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A UNIFIED LOCAL CONVERGENCE STUDY OF MULTISTEP ALGORITHMS

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Резюме.У роботі проведено аналіз локальної збіжності узагальнених багатокрокових алгоритмів за єдиного набору критеріїв збіжності. Також розглянуто деякі відомі багатокрокові методи і отримано точніші оцінки похибок і більші радіуси областей збіжності. Проведено чисельні експерименти, які підтверджують отримані теоретичні результати.

Abstract. We provide a unified local convergence analysis of generalized multistep algorithms under the same set of convergence criteria. Some known multistep methods are also considered and tighter error estimates and larger radii of convergence domain are obtained. Numerical experiments, which confirm the obtained theoretical results, are carried out.

1. Introduction

Let B_1, B_2 denote Banach spaces, $\Omega \subset B_1$ stand for an open and nonempty set and $U(z,r), \bar{U}(z,r)$ be the open and closed balls in B_1 respectively centered at $z \in B_1$ and of radius $r > 0$.

We are interested in obtaining a solution x^* of equation

$$F(x) = 0, \tag{1}$$

by using iterative algorithms, since the closed form of it is attainable only in special cases. There is a plethora of applications from diverse disciplines that using mathematical modeling can be written in the form of equation (1) [8,9,17]. But the convergence order of these algorithms is found using Taylor series with higher than ones derivatives not appearing on the algorithm. This is a setback restricting the applicability of algorithms. As an academic and motivational example consider $B_1 = B_2 = \mathrm{R}$, $\Omega = \left[-\frac{3}{2}, \frac{3}{2}\right]$, $t^* = 1$ and define function h on Ω by

$$h(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, \ t \neq 0 \\ 0, \quad t = 0. \end{cases}$$

Then, we get $h'''(t) = 6 \ln t^2 + 60t^2 - 24t + 22$, so the third derivative is unbounded on Ω. Hence, convergence results based on the third derivative (or higher) cannot guarantee the convergence of the algorithm. That is why there is a need for convergence results with as weak conditions on F as possible. In particular, there is a need for a unified study of multistep algorithms (MA) of the form

Key words. Multistep algorithm; frozen derivative; local convergence; Banach space; radius of convergence.

$$
\begin{aligned}
x_{-2} &= x_{-2}^{(0)},\ x_{-1} = x_{-1}^{(0)},\ x_i = x_i^{(0)},\ i = -2, -1, ...,\\
x_n^{(0)} &= \varphi^{(0)}(x_{n-1}^{(0)}, x_{n-2}^{(0)}) = x_n,\\
x_n^{(1)} &= \varphi^{(1)}(x_n^{(0)}, x_{n-1}^{(0)}) = x_n^{(0)} - \tilde{\varphi}^{(1)}(x_n^{(0)}, x_{n-1}^{(0)})F(x_n^{(0)}),\\
x_n^{(2)} &= \varphi^{(2)}(x_n^{(0)}, x_{n-1}^{(0)}) = x_n^{(1)} - \tilde{\varphi}^{(2)}(x_n^{(0)}, x_{n-1}^{(0)})F(x_n^{(1)}),\\
\dots & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2)\\
x_n^{(m-1)} &= \varphi^{(m-1)}(x_n^{(0)}, x_{n-1}^{(0)}) = x_n^{(m-2)} - \tilde{\varphi}^{(m-1)}(x_n^{(0)}, x_{n-1}^{(0)})F(x_n^{(m-2)}),\\
x_{n+1} = x_n^{(m)} &= \varphi^{(m)}(x_n^{(0)}, x_{n-1}^{(0)}) = x_n^{(m-1)} - \tilde{\varphi}^{(m)}(x_n^{(0)}, x_{n-1}^{(0)})F(x_n^{(m-1)}),\\
n &= 0, 1, 2, ...,
\end{aligned}
$$

m is a given natural number and $\varphi^{(j)} : \Omega \times \Omega \to B_1,\ j = 0, 1, 2, ..., m$ are continuous operators related to F such that $\lim_{n\to\infty} x_n = x^*$. A plethora of specializations of operators $\tilde{\varphi}$ lead to well studied algorithms (or new methods):

(1) m-step Newton's algorithm [1, 17]:

$$\tilde{\varphi}^{(j)}(x, y) = F'(x)^{-1};$$

(2) m-step simplified Newton's algorithm:

$$\tilde{\varphi}^{(j)}(x, y) = F'(x_0)^{-1};$$

(3) m-step Secant algorithm:

$$\tilde{\varphi}^{(j)}(x, y) = [y, x; F]^{-1};$$

(4) m-step Steffensen's-like algorithm:

$$\tilde{\varphi}^{(j)}(x, y) = [x - \lambda_n F(x), x + \lambda_n F(x); F]^{-1},\ \lambda_n \in \mathrm{R};$$

(5) m-step Kurchatov's algorithm [5]:

$$\tilde{\varphi}^{(j)}(x, y) = [2x - y, y; F]^{-1};$$

(6) m-step Stirling's algorithm:

$$\tilde{\varphi}^{(j)}(x, y) = F'(x - F(x))^{-1};$$

(7) m-step Newton's-like algorithm:

$$\tilde{\varphi}^{(j)}(x, y) = A(x)^{-1},\ A : \Omega \to L(B_1, B_2).$$

Many other choices for the φ (or $\tilde{\varphi}$) functions are possible [2–4, 6, 7, 10–12, 15, 16]. Therefore it is important to study the local convergence of MA under the same set of conditions. This is done in Section 2, where as applications and the conclusions appear in Section 3 and Section 4, respectively.

2. Local Convergence

It is convenient for the local convergence that follows in this section to first define some parameters and scalar functions. Set $S = [0, \infty)$. Let $\psi^{(j)} : S \times S \to S$ be continuous and nondecreasing functions, $j = 0, 1, 2, ..., m$. Consider functions $g_j : S \to S$ defined as

$$g_j(t) = \psi^{(j)}(t,t)\psi^{(j-1)}(t,t)...\psi^{(0)}(t,t). \tag{3}$$

Suppose equations

$$g_j(t) - 1 = 0 \tag{4}$$

have smallest solutions $\rho^{(j)} \in S - \{0\}$, respectively. Set

$$\rho = \min\{\rho^{(j)}\}. \tag{5}$$

We shall show that parameter ρ is a radius of convergence for MA. It follows by these definitions that

$$0 \leq g_j(t) \leq c < 1 \tag{6}$$

for all $t \in [0, \rho)$ and some $c \in [0, 1)$.

The conditions (A) are needed:

(A_1) There exist $F : \Omega \to B_2$, $\Omega_0 \subseteq \Omega$, $\varphi^{(j)} : \Omega_0 \times \Omega_0 \to B_1$ continuous operators, and $x^* \in \Omega_0$, such that $F(x^*) = 0$.

(A_2) $\rho^{(j)}$ given by (3) and (4) exist, $j = 0, 1, 2, ..., m$.

(A_3) There exist functions $\psi^{(j)} : S\times S \to S$ are continuous and nondecreasing such that for each $x \in \Omega_0, y \in \Omega_0$

$$\|\varphi^{(j)}(y,x) - x^*\| \leq \psi^{(j)}(\|y - x^*\|, \|x - x^*\|)\|y - x^*\|.$$

(A_4) $\bar{U}(x^*, \rho) \subset \Omega$, where ρ is given in (5).

Next, the main local convergence result for MA is presented using the conditions (A) and the introduced terminology.

Theorem 1. *Suppose the conditions (A) hold. Choose $x_{-2}, x_{-1}, x_0 \in \bar{U}(x^*, \rho) - \{x^*\}$. Then, sequence $\{x_n\}$ starting from x_0 and generated by MA (2) exists, remains in $\bar{U}(x^*, \rho)$ for each $n = 0, 1, 2, ...$ and converges to x^*.*

Proof. By the choice of x_{-2}, x_{-1}, x_0, conditions $(A_1) - (A_3)$ and (6) we have in turn that

$$\|x_n^{(0)} - x^*\| = \|\varphi^{(0)}(x_{n-1}, x_{n-2}) - x^*\|$$
$$\leq \psi^{(0)}(\|x_{n-1} - x^*\|, \|x_{n-2} - x^*\|)\|x_{n-1} - x^*\| \leq c\|x_{n-1} - x^*\| < \rho,$$

$$\|x_n^{(1)} - x^*\| = \|\varphi^{(1)}(x_n, x_{n-1}) - x^*\|$$
$$\leq \psi^{(1)}(\|x_n - x^*\|, \|x_{n-1} - x^*\|)\|x_n - x^*\|$$
$$\leq \psi^{(1)}(\rho,\rho)\psi^{(0)}(\rho,\rho)\|x_n - x^*\| \leq c^2\|x_{n-1} - x^*\| < \rho,$$
$$\cdots$$
$$\|x_{n+1} - x^*\| = \|x_n^{(m)} - x^*\| \leq c^{m+1}\|x_0 - x^*\| < \rho,$$

so $x_{n+1} \in U(x^*, \rho)$ and $\lim_{n\to\infty} x_n = x^*$. □

Remark 1. (i) *The condition $F(x^*) = 0$ is not needed to show the convergence of sequence $\{x_n\}$ to x^*.*

(ii) *The convergence is shown to be only linear at this generality. But if iteration functions are specialized as in the examples of the introduction, then the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [6] can be used given, respectively by*

$$\mu = \frac{\ln \dfrac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}}{\ln \dfrac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}}, \text{ for each } n = 1, 2, ...,$$

$$\mu_0 = \frac{\ln \dfrac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}{\ln \dfrac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}}, \text{ for each } n = 2, 3, ...$$

Note that the computation of these parameters does not require high order derivatives or even knowledge of the solution x^* (in the case of μ_0).

(iii) *In the case of Newton's algorithm*

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le l_0\|x - x^*\| \text{ for all } x \in \Omega,\ l_0 > 0$$

and

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \le l\|y - x\| \text{ for all } x, y \in \Omega_0 = \Omega \cap U(x^*, \tfrac{1}{l_0}),\ l > 0.$$

Moreover, using the estimate

$$x_{n+1} - x^* =$$

$$= -F'(x_n)^{-1}F'(x^*) \int_0^1 F'(x^*)^{-1}\Big[F'(x^* + \theta(x_n - x^*)) - F'(x_n)\Big] d\theta(x_n - x^*),$$

we see that

$$\psi^{(i)}(t, t) = \frac{lt}{2(1 - l_0 t)}.$$

Similar choices for the other algorithms listed in the introduction.

The old error bounds are

$$\|x_n^{(i+1)} - x^*\| \le e_n\|x_n^{(i)} - x^*\|,$$

whereas the new are

$$\|x_n^{(i+1)} - x^*\| \le \bar{e}_n\|x_n^{(i)} - x^*\|,$$

where

$$\bar{e}_n = \frac{l\|x_n^{(i)} - x^*\|}{2(1 - l_0\|x_n^{(i)} - x^*\|)} \le e_n = \frac{l_1\|x_n^{(i)} - x^*\|}{2(1 - l_1\|x_n^{(i)} - x^*\|)}$$

and

$$\rho_N \le \rho_N^1,$$

since

$$l_0 \leq l_1$$

and

$$l \leq l_1,$$

where l_1 *is the Lipschitz constant on* Ω *used in earlier studies [13, 14].*

(*iv*) *Consider Schmidt-Schwetlick method (14) in [11] defined as* x_n, $z_n \in \Omega$, *for* $0 \leq i \leq m-1$, $n = 0, 1, \ldots,$ *and*

$$\begin{aligned} x_n^{(0)} &= z_n, \\ x_n^{(i+1)} &= \Psi_{2,i+1}(x_n, x_{n-1}) = x_n^{(i)} - [x_n, z_n; F]^{-1} F(x_n^{(i)}), \\ x_{n+1} &= \Psi_{2,m-1}(x_n, x_{n-1}) = x_n^{(m-1)}, \\ z_{n+1} &= \Psi_{2,m}(x_n, x_{n-1}) = x_n^{(m)}. \end{aligned} \tag{7}$$

Then, in Theorem 3.2 under the condition

$$\|F'(x^*)^{-1}([x, y; F] - [u, v; F])\| \leq K(\|x - u\| + \|y - v\|) \tag{8}$$

for each $x, y, u, v \in \Omega$, *the radius*

$$\rho_{ss} = \frac{1}{5K}$$

was found. But under our idea [6, 8, 9] the condition

$$\|F'(x^*)^{-1}([x^*, x^*; F] - [x, y; F])\| \leq K_1(\|x^* - x\| + \|x^* - y\|)$$

for each $x, y \in \Omega$ *was used (implied by (8)), and the radius*

$$\rho_{ss}^1 = \frac{1}{2K_1 + 3K} \geq \rho_{ss},$$

$$K_1 \leq K$$

was given. But under our new technique a further enlargement can be found. Indeed, let $\Omega_0 = \Omega \cap U(x^*, \frac{1}{2K_1})$. *Notice that*

$$\Omega_0 \subseteq \Omega.$$

Consider instead of (8) the actually needed condition

$$\|F'(x^*)^{-1}([x, y; F] - [u, v; F])\| \leq \bar{K}(\|x - u\| + \|y - v\|)$$

for each $x, y, u, v \in \Omega_0$. *Then, we have*

$$\bar{K} \leq K,$$

so, the radius is

$$\rho_{ss}^2 = \frac{1}{2K_1 + 3\bar{K}} \geq \rho_{ss}^1.$$

These advantages are obtained under the same computational cost, since $\bar{K}$ *is a specialization of* K. *Moreover, the error bounds on* $\|x_n^{(i)} - x^*\|$ *become tighter, since the ones in [11] are given by*

$$\|x_n^{(i+1)} - x^*\| \leq \delta_n \|x_n^{(i)} - x^*\|,$$

whereas in our setting

$$\|x_n^{(i+1)} - x^*\| \le \bar{\delta}_n \|x_n^{(i)} - x^*\|,$$

where

$$\delta_n = \frac{K(\|x_n - x^*\| + \|z_n - x^*\| + \|x_n^{(i)} - x^*\|)}{1 - K_1(\|x_n - x^*\| + \|z_n - x^*\|)},$$

$$\bar{\delta}_n = \frac{\bar{K}(\|x_n - x^*\| + \|z_n - x^*\| + \|x_n^{(i)} - x^*\|)}{1 - K_1(\|x_n - x^*\| + \|z_n - x^*\|)}$$

and

$$\bar{\delta}_n \le \delta_n.$$

(v) *Similarly, we can extend the results given in [11] for the Kurchatov-Schmidt-Schwetlick method (26) defined for each* $n = 0, 1, \ldots$ *and* $0 \le i \le m-1$, $x_n,\ z_n \in \Omega$

$$\begin{aligned} x_n^{(i+1)} &= \Psi_{4,i+1}(x_n, x_{n-1}) = x_n^{(i)} - [z_n, 2x_n - z_n; F]^{-1} F(x_n^{(i)}), \\ x_n^{(0)} &= x_n,\ z_{n+1} = \Psi_{4,m-1}(x_n, x_{n-1}) = x_n^{(m-1)}, \\ x_{n+1} &= \Psi_{4,m}(x_n, x_{n-1}) = x_n^{(m)}. \end{aligned} \tag{9}$$

Then, in Theorem 3.5 in [11] under conditions

$$\|F'(x^*)^{-1}([x, y; F] - [u, v; F])\| \le K(\|x - u\| + \|y - v\|)$$

and

$$\|F'(x^*)^{-1}([x, x; F] - [y, 2x - y; F])\| \le L\|y - x\|^2$$

for each $x, y, u, v \in \Omega$, *the radius*

$$\rho_{kss} = \frac{2}{5K + \sqrt{25K^2 + 32L}}$$

and error bounds

$$\|x_n^{(i+1)} - x^*\| \le \sigma_n \|x_n^{(i)} - x^*\|$$

were given, where

$$\sigma_n = \frac{K(\|x_n^{(i)} - x_n\| + \|x_n - x^*\|) + L\|x_n - z_n|^2}{1 - 2K\|x_n - x^*\| - L\|x_n - z_n\|^2}.$$

But if we consider the actually needed conditions

$$\|F'(x^*)^{-1}([x^*, x^*; F] - [x, x; F])\| \le 2M_0\|x^* - x\|$$

and

$$\|F'(x^*)^{-1}([x, x^*; F] - [y, y; F])\| \le M_1(\|x - y\| + \|x^* - y\|),$$

then, we will have instead

$$\|x_n^{(i+1)} - x^*\| \le \bar{\sigma}_n \|x_n^{(i)} - x^*\|,$$

where

$$\bar{\sigma}_n = \frac{M_1(\|x_n^{(i)} - x_n\| + \|x_n - x^*\|) + L\|x_n - z_n\|^2}{1 - 2M_0\|x_n - x^*\| - L\|x_n - z_n\|^2}$$

and

$$\bar{\sigma}_n \leq \sigma_n,$$

since

$$M_0 \leq K$$

and

$$M_1 \leq K.$$

Moreover, the new radius is

$$\rho_{kss}^1 = \frac{2}{2M_0 + 3M_1 + \sqrt{(2M_0 + 3M_1)^2 + 32L}} \geq \rho_{kss}.$$

Notice that if $M_0 = M_1 = K$*, then*

$$\rho_{kss}^1 = \rho_{kss}.$$

Otherwise, we have

$$\rho_{kss}^1 > \rho_{kss}.$$

Examples, where the new constants are smaller can be found in the numerical section and in [4–6, 8, 9].

3. Numerical Examples

Let us consider the nonlinear integral equation [6]

$$F(x)(s) = x(s) - 5s \int_0^1 tx(t)^3 dt, \ \ s, t \in [0, 1],$$

where $x \in C[0,1]$, $F : C[0,1] \to C[0,1]$. This equation has two solutions: $x_1^*(s) = 0$ and $x_2^*(s) = s$.

It is easy to see that the derivative and divided difference of operator F are defined by formulas

$$F'(x)h(s) = h(s) - 15s \int_0^1 tx(t)^2 h(t) dt,$$

and

$$[x, y; F]h(s) = h(s) - 5s \int_0^1 t[x(t)^2 + x(t)y(t) + y(t)^2]h(t)dt,$$

respectively.

We have that

$$F'(x)h(s) - F'(y)h(s) = 15s \int_0^1 t(y(t) + x(t))(y(t) - x(t))h(t)dt,$$

$$[x, y; F]h(s) - [u, v; F]h(s) =$$

$$= 5s \int_0^1 t[(u(t)+y(t)+x(t))(u(t)-x(t))+(v(t)+y(t)+u(t))(v(t)-y(t))]h(t)dt.$$

Let's compute convergence radii of methods from Remark 1 for solution $x^*(s) = 0$. Given the equations written above, we get

$$l_1 = 15, \, l_0 = 7.5, \, l = 2;$$

$$K = 7.5,\ K_1 = 5,\ \bar{K} = 0.75;$$
$$K = 7.5,\ L = 2.5,\ M_0 = 2.5,\ M_1 = 5.$$

Obtained results are shown in Table 1 and confirm theoretical results.

Tabl. 1. The radii of converence domains

Multistep method	Old radius	New radius
Newton	$\rho_N \approx 0.0444$	$\rho_N^1 \approx 0.1176$
Schmidt-Schwetlick	$\rho_{ss} \approx 0.0267$	$\rho_{ss}^1 \approx 0.0308,\ \rho_{ss}^2 \approx 0.0816$
Kurchatov-Schmidt-Schwetlick	$\rho_{kss} \approx 0.0263$	$\rho_{kss}^1 \approx 0.0477$

Next, we give the number of iterations for which an approximate solution is obtained. To approximate an integral we use the trapezoidal rule. As result, we obtain the following system of nonlinear equations:

$$\xi_i - 5hs_i\Big(\frac{1}{2}t_0\xi_0^3 + \sum_{j=1}^{k-1} t_j\xi_j^3 + \frac{1}{2}t_k\xi_k^3\Big) = 0,\ i = 0, \dots, k,$$

where $\xi_i \approx x(s_i)$, $s_i = t_i = ih$, $i = 0, \dots, k$, $h = \frac{1}{k}$.

In Tables 2 and 3 there are number of iterations for such initial approximation x_0: $x^I = (0.25, \dots, 0.25)^T$ and $x^{II} = (10, \dots, 10)^T$. In the first case we get $x_1^*(s)$ and in the second case – $x_2^*(s)$. Additional initial approximation x_{-1} was computed by formula: $x_{-1} = x_0 + 0.0001$. To stop the iterative process we used the condition $\|x_{n+1} - x_n\| \leq 10^{-10}$. In the trapezoidal rule $k = 50$.

Tabl. 2. The number of iterations for x^I

Multistep method	$m = 1$	$m = 2$	$m = 3$
Newton	5	4	3
Secant	6	5	4
Kurchatov	6	5	4

Tabl. 3. The number of iterations for x^{II}

Multistep method	$m = 1$	$m = 2$	$m = 3$
Newton	12	9	8
Secant	17	12	10
Kurchatov	13	10	8

In case if the initial approximation x_0 is far from x^*, we have the advantage of the multi-step methods. But it is advisable to use these methods with small m.

4. Conclusions

Local convergence analysis of the generalized multistep method for solving nonlinear equation are provided. The convergence orders of some multistep methods are established using the approach of restricted convergence regions. The advantages of this approach are confirmed in practice.

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