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ON ACOUSTIC EQUILIBRIA

E. V. Tkachenko, A. N. Timokha

Резюме. Стаття узагальнює математичну теорію віброрівноваги на випадок акустично-керованої поверхні розділу між випаровним газом та рідиною в контейнері.

Abstract. The present paper generalises mathematical theory of vibroequilibria onto the case of the acoustically-driven interface between ullage gas and liquid in a container.

1. Introduction

Using high-frequency vibrations and acoustic waves for the contactless control of a limited liquid volume is a relatively-old technologic idea coming from the 70-90's. In this context, one should mention the so-called acoustical levitation (of liquid drops) utilised in chemical and pharmaceutical industries as well as for getting ultra-pure (smart) materials [4, 6, 15]. A mathematical theory of acoustically-levitated liquid drops can be found in [5]. Other popular studies deal with mean (time-averaged) shapes of the contained liquid in tanks undergoing a high-frequency vibration. These are associated with novel microgravity technologies, whose fundamentals were recently developed in experiments [7,12] (see, also, references therein). To explain the experimental vibro-phenomena, the authors extensively employ theoretical concept of vibroequilibria, which were first considered and analysed in the applied mathematical works [1, 2, 8]. The vibroeqilibria (time-averaged, mean liquid shapes in vibrating containers) may dramatically differ from those caused by Newtonian gravitation and surface tension. The difference is clarified by vibrational forces introduced by Blekhman [3]. The extra (in addition to gravitation and surface tension) forces affect both the mean liquid shape and its hydrodynamic stability, i.e., the high-frequency tank vibrations may make the mean free surface unstable, or, contrary, stabilise it. Using the mathematical theory from [1, 2, 8], even though it was based on a rather simple hydrodynamic model of ideal compressible fluids, demonstrates a rather adequate prediction of the experimentally-observed vibrational phenomena.

Along with technologies of acoustical levitation and vibrational control of a limited liquid volume in a shaken tank, there exists another class of contactless (acoustic) techniques in microgravity, whose idea comes from famous experimental observations by Wesseln [16]. These experiments showed that generating an acoustic field in the ullage gas (vapour) makes it possible to destabilise

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(stabilise) the liquid-gas interface for certain input acoustic frequencies. For cryogenic two-layer fluids, the destabilisation leads to extensive evaporations of the condensed component, an increase of the mean pressure in the gas domain, and, thereby, it causes the so-called acoustic pumping. A physical theory of the acoustic pumping can be found in [10,13]. By utilising [5], the present paper develops elements of a mathematical theory of the acoustic destabilisation (stabilisation).

After formulating the non-dimensional mathematical statement of the considered hydrodynamic problem in § 2, which adopts the model of ideal compressible barotropic two-layer fluids, we introduce small parameters (and relations between them) in § 3 to apply the two-timing (separation of slow and fast time) asymptotic technique and derive the free-surface problem describing slow (modulated) motions of the liquid exposed to acoustic loads from the gas side. Mathematically, the latter problem looks identical to those appearing in the liquid sloshing dynamics for a motionless container when Newtonian gravitation, surface tension and acoustic radiation pressure become comparable on the introduced asymptotic scale. This makes it possible to generalise classical results [11] on sloshing of a capillary liquid. § 4 introduces acoustic equilibria (generalisation of capillary equilibria) and spectral theory of linear relative (natural) harmonic standing waves (natural sloshing modes and frequencies). Spectral criterion of stability for the acoustic equilibria is formulated and applied to show that acoustic field can destabilise the flat liquid-gas interface (if exists) for certain input acoustic frequencies. In § 5, we derive an analogy of (pseudo-)potential energy for the acoustic equilibria.

2. Statement of the problem

Following [1], we consider the rigid container

$$Q = Q_1(t) \cap Q_2(t) = \{(x, y, z)|W(x, y, z) < 0\},$$

which is filled by a two-layer fluid where the upper fluid is associated with the ullage (ideal compressible barotropic) gas (domain $Q_1(t)$) but the lower one is an ideal compressible barotropic liquid (domain $Q_2(t)$). The gas and liquid domains are time-dependent and the interface

$$\Sigma(t) = \partial Q_2(t) \cap \partial Q_1(t) = \{(x, y, z)|\xi(z, y, z, t) = 0\}$$

is implicitly specified by the preliminary unknown function ξ such that $\nabla\xi/|\nabla\xi|$ is the outer normal to $Q_2(t)$ on $\Sigma(t)$. The gravitational acceleration is directed downward, against the Ox-axis. Furthermore, we assume an acoustic field generated in $Q_1(t)$ by means of a vibrator on a piece of the time-independent gas boundary

$$S_0 \subset \partial Q_1(t), \quad S_0 \cap \Sigma(t) = \emptyset,$$

which is, in fact, a part of the tank wall contacting with $Q_1(t)$.

As in [1,5], the two-layer fluid dynamics is described by the velocity potentials $\varphi_i(x, y, z, t)$, the pressure $p_i(x, y, z, t)$ and density $\rho_i(x, y, z, t)$ fields in ullage gas ($i = 1$) and liquid ($i = 2$), respectively. Henceforth, the corresponding boundary value problem is considered in the non-dimensional statement, which appears after choosing the characteristic dimension (length) l and time $2\pi/\sigma$, where σ is the circular frequency of the acoustic field in the gas. This non-dimensional mathematical statement takes then the form [5]:

$$\rho_i \nabla \left(\dot{\varphi}_i + \frac{1}{2}(\nabla\varphi_i)^2 + \sigma_*^{-2}\mathrm{Bo}\ x \right) = -\nabla p_i; \quad \rho_i = \left(\frac{p_i}{p_{0i}}\right)^{1/\gamma_i} \quad \text{in} \quad Q_i(t), \quad (1)$$

$$\dot{\rho}_i + \mathrm{div}(\rho_i \nabla \varphi_i) = 0 \quad \text{in} \quad Q_i(t); \quad \int_{Q_i(t)} \rho_i dQ = m_i, \quad (2)$$

$$\partial_n \varphi_i = 0 \quad \text{on} \quad S_i(t); \qquad \partial_n \varphi_i = -\,\dot{\xi}/|\nabla\xi| \quad \text{on} \quad \Sigma(t), \quad (3)$$

$$-p_2 + \sigma_*^{-2}\,(K_1 + K_2) = -\delta_0\, p_1 \quad \text{on} \quad \Sigma(t), \quad (4)$$

$$-\frac{(\nabla W, \nabla \xi)}{|\nabla W||\nabla \xi|} = \cos\alpha \quad \text{on} \quad \partial\Sigma(t), \quad (5)$$

$$\rho_1\ \partial_n \varphi_1 = \varepsilon\, \mu_0\, k^{-1}\, V(x,y,z)\, \sin t \quad \text{on} \quad S_0, \quad (6)$$

where $S_i(t) = \partial Q \cap \partial Q_i\,(i=1,2)$ are the time-depending wetted (contacted) walls of Q by gas and liquid, respectively, $\partial\Sigma(t)$ is the contact (gas-liquid-tank) line (curve), α is the contact angle (we assume that $\alpha =$const), K_i are the main curvatures of $\Sigma(t)$, ρ_{0i} are the mean densities of gas and liquid, respectively, γ_i are the adiabatic indices for the barotropic fluids, p_{0i} are the non-dimensional mean (static) pressures in the fluids ($i = 1,2$), m_1 and m_2 are (constant) masses of gas and liquid, respectively; the dot implies the time-derivative and ∂_n is the (outer) normal derivative. Furthermore, $\sigma_* = \sigma\sqrt{\rho_{02} l/T_s}$ is the non-dimensional (normalised) acoustic frequency, where T_s is the surface tension, $\mathrm{Bo} = gl^2\rho_{02}/T_s$ is the Bond number, where $\boldsymbol{g}$ is the gravity acceleration, $k = \sigma l/c$ is the wave number of the acoustic field in the gas, where c is the sound speed in the gas, $\delta_0 = \rho_{01}/\rho_{02} \ll 1$ is the ratio between the mean densities.

Originally, $V_0(x,y,z)\sin(\sigma t)$ is the given dimensional distribution of the normal velocity on the acoustic vibrator $S_0 \subset S_1$ but the normalisation introduces the non-dimensional distribution $V = V_0/\sup|V_0|$, the small parameter $\varepsilon = \sup|V_0|/(c\mu_0) \ll 1$ (ratio of the maximum vibration velocity and the sound speed, an analogy of the Mach number) as well as the non-dimensional parameter $\mu_0 = O(1)$.

Remark 2. *Since the fluids (gas and liquid) are barotropic, equations (1) admit the Lagrange-Cauchy integral. However, this does not simplify the asymptotic procedure below.*

3. Asymptotic almost-periodic solution of (1)-(6)

The problem (1)-(6) contains two small parameters, one of which is associated with the density ratio $\delta_0 \ll 1$ but the second small parameter is the non-dimensional value $\sigma_*^{-2} \ll 1$, which physically implies that the sound frequency is much larger than the lowest eigenfrequency of the interfacial (sloshing) waves [11]. To construct an almost-periodic solution, we assume the following asymptotic relations between the two small parameters

$$\rho_{01}/\rho_{02} = \delta_0 = \mu_1\varepsilon, \quad \mu_1 = O(1); \qquad \sigma_*^{-2} = \mu\ \mu_1\varepsilon^3, \quad \mu = O(1). \quad (7)$$

The asymptotic procedure adopts the multi-timing technique of vibrational mechanics [3], which introduces fast and slow time scales such that the fast time is associated with the dimensionless time t appearing in the inhomogeneous condition (6) (expresses the input acoustic signal) and the slow time scale τ should be proportional to the square-root of the dimensionless forces of potential type (Newtonian gravitation and surface tension). The latter forces are of the order $O(\varepsilon^3)$; they appear in the dynamic interface condition (4) and the Euler equations (1). Therefore, the slow time is defined as $\tau = \varepsilon^{3/2}t$ and the non-dimensionless solution of (1)-(6), (7) can be posed in the following form

$$\begin{aligned} \varphi_i &= \sum_{k=0}^{\infty} \varepsilon^{k/3}\ \varphi_i^{(k)}(x,y,z,t,\tau), \quad p_i = \sum_{k=0}^{\infty} \varepsilon^{k/3}\ p_i^{(k)}(x,y,z,t,\tau), \\ \rho_i &= \sum_{k=0}^{\infty} \varepsilon^{(k/3)}\ \rho_i^{(k)}(x,y,z,t,\tau), \quad \xi = \sum_{k=0}^{\infty} \varepsilon^{k/3}\ \xi_k(x,y,z,t,\tau). \end{aligned} \tag{8}$$

Substituting (8) into (1)-(6) and using the standard multi-timing technique, which separates t and τ, derives the free-surface (sloshing-type) problem

$$\Delta\varphi = 0 \quad \text{in } \langle Q_2\rangle(\tau), \tag{9}$$

$$\partial_n\varphi = 0 \quad \text{on } \langle S_2\rangle(\tau), \tag{10}$$

$$\partial_n\varphi = -\partial_\tau\zeta/|\nabla\zeta| \quad \text{on } \langle\Sigma\rangle(\tau), \tag{11}$$

$$\begin{aligned} &\partial_\tau\varphi + \frac{1}{2}(\nabla\varphi)^2 + \mu\mu_1\,(\mathrm{Bo}\ x - (K_1+K_2)) + \\ &\quad + \frac{\mu_1}{4}\left(k^2\Phi^2 - (\nabla\Phi)^2\right) = \text{const} \quad \text{on } \langle\Sigma\rangle(\tau), \\ &\quad - \frac{(\nabla W, \nabla\zeta)}{|\nabla W||\nabla\zeta|} = \cos\alpha \quad \text{on } \partial\langle\Sigma\rangle(\tau); \qquad \int_{\langle Q_2\rangle} dQ = \text{const} \end{aligned} \tag{12}$$

subject to

$$\begin{aligned} &\Delta\Phi + k^2\Phi = 0 \ \ \text{in } \langle Q_1\rangle(\tau); \\ &\partial_n\Phi = 0 \ \ \text{on } \langle S_1\rangle(\tau) \cup \langle\Sigma\rangle(\tau); \\ &\partial_n\Phi = \mu_0\frac{V(x,y,z)}{k} \ \ \text{on } S_0, \end{aligned} \tag{13}$$

which describes the wave function Φ in the slowly changing gas domain $\langle Q_1\rangle(\tau)$.

Here, $\langle\cdot\rangle$ denotes averaging by the fast time t and, therefore, $\langle Q_2\rangle(\tau)$, $\langle S_2\rangle(\tau)$ and $\langle\Sigma\rangle(\tau)$ are the fast-time averaged liquid domain, wetted tank surface and interface, respectively. The boundary value problem (9)-(13) couples the main terms of the asymptotic representation (8)

$$\begin{aligned} &\varphi_2 = \varepsilon\ \varphi(x,y,z,\tau) + o(\varepsilon); \\ &\varphi_1 = \varepsilon^{2/3}\ \Phi(x,y,z,\tau)\sin t + O(\varepsilon); \\ &\xi = \zeta(x,y,z,\tau) + o(\varepsilon), \end{aligned} \tag{14}$$

which are also independent of t.

Remark 3. *The boundary value problem (9)-(12) is of the mathematically identical structure to the classical sloshing problem of a capillary liquid but with extra pseudo-differential terms in the dynamic boundary condition associated with Φ appearing as solution of the Neumann boundary value problem (13). These extra terms can be interpreted as the acoustic radiation pressure. The radiation pressure parametrically depends on the slowly-varying interface $\langle\Sigma\rangle(\tau)$.*

4. Acoustic equilibria and relative harmonic waves

If the -time averaged interface does not depend on the slow time τ, i.e.

$$\langle\Sigma\rangle = \Sigma_0 : \zeta_0 = \zeta_0(x,y,z) = 0, \langle Q_i\rangle = \langle Q_i\rangle_0 \ (i=1,2),$$
$$\varphi = 0, \quad \Phi = \Phi_0(x,y,z),$$

the problem (9)-(13) reduces to the stationary boundary problem

$$\begin{aligned} &-\mu(K_1+K_2) - \mu \text{ Bo } x + \frac{1}{4}\left(k^2 \Phi_0^2 - (\nabla\Phi_0)^2\right) = \text{const} \quad \text{on } \Sigma_0, \\ &-\frac{(\nabla W, \nabla\zeta_0)}{|\nabla W||\nabla\zeta_0|} = \cos\alpha \quad \text{on } \partial\Sigma_0; \quad \int_{\langle Q_2\rangle_0} dQ = \text{const}, \end{aligned} \tag{15}$$

where Φ_0 comes from the Newman boundary value problem

$$\begin{aligned} &\Delta\Phi_0 + k^2 \Phi_0 = 0 \quad \text{in } \langle Q_1\rangle_0; \\ &\partial_n\Phi_0 = 0 \quad \text{on } \langle S_1\rangle_0 \cup \Sigma_0; \\ &\partial_n\Phi_0 = \mu_0\frac{V(x,y,z)}{k} \quad \text{on } S_0, \end{aligned} \tag{16}$$

$(S_0 \cup \Sigma_0 \cup \langle S_1\rangle_0 = \partial\langle Q_1\rangle_0)$. Equality (15) expresses a balance between surface tension, gravitation and the Langevin acoustic radiation. Following [5], solution of (15), (16) (surface Σ_0 and wave function Φ_0) is called the *acoustic equilibrium.*

Remark 4. *For the introduced asymptotic relations (7), the time-averaged (mean) surface Σ_0 may dramatical differ from the capillary surface. The Langevin acoustic radiation can also influence stability of Σ_0 as well as the natural sloshing frequencies and modes by (9)-(13), which are, in fact, small harmonic waves relative to Σ_0.*

Suppose Σ_0 admits the singe-valued representation, $x = H_0(y,z)$, and linearise (9)–(13) relative to the acoustic equilibrium Σ_0. Furthermore, we consider the natural sloshing modes (H, ψ, Ψ) and frequencies (ω), which correspond to the harmonic solution

$$h = \exp(i\omega\tau)H(y,z); \quad \varphi = i\omega\exp(i\omega\tau)\psi(x,y,z), \quad \Phi = i\omega\exp(i\omega\tau)\Psi(x,y,z)$$

of the linearised problem. The result is the spectral boundary problem with respect to H and ψ

$$\Delta\psi = 0 \text{ in } \langle Q_2\rangle_0; \ \partial_n\psi = 0 \text{ on } \langle S_2\rangle_0; \ \partial_n\psi = \frac{H}{(1+(\nabla H_0)^2)^{1/2}} \text{ on } \Sigma_0, \tag{17}$$

$$-\omega^2 \psi + \mu_1\mu A H = 0 \quad \text{on} \quad \Sigma_0, \tag{18}$$

where ω^2 is the spectral parameter and the linear operator $A = A_1 + A_2$ takes the form

$$\begin{aligned} AH &= [A_1H] + [A_2H] = \\ &= \left[-\mathrm{div}\left\{\frac{\nabla H}{(1+(\nabla H_0)^2\ )^{1/2}} - \frac{(\nabla H, \nabla H_0)\nabla H_0}{(1+(\nabla H_0)^2\ )^{3/2}}\right\}\right] + \\ &+ \left[\frac{1}{2\mu}\left\{k^2\ \Phi_0\ \Phi_{0x}\ H - (\nabla\Phi_0, \nabla\Phi_{0x})\, H + \right.\right. \\ &\left.\left. + k^2\ \Phi_0\ \Psi - (\nabla\Phi_0, \nabla\Psi)\right\} + \mathrm{Bo}\, H\right], \end{aligned} \tag{19}$$

$$\begin{gathered} \frac{W_yH_y + W_zH_z}{|\nabla_2 W|} = \frac{W_yH_{0y} + W_zH_{0z}}{|\nabla_2 W|}\frac{(\nabla H, \nabla H_0)}{(1+(\nabla H_0)^2\ )^{1/2}} \text{ on } \partial\Sigma_0; \\ \int_{\Sigma_0} H dy dz = 0, \end{gathered} \tag{20}$$

$$\begin{gathered} \Delta\Psi + k^2\Psi = 0 \ \text{ in } Q_0; \ \ \partial_n\Psi = 0 \ \text{ on } \langle S_1\rangle_0 \cup S_0, \\ \partial_n\Psi = \frac{\Phi_{0xx}H - \Phi_{0z}H_z - \Phi_{0y}H_y - [\Phi_{0xy}H_{0y} + \Phi_{0xz}H_{0z}]H}{(1+(\nabla H_0)^2\ )^{1/2}} \ \text{ on } \Sigma_0. \end{gathered} \tag{21}$$

One can study spectral properties of the pseudo-differential operator A and show that it is self-conjugated and has a real pointer spectrum with only a finite set of negative eigenvalues. The following theorem establishes main properties of (17), (18) with the operator (19)–(21).

Theorem 1. *Let H_0, Φ_0 be a solution of the acoustic equilibria problem (9)-(13) such that $H_0 \in C^2(p\Sigma_0)$ and $\Phi_0 \in C^2(\langle Q_1\rangle_0 \cup \Sigma_0)$ (here, $p\Sigma_0$ is the projection of Σ_0 on the Oyz plane). Then*

1. *The spectral boundary problem (17)-(21) has a real pointer spectrum consisting of eigenvalues and $\{H_n\}$ is the functional basis in the factor-space $L_2(p\Sigma_0)/const$.*
2. *The set of negative eigevalues $\{n|\,\omega_n^2 < 0\}$ is finite.*

Proof. Introduce the auxiliary Steklov-Poincáre operator $T : H \to \psi|_{\Sigma_0}$, which is defined by the Neumann problem (17). This operator T is precompact and invertible on the dense set in the factor-space $L_2(p\Sigma_0)/\mathrm{const}$. The boundary condition (18) yields the spectral equation

$$C_0(\omega^2)H = (\mu\mu_1 A - \omega^2 T)H = 0. \tag{22}$$

Spectrum of (22) coincides with spectrum of the original problem (17)-(21).

Consider operator A_1, defined by formulas (19). It appears when analysing the eigenoscillations of the capillary liquid and is unbounded, self-conjugate and positive in $L_2(p\Sigma_0)/\mathrm{const}$. Let us introduce the auxiliary operators C_1 and C_2 as

$$C_1(\omega^2) = \omega^2 A_1^{-1}T - \mu\mu_1(E + A_1^{-1}A_2) = C_2(\omega^2) - \mu\mu_1 E,$$

where C_1 is due to the action of A_1^{-1} from the left on operator C_0 in (22). The operator $C_2(\omega^2)$ is precompact in the factor-space $L_2(p\Sigma_0)/\mathrm{const}$. If ω^2 is the eigenvalue of (22), then $\mu\mu_1$ is the eigenvalue of the self-conjugate operator C_2, and, therefore, ω^2 is the eigenvalue of the original spectral problem (17)-(21). Because T and A are self-conjugate operators, their eigenvalues are real.

Regular set of the spectral boundary problem (17)-(21) is not empty and contains, at least, complex numbers with non-zero imaginary components. For a regular point ω_0^2, equation (22) is equivalent to the spectral equation

$$\left(C + (\omega^2 - \omega_0^2)^{-1} E\right) H = 0$$

where $C(\omega_0^2) = C_1(\omega_0^2)^{-1} A_1^{-1} T$ is the compact operator in $L_2(p\Sigma_0)$. Because C is compact, its pointer spectrum consists of eigenvalues. As a consequence, the first assertion of the theorem holds true.

All eigenvalues of $A_1^{-1}T$ are positive and follow from the spectral boundary problem on the natural sloshing modes and frequencies of the capillary liquid, i.e. for all admissible H, the inequality

$$(A_1^{-1} T H, H) > 0$$

holds true. Therefore,

$$\omega_n^2 = \mu\mu_1((H_n, H_n) + (A_1^{-1} A_2 H_n, H_n))/(A_1^{-1} T H_n, H_n),$$

where $(H_n, H_n) = 1$, $(A_1^{-1} T H_n, H_n) > 0$. Because $A_1^{-1} A_2$ is compact and $\{H_n\}$ is the functional base in $L_2(p\Sigma_0)$, then $(A_1^{-1} A_2 H_n, H_n) \to 0, n \to \infty$. Therefore, the second assertion holds.

Corollary 4.2 a. The acoustic equilibria may blow up only due to a finite set of linearly-independent perturbations.
Corollary 4.2 b. The acoustic equilibria are stable, if and only if, all eigenvalues $\{\omega_n^2\}$ of A are positive.

The second corollary is the same as the so-called spectral stability criteria, which was already used in [11] for analysing the stability of the capillary equilibria. The stability was investigated by studying the spectrum of the A_1-type operator.

Example 1. (The flat acoustic equilibrium.) The flat capillary surface in an upright cylindrical tank is realised for the contact angle $\alpha = \pi/2$. The flat Σ_0 is also possible for the acoustic equilibria when acoustic vibrator on S_0 generates a planar standing wave, namely, when

$$V_0(x, y, z) = V_0 = \text{const} \quad \left(\varepsilon = -\frac{V_0}{c \ \sin(kh_1)}, \mu_0 = -\sin(kh_1), V(y, z) = 1\right).$$

The acoustic equilibrium is then associated with the following solution

$$H_0(y, z) \equiv 0; \quad \Phi_0(x, y, z) = k^{-2} \cos(kx). \tag{23}$$

According to [11, 14], the flat capillary surface corresponds to a unique solution of the capillary problem in an upright circular cylinder, if and only if, $\text{Bo} > \kappa_{11}^2$, where κ_{11} is the minimum root of $J_1'(\kappa_{11}) = 0$ ($J_p(\cdot)$ is the Bessel function of the first kind). Let us pose solutions of the nonlinear boundary value problem (15), (16) as the Fourier series by

$$h_{pq}(r, \theta) = J_p(\kappa_{pq} r) \genfrac{}{}{0pt}{}{\sin}{\cos}(p\theta)$$

in the cylindrical coordinate system, i.e.

$$H_0(r,\theta) = \sum_{pq\neq 00} \eta_{pq}\; h_{pq}(r,\theta), \tag{24}$$

and

$$\Phi_0(x,y,z) = k^{-2}\cos(kx) + \sum_{pq\neq 00} \chi_{pq}\, b_{pq}(x)\; h_{pq}(r,\theta) + \chi_{00}\;\cos(k(x-h_1)), \tag{25}$$

where

$$b_{pq}(x) = \begin{cases} -\dfrac{\cosh(\phi(x-h_1))}{\cosh(\phi h_1)\;\phi\;\tanh(\phi h_1)}, & \kappa_{pq} > k, \\ -\dfrac{\cos(\phi(x-h_1))}{\cos(\phi h_1)\;\phi\;\tan(\phi h_1)}, & \kappa_{pq} < k, \end{cases} \qquad \phi = \sqrt{|\kappa_{pq}^2 - k^2|},$$

in which η_{pq}, χ_{pq} are the unknown coefficients.

Each index pq corresponds to two unknown coefficients for asymmetric solutions and one for symmetric ones $h_{pq}(r,\theta)$, namely,

$$\eta_{pq}\; h_{pq}(r,\theta) = \begin{cases} \eta'_{pq}\; J_p(\kappa_{pq} r)\sin p\theta + \eta''_{pq}\; J_p(\kappa_{pq} r)\cos p\theta, & p\neq 0, \\ \eta_{0q}\; J_0(\kappa_{0q}), & p = 0. \end{cases} \tag{26}$$

Inserting (24) and (25) into equations (15) and (16) and using the Fredholm alternative leads to an infinite system of nonlinear equations with respect to $\eta = \{\eta_{pq}\}$. To within the $o(||\eta||)$-quantities, we have the equalities

$$G_{\alpha\beta} = C_{\alpha\beta}\;\eta_{\alpha\beta} + o(||\eta||) = 0, \tag{27}$$

where

$$C_{pq} = \mu(\mathrm{Bo}\; + \kappa_{pq}\;) + \frac{1}{2}\; b_{pq}(0), \quad p = 0,1,...;\; q = 1,2,... \tag{28}$$

(C_{pq} are the eigenvalues of the operator A).

The system (27) admits the trivial solution $\eta = 0$, which corresponds to the flat acoustic equilibrium. Trivial solution is stable as $C_{pq} > 0$. When there is an index pq, such that $C_{pq}(k) = 0$, the trivial solution may not become unique. For the eigenvalues with $p \neq 0$, two equations in (27) do not have linear components at η_{pq} but the eigenvalues $C_{0q},\; q = 1,2,\ldots$ have the single multiplicity. In the latter case, the Krasnoselsky theorem [9] gives the sufficient condition of bifurcation of the trivial solution.

5. Pseudo-potential energy of acoustic equilibria

The above example shows that finding the stable acoustic equilibria from its differential statement (15), (16) can be efficient when interface Σ_0 coincides with the capillary surface. If the acoustic equilibrium Σ_0 differs from the capillary surface, identifying solutions of (15), (16) and studying their stability may become a rather complicated task. For the capillary surface, this task sufficiently simplifies by employing the potential energy functional whose minima correspond to the stable liquid shapes. Finding these shapes reduces to a direct numerical minimisation of the potential energy functional.

Theorem 1 in [1] states that the smooth solution of (1)-(6) can follow from necessary extrema condition of the functional

$$
\begin{aligned}
G(\xi,\varphi_i,\rho_i) = \int_{t_1}^{t_2} \Bigg\{ & \int_{Q_2} \rho_2 \left(\frac{(\nabla\varphi_2)^2}{2} - U_2(\rho_2) - \mu\mu_1\, \varepsilon^3 \,\mathrm{Bo}\, x \right) dQ - \\
& - \mu\mu_1\, \varepsilon^3 \, \left(|\Sigma| - \cos\alpha |S_2|\right) + \\
& + \varepsilon \int_{Q_1} \rho_1 \left(\frac{(\nabla\varphi_1)^2}{2} - U_1(\rho_1) - \mu\mu_1\, \varepsilon^3 \,\mathrm{Bo}\, x \right) dQ \Bigg\} dt
\end{aligned}
\tag{29}
$$

subject to (1)-(3), (6) and for small variations

$$
\delta\xi|_{t_1,t_2} = 0, \qquad \delta\rho_i|_{t_1,t_2} = 0 \tag{30}
$$

where $p_i = \rho_i^2 \, dU_i \,/d\rho_i$.

By using the multi-timing technique, one can show that

$$
\langle G(\xi,\varphi_i,\rho_i)\rangle = \mathrm{const} + \varepsilon\,\mathcal{G}(\zeta,\varphi) + \mathrm{O}(\varepsilon^{4/3}),
$$

where

$$
\begin{aligned}
\mathcal{G}(\zeta,\varphi) = \int_{\tau_1}^{\tau_2} \Bigg\{ & \int_{\langle Q_2\rangle} \left(\frac{(\nabla\varphi)^2}{2} - \mu\mu_1 \mathrm{Bo}\, x \right) dQ - \\
& - \mu\mu_1 \left(|\langle\Sigma\rangle| - \cos\alpha |\langle S_2\rangle|\right) + \\
& + \frac{\mu_1}{4} \int_{\langle Q_1\rangle} \left(k^2\Phi^2 - (\nabla\Phi)^2\right) dQ - \frac{\mu_0\mu_1}{2k} \int_{S_0} \Phi V(x,y,z) dS \Bigg\} d\tau,
\end{aligned}
\tag{31}
$$

where

$$
\int_{\langle Q_2\rangle} \frac{(\nabla\varphi)^2}{2} dQ
$$

implies the pseudo-kinetic energy for the sloshing problem (9)–(13) but the remaining quantities can be interpreted as the minus pseudo-potential energy.

Theorem 2. *The problem on the stable acoustic equilibria* $\Sigma_0 : \ \zeta_0 = 0$ *is equivalent to identifying the minima of the functional*

$$
\begin{aligned}
\Pi(\zeta_0) = \mu & \left(|\Sigma_0| + \cos\alpha |\langle S_1\rangle| + \int_{\langle Q_2\rangle_0} Bo\, x dQ \right) + \\
& + \left(\frac{1}{4} \int_{\langle Q_1\rangle_0} (k^2\Phi_0^2 - (\nabla\Phi_0)^2) dQ + \frac{\mu_0}{2k} \int_{S_0} V(x,y,z)\Phi_0 dS \right) = \\
& = -\mathcal{G}(\zeta_0(x,y,z), \Phi_0(x,y,z)),
\end{aligned}
\tag{32}
$$

where Φ_0 *is the solution of (16) subject to the volume conservation condition*

$$
\int_{\langle Q_2\rangle_0} dQ = const.
$$

The proof comes from computing the second variation by H_0 of the functional $\Pi(x - H_0)$. The second variation by Σ_0 for the surface tension quantities was already derived in [11] (chapter 1). The first variation by Φ_0 is equal to zero restricted to (16) but the first variation by H_0 leads to equation (15), which links Φ_0 and H_0. Furthermore,

$$\delta^2\Pi = \mu^{-1}\int_{p\Sigma_0}(A\delta H, \delta H)dydz,$$

where A is the operator by (19)-(21). Condition $\delta^2\Pi > 0$ is equivalent to the spectral stability criteria 4.2 a.

6. Conclusions

By applying the fast-time averaging of the non-dimensional free-interface problem for two compressible fluids, the mathematical theory of levitating drops in [5] is generalised to study how acoustic field in the ullage gas may affect the mean (time-averaged) liquid-gas interface (called the acoustic equilibrium) and its stability. The theory includes a spectral theorem on the natural frequencies and modes and a pseudo-potential energy introduced for the acoustic equilibria.

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Bibliography

1. Beyer K. Compressible potential flows with free boundaries. Part I: Vibrocapillary equilibria / K. Beyer, M. Gunther, I. Gawrilyuk, I. Lukovsky, A. Timokha // ZAMM. – 2001. – Vol. 81. – P. 261-271.
2. Beyer K. Variational and finite element analysis of vibroequilibria / K. Beyer, M. Guenther, A. Timokha // Comput. Methods Appl. Math. – 2004. – Vol. 4., No 3. – P. 290-323.
3. Blekhman I.I. Vibrational Mechanics. Nonlinear Dynamic Effects, General Approach, Applications / I.I. Blekhman. – Singapore: World Scientific, 2000.
4. Brandt E.H. Suspended by sound / E.H. Brandt // Nature. – 2001. – Vol. 413. – P. 474-475.
5. Chernova M. Differential and variational formalism for an acoustically levitating drop / M.O. Chernova, I.A. Lukovsky, A.N. Timokha // Journal of Mathematical Sciences. – 2017. – Vol. 220, Issue 3. – P. 359-375.
6. Eberhardt R. Acoustic levitation device for sample pretreatment in microanalysis and trace analysis / R. Eberhardt, B. Neidhart // Fresenius' J. Anal. Chem. – 1999. – Vol. 365. – P. 475–479.
7. Fernandez J. The CFVib experiment: control of fluids in microgravity with vibrations / J. Fernandez, P. Salgado Sanchez, I. Tinao, J. Porter, J.M. Ezquerro // Microgravity Science and Technology. – 2017. – Vol. 29, Issue 5. – P. 351-364.
8. Gavrilyuk I. Two-dimensional variational vibroequilibria and Faraday's drops / I. Gavrilyuk, I. Lukovsky, A. Timokha // ZAMP. – 2004. – Vol. 55. – P. 1015-1033.
9. Krasnoselsky M.A. Topological Methods in the Theory of Nonlinear Integral Equations / M.A. Krasnoselsky. – New-York: Pergamon Press, 1964.
10. Lukovskii I.A. Stabilization of liquid-gas interface in the presence of interaction with acoustic fields in the gas / I.A. Lukovskii, A.N. Timokha // Fluid Dynamics. – 1991. – Vol. 26, Issue 3. – P. 382-388.
11. Myshkis A. Low-gravity Fluid Mechanics: Mathematical Theory of Capillary Phenomena / A. Myshkis, V. Babskii, N. Kopachavskii, L. Slobozhanin, A. Tiuptsov. – Berlin: Springer-Verlag, 1987.
12. Sanchez P.S. Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity / P. Salgado Sanchez, V. Yasnou, Y. Gaponenko, A. Mialdun, J. Porter, V. Shevtsova // J. Fluid Mechanics. – 2019. – Vol. 865. – P. 850-883.

13. Timokha A. Influence of sound on the normal modes of oscillation of a liquid-gas interface in a bounded volume / A. Timokha // Acoustical Physics. – 1993. – Vol. 39. – P. 187-189.
14. Ural'tseva N.N. Solvability of the capillary problem II / N.N. Ural'tseva // Vestn. Leningr. Univ. Math. – 1980. – Vol. 8. – P. 151-158.
15. Weber R.J.K. Acoustic levitation: recent developments and emerging opportunities in biomaterials research / R.J.K. Weber, C.J. Benmore, S.K. Tumber, A.N. Tailor, C.A. Rey, L.S. Taylor, S.R. Byrn // Eur. Biophys. J. – 2012. – Vol. 41, No. 4. – P. 397-403.
16. Wesseln Ph.S Acoustic pumping in cryogenic liquids / Ph.S. Wesseln // Design News. – 1967. – Vol. 22. – P. 96-101.

E. V. Tkachenko,
Institute of Mathematics of NAS of Ukraine,
3, Tereschenkivs'ka st., Kyiv-4, 01004, Ukraine;

A. N. Timokha,
Centre for Autonomous Marine Operations and Systems,
Department of Marine Technology,
Norwegian University of Science and Technology,
NO-7491, Trondheim, Norway.

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