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APPROXIMATION AND ESTIMATES IN THE PERIODIC REPRESENTATION OF REAL NUMBERS OF THE CLOSED INTERVAL $[0,5;1]$ BY A_2-CONTINUED FRACTIONS

M. V. Pratsiovytyi, O. P. Makarchuk, A. S. Chuikov

Резюме. В роботі знайдено оцінки наближень дійсних чисел відрізка $[0,5;1]$ ланцюговими A_2-дробами, елементи яких належать множині $\{\frac{1}{2},1\}$. Доведено, що A_2-раціональні числа (числа, що мають два різних нескінченних A_2-зображення) крім двох нескінченних, мають зліченну множину різних скінченних зображень. Спростовується гіпотеза, що кожне раціональне число є A_2-раціональним і обговорюється проблема критерія раціональності числа за його ланцюговим A_2-зображенням.

Abstract. The paper investigates the estimates of the approximations of real numbers of the closed interval $[0,5;1]$ of the A_2-continued fractions whose elements belong to set $\{\frac{1}{2},1\}$. It is proved that A_2-rational numbers (i.e. numbers that have two different infinite A_2-continued fraction representation) except two endless A_2-continued fraction representation have a countable set of different finite ones. We refute the hypothesis that every rational number is A_2-rational numbers and discuss the criterion of rationality of numbers according to its A_2-continued fraction representation.

1. Introduction

The role and importance of continued fractions in mathematics and its applications are well-known [7–9, 11, 17, 19, 20]. They are also used to develop a metric [5, 10] and probabilistic number theory [1, 6, 15, 16], the theory of dynamical systems [12], fractal geometry and fractal analysis [2, 4]. Especially well developed is the theory of elementary continued fractions whose elements are natural numbers [20]. Relatively recently, the theory of simple infinite A_2-continued fractions whose elements are positive real numbers α_0 and α_1 was created [10, 13]. It is proved that at $\alpha_0\alpha_1 = \frac{1}{2}$ the system of representation of numbers of a certain closed interval by such continued fractions, being two-character, has zero redundancy. Particular attention deserves a case when $\alpha_0 = \frac{1}{2}, \alpha_1 = 1$. We continue to develop this theory, in particular, supplement it with finite decompositions, and we focus on the interconnections of finite and infinite continued A_2-decomposers of numbers.

Let $A_2 \equiv \{\frac{1}{2}, 1\}$ be a two-character alphabet. Infinite continued fraction

Key words. A_2-continued fraction; A_2-rational number; criterion of rationality of number; left shift operator of digits of the A_2-continued fraction representation of number; algorithm for decomposing of rational numbers into finite A_2-continued fractions.

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots+\cfrac{1}{a_n+\ddots}}} \equiv [0; a_1, a_2, \ldots, a_n, \ldots] \equiv \Delta^{A_2}_{a_1 a_2 \ldots a_n \ldots},$$

where $a_n \in A_2$, is called [5, 10] A_2-*continued fraction.*

Because $\sum\limits_{n=1}^{\infty} a_n = \infty$ then each A_2-continued fraction is convergent. Remind [20] that convergents of order n of the continued fraction $[0; a_1, a_2, \ldots, a_n, \ldots]$ is called the number $\frac{p_n}{q_n}$ which is the value of a finite continued fraction $[0; a_0; a_1, a_2, \ldots, a_n]$, that is a segment of the continued fraction moreover:

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, \\ q_n = a_n q_{n-1} + q_{n-2}, \qquad n = 2, 3, \ldots; \end{cases}$$

where $p_0 = a_0, q_0 = 1, p_1 = a_1 a_0 + 1, q_1 = a_1$.

For convergents of the continued fraction the following properties are performed [20]:

1. $q_k p_{k-1} - p_k q_{k-1} = (-1)^k, \quad \forall k \in N;$
2. $\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}, \quad \forall k \in N;$
3. $q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k, \quad \forall k \in N;$
4. $\frac{q_k}{q_{k-1}} = [a_k; a_{k-1}, \ldots, a_1], \quad \forall k \in N.$

From the property (4) for A_2-continued fractions it follows that $\frac{1}{2} < \frac{q_{n-1}}{q_n} < 1$ at $n = 2, 3, \ldots$.

Theorem 1. *[10] For any $x \in [0,5; 1]$ there exists a sequence $(a_n) \in L$ such that*

$$x = [0; a_1, a_2, \ldots, a_n, \ldots], \tag{1}$$

and the numbers of a countable set can be represented as two different A_2-continued fractions:

$$x = [0; a_1, a_2, \ldots, a_n, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, a_2, \ldots, a_n, 1, (1, \frac{1}{2})], \tag{2}$$

here the round brackets mean the period.

Those numbers of the closed interval $[0,5; 1]$ having two representation of A_2-continued fractions are called A_2-*rational numbers.* The rest of the numbers in this closed interval have only one representation and are called A_2-*irrational numbers.* The task of finding a criterion (necessary and sufficient conditions) for rationality of a number by its representation in a given coding system is traditional and for many representations is solved. Consider it for this representation.

2. Conditions of the rationality of the number by its A_2-continued fraction representation

Let denote $t = \big[0; \big(\frac{1}{2}, 1\big)\big]$ then from equality

$$t = \frac{1}{\frac{1}{2} + \frac{1}{1+t}}$$

it is easy to get equality $0{,}5t^2 + 0{,}5t - 1 = 0$ and solution of the equation $t = 1 = \big[0; \big(\frac{1}{2}, 1\big)\big]$. Similarly

$$\left[0; \left(1, \frac{1}{2}\right)\right] = \frac{1}{2}.$$

Lemma 1. *Each A_2-rational number has at least two different finite A_2-continued fraction representations that is*

$$x = [0; a_1, \ldots, a_m, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, \ldots, a_m, \frac{1}{2} + 1] = [0; a_1, \ldots, a_m, \frac{1}{2}, 1] =$$

$$= [0; a_1, \ldots, a_m, 1, (1, \frac{1}{2})] = [0; a_1, \ldots, a_m, 1 + \frac{1}{2}] = [0; a_1, \ldots, a_m, 1, 1, 1],$$

and hence it is a rational number.

Proof. Because

$$\frac{1}{2} = \left[0; \left(1, \frac{1}{2}\right)\right] \qquad \text{i} \qquad 1 = \left[0; \left(\frac{1}{2}, 1\right)\right],$$

then

$$[0; a_1, \ldots, a_m, \frac{1}{2}, (\frac{1}{2}, 1)] = [0; a_1, \ldots, a_m, \frac{1}{2} + 1] =$$

$$= \frac{1}{a_1 + \ddots + \frac{1}{a_m + \frac{1}{0{,}5 + \frac{1}{1}}}} = [0; a_1, \ldots, a_m, \frac{1}{2}, 1],$$

$$[0; a_1, \ldots, a_m, 1, (1, \frac{1}{2})] = [0; a_1, \ldots, a_m, 1 + \frac{1}{2}] =$$

$$= \frac{1}{a_1 + \ddots + \frac{1}{a_m + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}} = [0; a_1, \ldots, a_m, 1, 1, 1].$$

Then equalities which are indicated in the formulation of the lemma follow from the fact that equality

$$[0; a_1, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots] = [0; a_1, \ldots, a_n, a'_{n+1}, a'_{n+2}, \ldots]$$

is executed then and only then

$$[0; a_{n+1}, a_{n+2}, \ldots] = [0; a'_{n+1}, a'_{n+2}, \ldots].$$

The value of each finite A_2-continued fraction is the result of a finite number of rational actions on rational numbers. So each A_2-rational number is a rational number. □

Theorem 2. *Each A_2-rational number has a countable set of different finite A_2-continued fraction representations, in particular*

$$\frac{1}{2} = \left[0; 1, \underbrace{\frac{1}{2}, 1, \ldots, \frac{1}{2}}_{2m}, 1, 1\right], 1 = \left[0; \underbrace{\frac{1}{2}, 1, \ldots, \frac{1}{2}}_{2m}, 1, 1\right].$$

Proof. Indeed, from equality

$$1 = \frac{1}{1} = \frac{1}{\frac{1}{2} + \frac{1}{1+\frac{1}{1}}} \tag{3}$$

we have the following

$$1 = [0; 1] = [0; \frac{1}{2}, 1, 1] = [0; \frac{1}{2}, 1, \frac{1}{2}, 1, 1] = [0; \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, 1] = \ldots$$

Then $\frac{1}{2} = [0; 1, 1] = [0; 1, \frac{1}{2}, 1, 1] = [0; 1, \frac{1}{2}, 1, \frac{1}{2}, 1, 1] = \ldots$.

Representations that are indicated in the lemma have the last element which is equal to 1, and hence, taking into account equality (3), we get countable set of finite representations of A_2-rational number. □

The question whether every rational number of a $[0, 5; 1]$ is A_2-rational is interesting. The answer to this question is directly related to another question. Is every rational number decomposed into a finite continued fraction? Let us give some examples of such expansions. But first we give the algorithm for decomposing a rational number $\frac{a}{b}$ into a A_2-continued fraction.

1. The first element a_1 of the expansion of number $x = \frac{a}{b}$ is based on the formula:

$$a_1 = \varphi\left(\frac{a}{b}\right) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq \frac{a}{b} \leq \frac{2}{3}, \\ \frac{1}{2}, & \text{if } \frac{2}{3} \leq \frac{a}{b} \leq 1. \end{cases}$$

2. The following elements a_i are determined from equalities:

$$x_1 = \frac{1}{x} - \varphi(x) = \frac{1}{x} - \frac{1}{2}\varepsilon_1 = \frac{2b - a\varepsilon_1}{2a},$$
$$x_2 = \frac{1}{x_1} - \varphi(x_1) = \frac{1}{x_1} - \frac{1}{2}\varepsilon_2,$$
$$\ldots$$
$$x_{n+1} = \frac{1}{x_n} - \varphi(x_n) = \frac{1}{x_n} - \frac{1}{2}\varepsilon_{n+1},$$

where $a_n = \varphi(x_{n-1}) = \frac{1}{2}\varepsilon_n, \varepsilon_n \in \{1, 2\}$.

3. The process ends if x_n becomes equal to 1 or $\frac{1}{2}$ or $\frac{1}{3}$.

- In the first case number x has $n+1$ digits, and $a_{n+1}(x) = 1$.
- In the first case number x has $n+2$ digits, and $a_{n+1}(x) = 1, a_{n+2}(x) = 1$.
- In the first case number x has $n+2$ digits, and $a_{n+1}(x) = 1, a_{n+2}(x) = \frac{1}{2}$.

The following expansions are performed:

$$\begin{aligned}
\frac{2}{3} &= [0;1,1,1] = \left[0;\frac{1}{2},1\right], \frac{3}{4} = \left[0;1,1,\frac{1}{2}\right], \frac{4}{5} = \left[0;\frac{1}{2},1,1,\frac{1}{2}\right],\\
\frac{5}{6} &= \left[0;\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,1,1,1,\frac{1}{2}\right].\\
\frac{6}{7} &= \left[0;\frac{1}{2},1,1,1\right] = \left[0;\frac{1}{2},\frac{1}{2},1\right],\\
\frac{7}{8} &= \left[0;\frac{1}{2},1,1,\frac{1}{2},1,1,\frac{1}{2},\right], \ \frac{8}{9} = \left[0;\frac{1}{2},1,1,1,1,1\right],\\
\frac{9}{10} &= \left[0;\frac{1}{2},1,1,1,1,1,\frac{1}{2}\right],\\
\frac{5}{6} &= \left[0;\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1,1,1,\frac{1}{2}\right)\right].
\end{aligned}$$

Notation. The number $x = \frac{5}{6}$ has both finite and periodic A_2-expansion and the later does not satisfy the definition of a A_2-rational number. This fact refutes the hypothesis that every rational number is A_2-rational number. It remains neither proven nor disproved the hypothesis that every rational number has a finite A_2-continued fraction expansion.

3. Left shift operator on digits of A_2-continued fraction representation of number

In the space of A_2-continued fraction representations we define operator ω by equality

$$\omega(\Delta^{A_2}_{a_1a_2\ldots}) = \Delta^{A_2}_{a_2a_3\ldots}, \tag{4}$$

called *left shift operator on digits of A_2-continued fraction representation of number.*

Let us use only the first of the two existing representations (2) of A_2-rational number. Then from equality (4) we get well-defined function of number $x = [0;a_1,a_2,\ldots]$ that has the following analytical form

$$\omega(x) = \frac{1}{x} - a_1(x) = \frac{1-a_1x}{x}.$$

Let

$$\omega^n(x) = \underbrace{\omega(\omega(\ldots\omega}(x))) = \frac{u_nx+v_n}{c_nx+d_n},$$

then

$$\omega^n(x) = \frac{1}{\omega^{n-1}(x)} - a_n(x) = \frac{1-\omega^{n-1}a_n(x)}{\omega^{n-1}}.$$

Then $u_0 = 1, \quad v_0 = 0, \quad c_0 = 0, \quad d_0 = 1$ and at $a_n = \frac{1}{2}$ we have

$$\frac{u_{n+1}x+v_{n+1}}{c_{n+1}x+d_{n+1}} = \frac{c_nx+d_n}{u_nx+v_n} - \frac{1}{2} = \frac{(2c_n-u_n)x+2d_n-v_n}{2u_nx+2v_n},$$

hence the following

$$\begin{cases} u_{n+1} = 2c_n - u_n, \\ v_{n+1} = 2d_n - v_n, \\ c_{n+1} = 2u_n, \\ d_{n+1} = 2v_n. \end{cases}$$

If $a_n = 1$ then

$$\frac{u_{n+1}x + v_{n+1}}{c_{n+1}x + d_{n+1}} = \frac{c_n x + d_n}{u_n x + v_n} - 1 = \frac{(c_n - u_n)x + d_n - v_n}{u_n x + v_n}.$$

We have

$$\begin{cases} u_{n+1} = c_n - u_n, \\ v_{n+1} = d_n - v_n, \\ c_{n+1} = u_n, \\ d_{n+1} = v_n. \end{cases}$$

Let's estimate the value of $|u_n|$ above.

Theorem 3. *The following inequality is being performed:*

$$c_n \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^n - \left(\frac{1-\sqrt{17}}{4} \right)^n \right), \quad \forall n \in Z_+.$$

Proof. It is clear that $(u_0; v_0; c_0; d_0) = (1; 0; 0; 1)$. Possible options for $(u_1; v_1; c_1; d_1)$ are $(-1; 2; 2; 0)$ and $(-1; 1; 1; 0)$ for $a_1 = \frac{1}{2}$ or 1 respectively. The following cases are possible:

$$u_{n+1} = 2c_n - u_n = \begin{cases} 2u_{n-1} - u_n, \\ 4u_{n-1} - u_n, \end{cases}$$

and

$$u_{n+1} = c_n - u_n = \begin{cases} 2u_{n-1} - u_n, \\ u_{n-1} - u_n. \end{cases}$$

So

$$u_{n+1} = ku_{n-1} - u_n,$$

where $k \in \{ 1; 2; 4\}$.

We got

$$|u_{n+1}| = |ku_{n-1} - u_n| \leq |ku_{n-1}| + |u_n| \leq 4|u_{n-1}| + |u_n|.$$

Let (s_n) such a sequence that

$$s_{n+1} = s_n + 4s_{n-1}, \quad \forall n \in N,$$

$$s_0 = 1, \quad s_1 = 1.$$

It is inductively easy to show that

$$u_n \leq s_n, \quad \forall n \in Z_+.$$

Because

$$s_n = \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^n - \left(\frac{1-\sqrt{17}}{4} \right)^n \right),$$

then

$$u_n \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^n - \left(\frac{1-\sqrt{17}}{4} \right)^n \right), \quad \forall n \in Z_+.$$

It is clear that $c_n = ku_n$, where $k \in \{ 1; \frac{1}{2}\}$, then

$$c_n \leq u_n \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^n - \left(\frac{1-\sqrt{17}}{4} \right)^n \right), \quad \forall n \in Z_+. \qquad \square$$

4. Properties of numbers with periodic A_2-continued fractions representation

Theorem 4. *If number y has a period in its A_2-continued fractions representation then it looks $y = \alpha + \sqrt{\gamma}$ where $\alpha, \gamma \in \mathbb{Q}$.*

Proof. Let $y = [0, \alpha_1, \alpha_2, \ldots, \alpha_k, (\beta_1, \ldots \beta_l)]$, then we have

$$[0, (\beta_1, \ldots, \beta_l)] = \omega^k(y) = \frac{u_k y + v_k}{c_k y + d_k},$$

$$[0, (\beta_1, \ldots, \beta_l)] = \omega^l \left(\frac{u_k y + v_k}{c_k y + d_k} \right).$$

So,

$$\frac{u_{k+l} y + v_{k+l}}{c_{k+l} y + d_{k+l}} = \frac{u_k y + v_k}{c_k y + d_k},$$

hence the following

$$y^2(c_{k+l}u_k - u_{k+l}c_k) + y(u_k c_{k+l} + v_k d_{k+l} - u_{k+l}d_k - v_{k+l}c_k) + v_k d_{k+l} - v_{k+l}d_k = 0,$$

which proves necessary. $\square$

Theorem 5. *If equation $ax^2 + bx + c = 0, \quad (a, b, c \in \mathbb{Z}, \quad a \neq 0)$ has a solution $x_1 = \alpha + \sqrt{\gamma}$, where $\alpha, \gamma \in \mathbb{Q}, \quad \sqrt{\gamma} \notin \mathbb{Q}$, then it has a solution $x_2 = \alpha - \sqrt{\gamma}$.*

Proof. It is clear that

$$a(\alpha^2 + 2\alpha\sqrt{\gamma} + \gamma) + b(\alpha + \sqrt{\gamma}) + c = 0,$$

$$\sqrt{\gamma}(2a\alpha + b) + a\alpha^2 + a\gamma + c + b\alpha = 0.$$

If $2a\alpha + b \neq 0$, then $\sqrt{\gamma} \in \mathbb{Q}$ and we get a contradiction.

So,

$$\begin{cases} 2a\alpha + b = 0, \\ a\alpha^2 + a\gamma + c + b\alpha = 0. \end{cases}$$

Hence we get

$$\begin{aligned} ax_2^2 + bx_2 + c &= a(\alpha^2 - 2\alpha\sqrt{\gamma} + \gamma) + b(\alpha - \sqrt{\gamma}) + c = \\ &= -\sqrt{\gamma}(2\alpha a + b) + a\alpha^2 + a\gamma + c + b\alpha = 0. \end{aligned} \qquad \square$$

Theorem 6. *If number*

$$y = \frac{e}{f} + \sqrt{\frac{g}{h}} \quad \in [0{,}5; 1],$$

where $l, f, g, h \in \mathbb{N}$, $(g; h) = (f; h) = 1$, $\sqrt{\frac{g}{h}} \notin \mathbb{Q}$, *has* A_2*-continued fractions representation of the form*

$$y = [0, (\beta_1, \ldots, \beta_l)],$$

then the following inequality is being performed:

$$h \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^l - \left(\frac{1-\sqrt{17}}{4} \right)^l \right).$$

Proof. We have

$$\frac{u_l y + v_l}{c_l y + d_l} = y$$

hence

$$c_l y^2 + (d_l - u_l)y - v_l = 0.$$

By the theorem 5 the last equation also has a root

$$\tilde{y} = \frac{e}{f} - \sqrt{\frac{g}{h}},$$

then

$$-\frac{b_l}{c_l} = y\tilde{y} = \frac{e^2}{f^2} - \frac{g}{h} = \frac{l^2 h - g f^2}{f^2 h}$$

hence

$$-b_l f^2 h = c_l(l^2 h - g f^2).$$

The left side of the last equality is divided by h hence c_l is divided by h.

Taking into account theorem 3, we have

$$h \leq |c_l| \leq \frac{1}{\sqrt{17}} \left(\left(\frac{1+\sqrt{17}}{4} \right)^l - \left(\frac{1-\sqrt{17}}{4} \right)^l \right). \qquad \square$$

5. Approximation of real number of the closed interval $[0,5;1]$ by A_2-continued fractions

Let $\nu_1(x, n) = \frac{l_n}{n}, \nu_{\frac{1}{2}}(x, n) = \frac{k_n}{n}$, where l_n i k_n is the number of elements 1 i $\frac{1}{2}$ respectively among $(a_1, \ldots, a_n)$ in A_2-continued fraction representation of number $x = [0; a_1, \ldots, a_n, \ldots]$.

Let's call the values $\lim\limits_{n\to\infty} \nu_{\frac{1}{2}}(x, n) = \nu_{\frac{1}{2}}$ and $\lim\limits_{n\to\infty} \nu_1(x, n) = \nu_1$ by the frequencies of the digits 1 and $\frac{1}{2}$ in A_2-continued fraction representation of x, provided that these limits exist.

Lemma 2. *Let B be a set of sets of numbers $(\alpha_1, \dots, \alpha_{k+l})$ among which l elements are equal to 1 and k elements are equal to $\frac{1}{2}$, and let $q((\alpha_1, \dots, \alpha_{k+l}))$ be a number that is defined by the following recurrence formula:*

$$q_0 = 1, \quad q_1 = \alpha_1, \quad q_n = \alpha_n q_{n-1} + q_{n-2}, \quad n = 2, 3, \dots, k + l,$$

$$q_{k+l} = q((\alpha_1, \dots, \alpha_{k+l})).$$

Then there exist such constants $D_j, \tilde{D}_j (j \in \{1, 2, 3, 4\})(D_j, \tilde{D}_j > 0)$, that do not depend k and l such that

$$\min_{(\beta_1,\dots,\beta_{k+l})} = D_1\delta_1^l\eta_1^k + D_2\delta_1^l\eta_2^k + D_3\delta_2^l\eta_1^k + D_4\delta_2^l\eta_2^k. \tag{5}$$

$$\max_{(\beta_1,\dots,\beta_{k+l})} = \tilde{D}_1\delta_1^l\eta_1^k + \tilde{D}_2\delta_1^l\eta_2^k + \tilde{D}_3\delta_2^l\eta_1^k + \tilde{D}_4\delta_2^l\eta_2^k. \tag{6}$$

where $\delta_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}, \eta_{1,2} = \dfrac{1 \pm \sqrt{17}}{4}$.

Proof. Let $q_k = \frac{1}{2}q_{k-1} + q_{k-2}, q_{k+1} = q_k + q_{k-1}$, then

$$q_{k+1} = \frac{1}{2}q_{k-1} + q_{k-2} + q_{k-1} = 1,5q_{k-1} + q_{k-2}.$$

If $q_k = q_{k-1} + q_{k-2}, q_{k+1} = \frac{1}{2}q_k + q_{k-1}$, then

$$q_{k+1} = \frac{1}{2}q_{k-1} + \frac{1}{2}q_{k-2} + q_{k-1} = 1,5q_{k-1} + 0,5q_{k-2}.$$

As we see, in the first case, the value of q_{k+1} is greater than in the second case.

Let $c_n(\beta_0, \beta_1), d_n(\gamma_0, \gamma_1)$ be such sequences that

$$c_{n+1}(\beta_0, \beta_1) = c_n(\beta_0, \beta_1) + c_{n-1}(\beta_0, \beta_1), \quad \forall n \in N,$$
$$c_0(\beta_0, \beta_1) = \beta_0, c_1(\beta_0, \beta_1) = \beta_1.$$
$$d_{n+1}(\gamma_0, \gamma_1) = \frac{1}{2}d_n(\gamma_0, \gamma_1) + d_{n-1}(\gamma_0, \gamma_1), \quad \forall n \in N,$$
$$d_0(\gamma_0, \gamma_1) = \gamma_0, d_1(\gamma_0, \gamma_1) = \gamma_1.$$

Inductively on n it is easy to show that

$$c_n(\tilde{\beta_0}, \tilde{\beta_1}) > c_n(\beta_0, \beta_1),$$
$$d_n(\tilde{\gamma_0}, \tilde{\gamma_1}) > d_n(\gamma_0, \gamma_1), \quad \forall n \in N,$$

if $\tilde{\beta_j} > \beta_j > 0,\ \tilde{\gamma_j} > \gamma_j > 0,\ \forall j \in \{0; 1\}$.

Considering all the above, we obtain that when replacing the neighboring elements $(\frac{1}{2}, 1)$ on $(1, \frac{1}{2})$ in a set $(\alpha_1, \dots, \alpha_{k+l})$ we will reduce the value of the expression $q(\alpha_1, \dots, \alpha_{k+l})$. We will make such a replacement as long as possible. As a result, we will come to the set

$$\left(\underbrace{1, 1, \dots, 1}_{l}, \underbrace{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}}_{k}\right).$$

So,

$$\min_{(\alpha_1,\ldots,\alpha_{k+l})\in B} q((\alpha_1,\ldots,\alpha_{k+l})) = q\left((\underbrace{1,1,\ldots,1}_{l},\underbrace{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}_{k})\right).$$

Easy to see that

$$d_n(\gamma_0;\gamma_1) = \frac{4\gamma_1 + (\sqrt{17}-1)\gamma_0}{2\sqrt{17}}\left(\frac{1+\sqrt{17}}{4}\right)^n + \\ + \frac{(\sqrt{17}+1)\gamma_0 - 4\gamma_1}{2\sqrt{17}}\left(\frac{1-\sqrt{17}}{4}\right)^n,$$

$$c_n(\beta_0;\beta_1) = \frac{2\beta_1 + (\sqrt{5}-1)\beta_0}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \\ + \frac{(\sqrt{5}+1)\beta_0 - 2\beta_1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad \forall \in N.$$

It is clear that $q\left(\left(\underbrace{1,1,\ldots,1}_{l},\underbrace{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}_{k}\right)\right)$ is determined by the system of equations

$$q_0 = 1, q_1 = 1, q_2 = q_1 + q_0, q_3 = q_2 + q_1, \ldots, q_l = q_{l-1} + q_{l-2},$$
$$q_{l+1} = \frac{1}{2}q_l + q_{l-1}, q_{l+2} = \frac{1}{2}q_{l+1} + q_l, \ldots, q_{k+l} = \frac{1}{2}q_{k+l-1} + q_{k+l-2}.$$

Then we have

$$q_l = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^l + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^l.$$

$$q_{l-1} = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{l-1} + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{l-1}.$$

$$q_{k+l} = \frac{4b_1^* + (\sqrt{17}-1)b_0^*}{2\sqrt{17}}\left(\frac{1+\sqrt{17}}{4}\right)^k + \frac{(1+\sqrt{17})b_0^* - 4b_1^*}{2\sqrt{17}}\left(\frac{1-\sqrt{17}}{4}\right)^k,$$

where $b_1^* = q_l, b_0^* = q_{l-1}$.

Similarly, from a system of equations

$$q_0 = 1, q_1 = \frac{1}{2}, q_2 = \frac{1}{2}q_1 + q_0, \ldots, q_k = \frac{1}{2}q_{k-1} + q_{k-2},$$
$$q_{k+1} = q_k + q_{k-1}, \ldots, q_{k+l} = q_{k+l-1} + q_{k+l-2}$$

we have

$$b_0^* = E_1\eta_1^k + E_1\eta_2^k, \qquad b_1^* = \tilde{E}_1\eta_1^k + \tilde{E}_2\eta_2^k,$$

$$\max_{(\beta_1,\ldots,\beta_{k+l})\in B} q((\beta_1,\ldots,\beta_{k+l})) =$$
$$= \frac{2b_1^* + (\sqrt{5}-1)b_0^*}{2\sqrt{5}}\delta_1^l + \frac{(\sqrt{5}+1)b_0^* - 2b_1^*}{2\sqrt{5}}\delta_2^l$$

for some constants $E_1, E_2, \tilde{E}_1, \tilde{E}_2$, which are easily determined, and from here we have (6). □

Lemma 3. *If number $x = [0; a_1, a_2, \ldots, a_n, \ldots]$ is A_2-rational number, then*

$$\overline{\lim_{n\to\infty}} \frac{q_n}{\delta_1^{l_n}\eta_1^{k_n}} \le \tilde{D}_1,$$
$$\underline{\lim_{n\to\infty}} \frac{q_n}{\delta_1^{l_n}\eta_1^{k_n}} \ge D_1.$$

Proof. It is clear that $\lim\limits_{n\to\infty} l_n = +\infty$, because otherwise number x will be A_2-rational number. The same is true for the (k_n). We get

$$\lim_{n\to\infty} \frac{\delta_1^{l_n}\eta_2^{k_n}}{\delta_1^{l_n}\eta_1^{k_n}} = \lim_{n\to\infty}\left(\frac{\eta_2}{\eta_1}\right)^{k_n} = 0,$$
$$\lim_{n\to\infty} \frac{\delta_2^{l_n}\eta_1^{k_n}}{\delta_1^{l_n}\eta_1^{k_n}} = \lim_{n\to\infty}\left(\frac{\delta_2}{\delta_1}\right)^{l_n} = 0,$$
$$\lim_{n\to\infty} \left|\frac{\delta_2^{l_n}\eta_2^{k_n}}{\delta_1^{l_n}\eta_1^{k_n}}\right| \le \lim_{n\to\infty}\frac{|\eta_2|^n}{\eta_1^n} = \lim_{n\to\infty}\left(\frac{|\eta_2|}{\eta_1}\right)^n = 0.$$

Taking into account lemma 2 we get what we need. □

Theorem 7. *If for A_2-continued fraction representation of irrational number x frequencies of digits $\frac{1}{2}$ and 1 exist, which are equal відповідно $\nu_{\frac{1}{2}}$ and ν_1 respectively then for any $\varepsilon > 0$ there is a number n_0 such that*

$$|x - \frac{p_n}{q_n}| < \frac{1}{(\delta_1^{\nu_1}\eta_1^{\nu_{\frac{1}{2}}} - \varepsilon)^{2n+1}}, \quad \forall n \ge n_0,$$

in particular for any irrational number $y \in [0,5;1]$ there exist a number n_1 and constant C such that

$$|y - \frac{p_n}{q_n}| < \frac{C}{(\frac{1+\sqrt{17}}{4})^{2n+1}}, \quad \forall n \ge n_1.$$

Proof. Taking into account lemma 2 we get $\sqrt[n]{q_n} \to \delta_1^{\nu_1}\eta_1^{\nu_{\frac{1}{2}}}\,(n\to+\infty)$. Then for any for any sufficiently small $\varepsilon > 0$ we get $q_n > (\delta_1^{\nu_1}\eta_1^{\nu_{\frac{1}{2}}} - \varepsilon)^n$ starting with a certain number n_0.

Given the inequality

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

we have the required inequality.

Let us consider the function $g(x) = \left(\frac{1+\sqrt{5}}{2}\right)^x\left(\frac{1+\sqrt{17}}{4}\right)^{1-x}$ interval $[0;1]$. It is obvious that the function $g(x)$ continuous on $[0;1]$. Since the function

$\ln(g(x)) = x\ln\left(\frac{1+\sqrt{5}}{2}\right) + (1-x)\ln\left(\frac{1+\sqrt{17}}{4}\right)$ is increasing $\left(\frac{1+\sqrt{5}}{2} > \frac{1+\sqrt{17}}{4}\right)$ then $g(x)$ is increasing too.

Taking into account lemma 2 we get that

$$q_n \geq D_1(\delta_1^{\nu_1 n}\eta_1^{\frac{\nu_1}{2}n})^n \geq D_1(g(0))^n = D_1\left(\frac{1+\sqrt{17}}{4}\right)^n,$$

starting with a certain number n_1. This implies that we need. □

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M. V. Pratsiovytyi, A. S. Chuikov
Institute of Mathematics of NAS of Ukraine,
National Pedagogical Mykhailo Drahomanov University,
3, Tereschenkivs'ka st., Kyiv-4, 01004, Ukraine;

O. P. Makarchuk
Volodymyr Vynnychenko Central Ukrainian State Pedagogical University,
1, Shevchenka st., Kropyvnytskyi, 25006, Ukraine.

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