

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