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REPLACEMENTS IN THE FINITE ELEMENT METHOD FOR THE PROBLEM OF ADVECTION-DIFFUSION-REACTION

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Резюме. У даній роботі запропоновано новий підхід до числового розв'язування сингулярно-збурених задач адвекції-дифузії-реакції (АДР). Цей підхід базується на експоненціальних прямій і зворотній замінах до і після варіаційного формулювання, відповідно. Одержано теоретичні результати існування розв'язку та порядку збіжності. Проведено числові експерименти для сингулярно-збурених задач АДР. Наведено графіки одержаних розв'язків у стаціонарному та нестаціонарному випадках, таблиці похибок та експериментальний порядок збіжності запропонованого методу.

Abstract. In this work, a new approach for the numerical approximation of the solution for the initial-boundary problem of advection-diffusion-reaction (ADR) is proposed. This approach is based on exponential direct and inverse replacements, before and after variation formulations, respectively. Theoretical results of the existence of the solution and of the order of convergence are obtained. Numerical experiments are conducted for singularly perturbed ADR problems. Graphs of the obtained results for stationary and non-stationary problems, table of errors and experimental orders of convergence are presented.

1. Introduction

The mathematical modeling of processes of advection-diffusion-reaction (ADR) is the relevant area of research. However, in the case of large advantage of advection coefficients over diffusion coefficients, the standard approach based on the finite element method (FEM) leads to the loss of stability of the approximation. Nowadays, many approaches to solving singularly perturbed ADR problems might be found in works of M. Ainsworth, N. Bahvalov, I. Babuska, G. Marchuk, Ya. Savula, G. Shynkarenko, S. Wang and others. In particular, among the approaches well known are an application of the exponential basis and exponential weights [6], [9], functions bubbles basis [5] in the FEM. Among the well-known approaches, there are also adaptive schemes of FEM [1], [10].

The problem of improving the stability of FEM to solve the problem of ADR, despite a large number of publications, is still opened. Among a large number of existing methods, there is a question of choosing the optimal method for improving sustainability. This fact may be the subject of another review publication. The authors propose a new approach to solving this actual problem,

Key words. Advection-diffusion-reaction; finite element method; exponential replacement.

which does not require the use of irregular grids, h-p adaptive grids, counter-flow schemes, etc., which might greatly complicate the programming of the method.

Let there Ω is a bounded limited area in R^2 with a Lipschitz boundary Γ. The problem is to find c — an unknown concentration, which satisfies a differential equation

$$\frac{\partial c}{\partial t} + \nabla \cdot (Vc) - \nabla \cdot (K \cdot \nabla c) + \sigma c = f(x,t); \quad x \in \Omega, \quad t \in (0,T] \tag{1}$$

an initial condition

$$c(x,0) = 0; \quad x \in \bar{\Omega} \tag{2}$$

and a boundary condition

$$\nu \cdot (K \cdot \nabla c) + \lambda c = \psi; \quad x \in \Gamma, \quad t \in (0,T]. \tag{3}$$

In (1),(3) $V = (V_1,\, V_2)$ is a velocity vector of constant values $V_1 > 0$, $V_2 > 0$, K is a diffusivity coefficient, σ is a coefficient of reaction, λ is a constant value, f is a function of external sources, ψ is a function defined on the boundary Γ and $\nu = (l_1,\, l_2)$ is a directed vector to Γ. Coefficients are positive, constant and dimensionless and, because V_1, V_2 are constant, environment is incompressible $\nabla \cdot (V) = 0$.

An operator of the problem was considered

$$Ac = \nabla \cdot (V \cdot c) - \nabla \cdot (K \cdot \nabla c) + \sigma c.$$

Therefore, the following equation has been considered

$$\frac{\partial c}{\partial t} + Ac = f$$

with initial and boundary conditions (2), (3), respectively.

2. FEM with exponential replacement

Previously, using a numerical experiment, it was found that the solution obtained by the standard FEM with linear and quadratic basis functions [1,5–10] is unstable in the case of a singular perturbed problem. In this paper, a new alternative approach to solving the singular perturbed ADR problems is proposed.

In (1)-(3) the following replacement [4] was applied

$$c = u \exp\left(\frac{V_1 x_1 + V_2 x_2}{2K}\right). \tag{4}$$

Therefore, the problem (1)-(3) will be equivalent to the following problem

$$\begin{gathered} \frac{\partial u}{\partial t} - K\left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2}\right) + \left(\frac{V_1^2 + V_2^2}{4K} + \sigma\right) u = \\ = f \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right), \quad x \in \Omega; \end{gathered} \tag{5}$$

$$K\frac{\partial u}{\partial \nu} + \left(\left(\frac{V_1}{2} l_1 + \frac{V_2}{2} l_2\right) + \lambda\right) u = \psi \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right), \quad x \in \Gamma; \tag{6}$$

$$u(x,0)=0; \quad x\in\bar{\Omega}.$$

The next step is a variation formulation of the resulting problem. To do this, space $W=W_2^{(1)}(\Omega)$ was introduced. Then, equation (5) was multiplied on arbitrary function $w\in W$ and integrated over the area Ω

$$\begin{gathered}\int\limits_{\Omega}\frac{\partial u}{\partial t}wd\Omega-K\int\limits_{\Omega}\Delta uwd\Omega+\left(\frac{V_1^2+V_2^2}{4K}+\sigma\right)\int\limits_{\Omega}uwd\Omega=\\=\int\limits_{\Omega}fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.\end{gathered}\tag{7}$$

To the first term of the equation (7) the Green's formula for Laplacian [2] was applied. Thus, the following expression was obtained

$$\begin{gathered}\int\limits_{\Omega}\frac{\partial u}{\partial t}wd\Omega+K\int\limits_{\Omega}\nabla u\nabla wd\Omega-K\int\limits_{\Gamma}\frac{\partial u}{\partial\nu}wd\Gamma+\\+\left(\frac{V_1^2+V_2^2}{4K}+\sigma\right)\int\limits_{\Omega}uwd\Omega=\int\limits_{\Omega}fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.\end{gathered}\tag{8}$$

According to the algorithm, the discretization of the problem based on the division of the area Ω by finite elements and then on the construction of approximations using a linear combination of basic functions might be the next step. However, after direct applying of the discretization, the initial system of linear algebraic equations (SLAE) will have different orders of the coefficients of right and left parts. That is due to the last integrant multiplier on the right side of (8). Therefore, an approximation of the solution might be unstable.

That is the main reason why a reverse replacement was proposed to be applied in (8)

$$u=c\,\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right).\tag{9}$$

Then, because

$$\frac{\partial u}{\partial x_i}=\frac{\partial c}{\partial x_i}\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)-\frac{V_i}{2K}c\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)$$

the following expression was obtained

$$\begin{gathered}K\int\limits_{\Omega}\nabla u\nabla wd\Omega=K\int\limits_{\Omega}\nabla c\nabla w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega-\\-\sum_{i=1,2}\frac{V_i}{2}\int\limits_{\Omega}c\frac{\partial w}{\partial x_i}\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega.\end{gathered}\tag{10}$$

The formula is known [2]

$$\int\limits_{\Omega}\left(\frac{\partial\varphi}{\partial x_1}+\frac{\partial\psi}{\partial x_2}\right)d\Omega=\int\limits_{\Gamma}\left(\varphi l_1+\psi l_2\right)d\Gamma,$$

then, taking: $\varphi = uv, \psi = 0$ and vice versa, it is easy to make sure that

$$\int_{\Omega} v \frac{\partial u}{\partial x_i} d\Omega = -\int_{\Omega} u \frac{\partial v}{\partial x_i} d\Omega + \int_{\Gamma} uvl_i d\Gamma.$$

Therefore, the following transformation was applied to the last two terms in expression (10)

$$\begin{aligned} -\frac{V_i}{2} \int_{\Omega} c \frac{\partial w}{\partial x_i} \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Omega = \\ = \frac{V_i}{2} \int_{\Omega} \frac{\partial c}{\partial x_i} w \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Omega - \\ - \int_{\Gamma} cwl_i \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) - \\ -\frac{V_i^2}{4K} \int_{\Omega} cw \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Omega. \end{aligned} \tag{11}$$

According to the boundary condition (6)

$$\begin{aligned} -K \int_{\Gamma} \frac{\partial u}{\partial \nu} w d\Gamma = \int_{\Gamma} \left(\frac{V_1}{2} l_1 + \frac{V_2}{2} l_2\right) uw d\Gamma + \int_{\Gamma} \lambda uw d\Gamma - \\ - \int_{\Gamma} \psi \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) w d\Gamma. \end{aligned}$$

Further, taking into account the inverse replacement (9), the following expression was obtained

$$\begin{aligned} -K \int_{\Gamma} \frac{\partial u}{\partial \nu} w d\Gamma = \int_{\Gamma} \left(\frac{V_1}{2} l_1 + \frac{V_2}{2} l_2\right) cw \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Gamma + \\ + \int_{\Gamma} \lambda cw \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Gamma - \int_{\Gamma} \psi \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) w d\Gamma. \end{aligned} \tag{12}$$

Finally, after combining expressions (7) - (12), the variation formulation of problem was obtained. To find such $c(x,t) \in L_2(0,T;W)$ that satisfies the following equation $\forall w \in W$

$$\int_{\Omega} \frac{\partial c}{\partial t} w \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Omega + K \int_{\Omega} \nabla c \nabla w \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Omega +$$

$$+\frac{V_1}{2} \int_{\Omega} \frac{\partial c}{\partial x_1} w \exp\left(-\frac{V_1 x_1 + V_2 x_2}{2K}\right) d\Omega +$$

$$+\frac{V_2}{2}\int\limits_{\Omega}\frac{\partial c}{\partial x_2}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+$$

$$+\int\limits_{\Gamma}\lambda cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma+\sigma\int\limits_{\Omega}cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega= \quad (13)$$

$$=\int\limits_{\Omega}fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+\int\limits_{\Gamma}\psi w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma.$$

It is important to notify that variation formulation (13) is significantly different from the formulation obtained by using the classical approach for obtaining variation formulation. Coefficients V_1 and V_2 at advection integral expressions are divided by 2. Integral expressions on the left and the right sides have the same order.

According to the procedure of FEM, the triangulation of the area Ω by finite elements $\Omega\approx\bigcup_{i=0}^{N}\Omega_i$ with boundary elements $\Gamma\approx\bigcup_{i=1}^{M}\Gamma_i$ was obtained. Then, on the each finite element Ω_e with vertices numbering i,j,k an approximation of the solution was built by using linear basic functions [8]:

$$c_h=c_i^h\varphi_i^{(e)}(x_1,x_2)+c_j^h\varphi_j^{(e)}(x_1,x_2)+c_m^h\varphi_m^{(e)}(x_1,x_2), \quad (14)$$

where $\varphi_i^{(e)}(x_1^{(i)},x_2^{(i)})=\frac{1}{\delta}\left(a_i+b_ix_1+c_ix_2\right)$ and $a_i=x_1^{(j)}x_2^{(m)}-x_1^{(m)}x_2^{(j)},\ b_i=x_2^{(j)}-x_2^{(m)},\ c_i=x_1^{(m)}-x_1^{(j)},\ \delta=2S_{ijm}$.

Then the following bilinear forms were introduced

$$m(c',w)=\int\limits_{\Omega}\frac{\partial c}{\partial t}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega;$$

$$a(c,w)=K\int\limits_{\Omega}\nabla c\nabla w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+$$

$$+\sum_{i=1,2}\frac{V_i}{2}\int\limits_{\Omega}\frac{\partial c}{\partial x_i}w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+$$

$$+\int\limits_{\Gamma}\lambda cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma+\sigma\int\limits_{\Omega}cw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega;$$

$$l(w)=\int\limits_{\Omega}fw\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Omega+\int\limits_{\Gamma}\psi w\exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right)d\Gamma.$$

Therefore, by application semi-discrete Galerkin's method with

$$c_h(x,t)=\sum_{j=1}^{N}c_j\left(t\right)\varphi_j^h\left(x\right)$$

the following Cauchy problem was formulated

$$\begin{cases} \sum\limits_{j=1}^{N} \left\{ m_{ij} C_j' \left(t \right) + a_{ij} C_j \left(t \right) \right\} = l_i \left(t \right), \ t \in \left(0, T \right], \ i = \overline{1, N}; \\ \sum\limits_{j=1}^{N} m_{ij} C_j \left(0 \right) = p_i, \ i = \overline{1, N} \end{cases} \tag{15}$$

where $m_{ij} = m \left(\varphi_i^h, \varphi_j^h \right)$; $a_{ij} = a \left(\varphi_i^h, \varphi_j^h \right)$; $l_i \left(t \right) = l \left(\varphi_i^h \right)$; $p_i = m \left(c_0, \varphi_i^h \right)$.

To discretize the problem (15) by time variable the Euler's method [8] was applied. Mesh partitioning step δ was introduced. Thus, the following recurrence scheme was obtained

$$\begin{gathered} \sum_{j=1}^{N} \left\{ m_{ij} C_j \left(t_{k+1} \right) \right\} = \sum_{j=1}^{N} \left\{ m_{ij} C_j \left(t_k \right) \right\} + \\ + \delta \left\{ l_i \left(t_k \right) - \sum_{j=1}^{N} a_{ij} C_j \left(t_k \right) \right\}, \ i = \overline{1, N}; \\ \sum_{j=1}^{N} m_{ij} C_j \left(t_0 \right) = p_i, \ i = \overline{1, N}; \end{gathered} \tag{16}$$

where $k = \overline{1, N_t}$, N_t is a number of subintervals by time variable.

It should be noted that, according to the specifics of the proposed approach, FEM ultimately leads to solving the SLAE with the specific coefficients. These coefficients are the sum of integrals, which will include exponential function. It is known that for such integrals using classic quadrature in practice gives a high error of the approximation. Therefore, we propose to use special IOST quadrature [3], which is an extended Gaussian quadrature. The proposed in [3] formula completely avoids the crowding of Gaussian points and allows to obtain approximate values of the integrals determined with the high accuracy. The last is shown in [3] for exponential integrant functions.

3. Convergence analysis and error estimate

For the purpose of theoretical study, a stationary problem with homogeneous Dirichlet boundary conditions was considered

$$\nabla \cdot (Vc) - \nabla \cdot (K \cdot \nabla c) + \sigma c = f(x); \quad x \in \Omega,$$

$$c = 0, \quad x \in \Gamma.$$

3.1. **Classical approach FEM (linear basis).** According to the classical approach, the following variation formulation was obtained: find $c \in W$ that

$$\begin{gathered} K \int\limits_{\Omega} \nabla c \nabla w d\Omega + V_1 \int\limits_{\Omega} \frac{\partial c}{\partial x_1} w d\Omega + V_2 \int\limits_{\Omega} \frac{\partial c}{\partial x_2} w d\Omega + \\ + \sigma \int\limits_{\Omega} c w d\Omega = \int\limits_{\Omega} f w d\Omega, \quad \forall w \in W. \end{gathered} \tag{17}$$

Bilinear form was defined

$$\tilde{a}(c,w) = K\int\limits_{\Omega} \nabla c \nabla w d\Omega + V_1 \int\limits_{\Omega} \frac{\partial c}{\partial x_1} w d\Omega + V_2 \int\limits_{\Omega} \frac{\partial c}{\partial x_2} w d\Omega + \sigma \int\limits_{\Omega} cw d\Omega.$$

Theorem 1. *The bilinear form $\tilde{a}(c,w)$ is continuous, i.e. $\exists M > 0$:*

$$\tilde{a}(c,w) \le M \|c\|_{W_2^{(1)}} \|w\|_{W_2^{(1)}}.$$

$M = \max\left\{\sqrt{3}K, \sqrt{3}\max\left\{V_1, V_2\right\}, \sqrt{3}\sigma, 1\right\}.$

Proof. Norm in Sobolev's space is $\|u\|^2_{W_2^{(1)}} = \int\limits_{\Omega} \left(u^2 + (\nabla u)^2\right) d\Omega$. Expression for $(\tilde{a}(c,w))^2$ was considered and evaluated by using elementary inequality $(q-p)^2 \ge 0 \quad \Rightarrow \quad 2qp \le q^2 + p^2$.

$$(\tilde{a}(c,w))^2 = \left(\int\limits_{\Omega} \left(K\nabla c \nabla w + \sum_i V_i \frac{\partial c}{\partial x_i} w + \sigma c w\right)\right)^2 \le$$

$$\le \int\limits_{\Omega} \left(3(K\nabla c\nabla w)^2 + 3(\max\left\{V_1, V_2\right\}\nabla c w)^2 + 3(\sigma c w)^2\right) d\Omega.$$

Let's reinforce inequality by adding an integral term

$$\int\limits_{\Omega} c^2 (\nabla w)^2 d\Omega \ge 0$$

$$(\tilde{a}(c,w))^2 \le \int\limits_{\Omega} \Big(3(K\nabla c\nabla w)^2 + 3(\max\left\{V_1, V_2\right\}\nabla c w)^2 +$$

$$+3(\sigma c w)^2 + (c\nabla w)^2\Big) d\Omega \le\le M^2 \|c\|^2_{W_2^{(1)}} \|w\|^2_{W_2^{(1)}}.$$ □

Obviously, in the case $V_1 >> K$ and(or) $V_2 >> K$, $M = \sqrt{3}\max\left\{V_1, V_2\right\}$.

Theorem 2. *The bilinear form $\tilde{a}(c,w)$ is V-elliptic, i.e. $\exists m > 0$: $\tilde{a}(c,c) \ge m\|c\|^2_{W_2^{(1)}}$.*

$$m = \min\left\{K, \sigma\right\}.$$

Proof. It is known [8] that a bilinear form $b(c,w) = \int\limits_{\Omega} \left(V_1 \frac{\partial c}{\partial x_1} w + V_2 \frac{\partial c}{\partial x_2} w\right) d\Omega$ is skew-symmetric, i.e. $b(c,w) = -b(w,c)$. Therefore, $b(c,c) = 0$. Then

$$(\tilde{a}(c,c)) = \int\limits_{\Omega} \left(K\nabla c\nabla c + \sum_i V_i \frac{\partial c}{\partial x_i} c + \sigma c^2\right) =$$

$$= \int\limits_{\Omega} \left(K(\nabla c)^2 + \sigma c^2\right) \ge m\|c\|^2_{W_2^{(1)}}.$$ □

Thus, the following two-sided estimate of bilinear form was obtained

$$m\|c\|^2_{W_2^{(1)}} \le \tilde{a}(c,c) \le M\|c\|^2_{W_2^{(1)}}.$$

Consequences If the function $f(x) \in L_2(\Omega)$, then, according to the Lax-Milgram's theorem [8], there is a single weak solution of the variation problem (17). In addition, by using Cea's lemma and theorem about the order of convergence proved in [8], applying the FEM with linear basis functions (14), a priori estimation of the error of approximate solution c_h to an exact solution c was obtained

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{M}{m} \|c\|_{W_2^{(2)}}.$$

3.2. **Method of exponential replacements.** According to the approach proposed in this paper, taking into account the homogeneous boundary condition

$$\begin{aligned} a(c,w) = K \int\limits_\Omega \nabla c \nabla w \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega + \\ + \sum_i \frac{V_i}{2} \int\limits_\Omega \frac{\partial c}{\partial x_i} w \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega + \\ + \sigma \int\limits_\Omega cw \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega. \end{aligned} \tag{18}$$

Theorem 3. *The bilinear form $a(c,w)$ is continuous, i.e. $\exists Q > 0:$*

$$a(c,w) \leq Q \|c\|_{W_2^{(1)}} \|w\|_{W_2^{(1)}}.$$

Proof. An expression for $(a(c,w))^2$ was considered and Cauchy-Schwarz's inequality was applied

$$\begin{aligned} (a(c,w))^2 = \\ = \left(\int\limits_\Omega \left(K\nabla c \nabla w + \sum_i \frac{V_i}{2} \frac{\partial c}{\partial x_i} w + \sigma cw \right) \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right) d\Omega \right)^2 \leq \\ \leq \int\limits_\Omega \left(K\nabla c \nabla w + \sum_i \frac{V_i}{2} \frac{\partial c}{\partial x_i} w + \sigma cw \right)^2 d\Omega \int\limits_\Omega \exp\left(-\frac{V_1x_1 + V_2x_2}{2K}\right)^2 d\Omega. \end{aligned} \tag{19}$$

Let's evaluate the last multiplier

$$\begin{aligned} \int\limits_\Omega \exp\left(-\frac{V_1x_1 + V_2x_2}{K}\right) d\Omega \leq \left\{ \max_\Omega \exp\left(-\frac{V_1x_1 + V_2x_2}{K}\right) \right\} S_\Omega = \\ = \left\{ \min_\Omega \exp\left(\frac{V_1x_1 + V_2x_2}{K}\right) \right\} S_\Omega, \end{aligned}$$

where S_Ω is a square of the area Ω. Let's evaluate the first multiplier of the right side of (19) by introducing notation $L = \frac{1}{2}\max\{V_1, V_2\}$ and using elementary inequality $2qp \le q^2 + p^2$.

$$\int_\Omega \left(K\nabla c\nabla w + \sum_i \frac{V_i}{2}\frac{\partial c}{\partial x_i} w + \sigma c w \right)^2 d\Omega \le$$
$$\le \int_\Omega (K\nabla c\nabla w + L\nabla c w + \sigma c w)^2 d\Omega \le$$
$$\le \int_\Omega \Big(3(K\nabla c\nabla w)^2 + 3(L\nabla c w)^2 + 3(\sigma c w)^2 \Big) d\Omega.$$

Let's reinforce inequality by adding an integral term

$$\int_\Omega c^2 (\nabla w)^2 d\Omega \ge 0$$

$$\int_\Omega \left(K\nabla c\nabla w + \sum_i \frac{V_i}{2}\frac{\partial c}{\partial x_i} w + \sigma c w \right)^2 d\Omega \le$$
$$\le \int_\Omega \Big(3(K\nabla c\nabla w)^2 + 3(L\nabla c w)^2 + (c\nabla w)^2 + 3(\sigma c w)^2 \Big) d\Omega \le$$
$$\le 3K^2 \int_\Omega (\nabla c)^2 d\Omega \int_\Omega (\nabla w)^2 d\Omega + 3L^2 \int_\Omega (\nabla c)^2 d\Omega \int_\Omega (w)^2 d\Omega +$$
$$+ \int_\Omega (c)^2 d\Omega \int_\Omega (\nabla w)^2 d\Omega + 3\sigma^2 \int_\Omega (c)^2 d\Omega \int_\Omega (w)^2 d\Omega \le$$
$$\le P^2 \int_\Omega \Big(c^2 + (\nabla c)^2 \Big) d\Omega \int_\Omega \Big(w^2 + (\nabla w)^2 \Big) d\Omega,$$

$P = \max\left\{ \sqrt{3}K, \sqrt{3}L, \sqrt{3}\sigma, 1 \right\}$. Obviously, in the case of the singularly perturbed problem $P = \frac{\sqrt{3}}{2} \max\{V_1, V_2\}$.

Therefore, the following evaluation was obtained

$$(a(c,w))^2 \le Q^2 \|c\|^2_{W_2^{(1)}} \|w\|^2_{W_2^{(1)}}$$

and

$$Q = \sqrt{\left\{ \min_\Omega \exp\left(\frac{V_1 x_1 + V_2 x_2}{K} \right) \right\} S_\Omega P}. \qquad \square$$

Theorem 4. *The bilinear form $a(c,w)$ is V-elliptic, i.e. $\exists q > 0 : a(c,c) \ge q \|c\|^2_{W_2^{(1)}}$.*

Proof. Let's investigate the bilinear form

$$b(c,w) = \frac{V_1}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_1} w \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega +$$
$$+\frac{V_2}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_2} w \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega.$$

Taking into account the homogeneous boundary conditions

$$\sum_i \frac{V_i}{2}\int\limits_{\Omega} \frac{\partial c}{\partial x_i} w \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega =$$
$$= \sum_i \frac{V_i}{2}\int\limits_{\Omega} \frac{\partial w}{\partial x_i} c \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega +$$
$$+\sum_i \frac{V_i{}^2}{4K}\int\limits_{\Omega} cw \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega.$$

Then

$$b(c,w) = -b(w,c) + \sum_i \frac{V_i{}^2}{4K}\int\limits_{\Omega} cw \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega.$$

Therefore,

$$b(c,c) = \sum_i \frac{V_i{}^2}{8K}\int\limits_{\Omega} c^2 \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega$$

and

$$a(c,c) = K\int\limits_{\Omega} (\nabla c)^2 \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega +$$
$$+\left(\frac{V_1^2+V_2^2}{8K}+\sigma\right)\int\limits_{\Omega} c^2 \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega \geq$$
$$\geq \mu \int\limits_{\Omega} \left((\nabla c)^2 + c^2\right) \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega,$$

$\mu = \min\left\{K, \left(\frac{V_1^2+V_2^2}{8K}+\sigma\right)\right\}$. Obviously, in the case of the singularly perturbed problem $\mu = K$.

$$\int\limits_{\Omega} \left((\nabla c)^2 + c^2\right) \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) d\Omega \geq$$

$$\geq \min_{\Omega} \exp\left(-\frac{V_1x_1+V_2x_2}{2K}\right) \int\limits_{\Omega} \left((\nabla c)^2 + c^2\right) d\Omega =$$

$$= \max_{\Omega} \exp\left(\frac{V_1x_1 + V_2x_2}{2K}\right) \int_{\Omega} \left((\nabla c)^2 + c^2\right) d\Omega.$$

Therefore,

$$q = \mu \max_{\Omega} \exp\left(\frac{V_1x_1 + V_2x_2}{2K}\right). \qquad \square$$

Consequences If the function $f(x) \in L_2(\Omega)$, then, according to the Lax-Milgram theorem [8], there is a single weak solution of the variation problem (19). In addition, by using Cea's lemma and theorem about the order of convergence proved in [8], applying the FEM with linear basis functions (14), a priori estimation of the error of the approximate solution c_h to the exact solution c was obtained

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{Q}{q} \|c\|_{W_2^{(2)}}.$$

In the case that $V_1 >> K$ and(or) $V_2 >> K$ **classical approach of FEM** gives an error

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{\sqrt{3} \max\{V_1, V_2\}}{\min\{K, \sigma\}} \|c\|_{W_2^{(2)}}. \tag{20}$$

And **method of exponential replacements** gives an error

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{\sqrt{3}}{2} \frac{\max\{V_1, V_2\} \sqrt{\left\{\min_{\Omega} \exp\left(\frac{V_1x_1+V_2x_2}{K}\right)\right\} S_\Omega}}{K \max_{\Omega} \exp\left(\frac{V_1x_1+V_2x_2}{2K}\right)} \|c\|_{W_2^{(2)}}.$$

Considering that the region Ω is in the first quarter of the coordinate system

$$\left\{\min_{\Omega} \exp\left(\frac{V_1x_1 + V_2x_2}{K}\right)\right\} = 1.$$

Therefore,

$$\|c - c_h\|_{W_2^{(1)}} \leq C_1 h \frac{\sqrt{3}}{2} \frac{\max\{V_1, V_2\} S_\Omega}{K \max_{\Omega} \exp\left(\frac{V_1x_1+V_2x_2}{2K}\right)} \|c\|_{W_2^{(2)}}. \tag{21}$$

On the right sight of evaluation (20), a maximum of advection coefficients appears, which in the case of singularly perturbed problems might be a high number. This is the main reason for the loss of stability by using the classical FEM approach. On the other hand, in the evaluation (21) the value in the denominator of the corresponding constant value is much higher than in the numerator and balances this issue.

Thus, the order of the convergence is preserved in both methods, but the constant at h in the method of exponential replacements is much smaller. Therefore, at the same value of step, an estimate of the error of the proposed method is much better than without replacements.

4. Numerical Results

Numerical experiments were conducted for different ADR problems. In this paper stationary and non-stationary cases were considered.

4.1. **Sationary problem.** For the purpose of study of experimental order of convergence, a stationary one-dimensional problem on $[0,1]$ with homogeneous Dirichlet boundary conditions was considered. In this case, the exact solution is known

$$c(x) = \frac{f}{\sigma}\left\{\left(\frac{e^{\alpha_2 b}-1}{e^{\alpha_1 b}-e^{\alpha_2 b}}\right)e^{\alpha_1 x}+\left(\frac{1-e^{\alpha_1 b}}{e^{\alpha_1 b}-e^{\alpha_2 b}}\right)e^{\alpha_2 x}+1\right\}, \qquad \alpha_{1,2} = \frac{-V \pm \sqrt{V^2+4K\sigma}}{-2K}. \tag{22}$$

The relative error of the method was calculated by the following formula

$$R_h = \max_i \frac{|c(x[i]) - c_h(x[i])|}{c(x[i])} * 100\%.$$

In the Table 1 we show relative errors with different advection coefficients and numbers of mesh points. For the rest of input parameters the following values were set $K = 1.0;\quad \sigma = 1.0;\quad f = 1.0$. As can be seen from Table 1 relative

Tabl. 1. Relative errors

N	$V = 70$	$V = 100$	$V = 150$
16	0.045904065	0.033463525	0.012854694
32	0.035734439	0.044157740	0.042586510
64	0.018337116	0.026518133	0.037617746
128	0.014365357	0.017101069	0.022781162

error of the exact and approximate solution is extremely small and decreases with an increase in the number of mesh points.

To calculate the experimental order of the convergence, the following scheme was applied. Approximations c_{h_1}, c_{h_2} were calculated on 2 grids for $h_1, h_2 = 0.5h_1$, respectively.

Denotation $e_i = \|c - c_{h_i}\|$, $i = 1,2$ was introduced. Then, orders of convergence in the output spaces $W_2^{(1)}(\Omega)$ and $L_2(\Omega)$ were calculated according to the formula

$$p \approx \frac{\ln e_1 - \ln e_2}{\ln h_1 - \ln h_2}.$$

Corresponding orders of convergence are not presented for $N = 20,\ V = 1$ and $N = 80,\ V = 100$ because results on 2 grids are needed to calculate the orders. From the results obtained, the experimental order of convergence coincides with the theoretical one obtained in the preceding paragraph of the article.

4.2. **Non-stationary problem.** The same area and boundary conditions as in the previous example were considered. The scheme (16) was applied. On the (Fig. 1) an exact solution and approximations of the solution of problem (1)-(3) in different moments of time are presented. The number of mesh points

Tabl. 2. Orders of convergence

V	N	$\|c_h - c\|_{W_2^{(1)}}$	$\|c_h - c\|_{L_2}$	order p in $W_2^{(1)}$	order p in L_2
1	10	$0,02764197$	$0,00081619$	–	–
	20	$0,01375605$	$0,00020139$	$1,0067934$	$2,0189291$
	40	$0,00681524$	$5,16259 \cdot 10^{-5}$	$1,0132307$	$1,9638124$
100	20	$0,0547536$	$0,0007214$	–	–
	40	$0,0398390$	$0,0002793$	$0,4587726$	$1,3689093$
	80	$0,0231478$	$8,26024 \cdot 10^{-5}$	$0,7833058$	$1,7576934$
	160	$0,0116643$	$1,95351 \cdot 10^{-5}$	$0,9887709$	$2,0801149$
	320	$0,0053174$	$3,88396 \cdot 10^{-6}$	$1,1332922$	$2,330465$

$N = 128$,mesh partitioning step by time variable $\delta = 0.05$. Input parameters were set into the following values

$$V = 100; \quad K = 1.0; \quad \sigma = 1.0; \quad f = 1 - e^{-t}.$$

It is obviously that solution coincides with an exact solution (22) at $t \to \infty$. Graphs 1, 2, 3 are approximated concentrations c_h in moments of time $t = 0.1, t = 0.2, t = 0.3$, respectively; Graphs 4, 5, 6 are approximated concentrations c_h in moments $t = 0.8, t = 1.0, t = 2$, respectively; Graphs 7, 8, 9 are approximated concentrations c_h in moments $t = 3, t = 4.5, t = 5$, respectively; Graph 10 is an exact solution (22) of the problem (1)-(3) at $t \to \infty$.

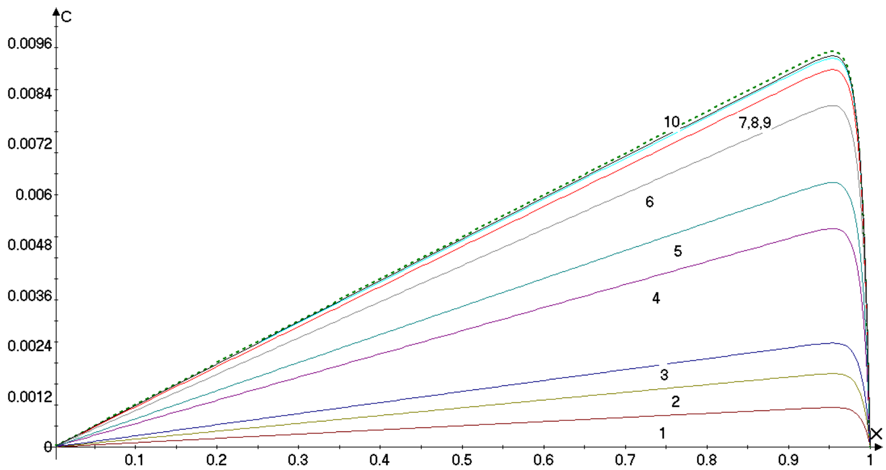

Fig. 1. Approximations in different moments of time and an exact solution

As can be seen from (Fig. 1), approximations of the unknown solution exactly coincide with the solution of a stationary problem with increasing moments of time.

The concentration closer to the end of interval $[0, 1]$ in the fixed point $x = 0.875$ is shown on the (Fig. 2). This is a point where, in fact, there is a problem in the case of significant advection coefficients, overcome by the method proposed in this paper. Coefficients of diffusion, reaction, right part f and the number of mesh points are the same as in the previous example.

On the graph 1 coefficient of advection $V = 70$, on graph 2 coefficient of advection $V = 100$, on graph 3 coefficient of advection $V = 150$.

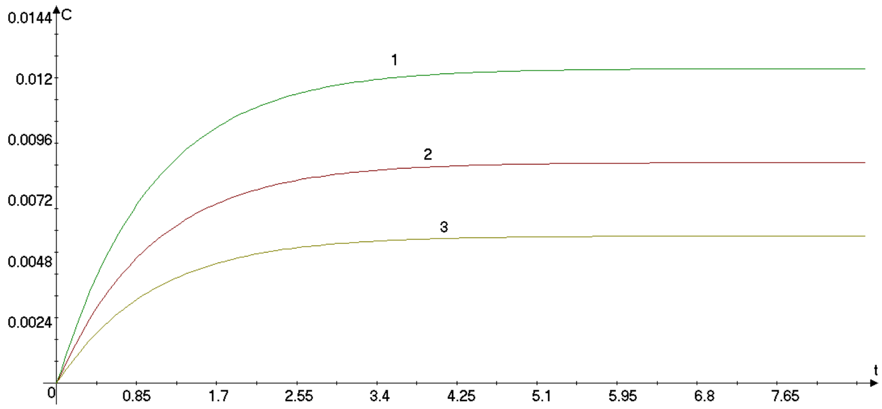

FIG. 2. Approximations in the fixed point $x = 0.875$

As can be seen from obtained results, the solution coincides with the solution of the stationary problem, that is, the process becomes stationary. It is also worth noting that the value of the desired concentration c at the fixed point $x = 0.875$ decreases with an increase in the advection coefficient, which corresponds to the nature of the phenomenon, as well as the fact that with an increase of V, obtained approximation reaches stationary behavior faster.

5. CONCLUSIONS

Thus, in this paper, a singular perturbed initial-boundary problem of ADR has been considered. A new alternative method based on exponential direct and reverse replacement in FEM for resolving singular-perturbed problems of ADR has been proposed.

The sequence of theorems have been proved and the existence of the solution and order of convergence of the proposed method have been shown.

Numerical experiments have been conducted and results have been compared with an exact solution, known in partial case. Obtained results have proved the effectiveness of the proposed method.

In the long term, it is planned to apply the proposed method to the mathematical models of the distribution of drugs and others in which the aforementioned specificity of the coefficients arises.

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