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ON THE METHOD OF FUNDAMENTAL SOLUTIONS FOR THE TIME DEPENDENT DIRICHLET PROBLEMS

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Резюме. Ми розглядаємо наближене розв'язування нестаціонарних задач Діріхле для хвильового рівняння і рівняння теплопровідності у двовимірних і тривимірних двозв'язних областях за методом фундаментальних розв'язків (МФР). Перетворення Лаггера і метод Губольта застосовані незалежно для дискретизації вихідної задачі по часовій змінній до послідовності еліптичних задач. В свою чергу стаціонарні задачі повністю дискретизовано до рекурентної системи алгебраїчних рівнянь, використовуючи МФР. Наведено алгоритм методу, вибір точок колокації і джерела для конкретних випадків границь, а також результати чисельних експериментів, які підтверджують застосовність даного підходу.

Abstract. We consider the approximation of the solution of non-stationary Dirichlet problems for the wave and heat equations in 2-dimensional and 3-dimensional double connected domains, by the method of fundamental solutions (MFS). The Laguerre transformation and the Houbolt method are applied independently to reduce the initial problem by a time to a sequence of elliptic problems. In turn, stationary problems are completely discretized to a recurrent system of algebraic equations using MFS. The algorithm of the method, the choice of collocation and source points for specific cases of boundaries, as well as the results of numerical experiments, that confirm the applicability of this approach, are provided.

1. Introduction

The method of fundamental solutions is one of the common-used approaches for numerical solving of the elliptic problems, see, for example, in [2,3, 10]. The case of the hyperbolic and parabolic problems is less studied. For that class of problems a few studies already developed, using the Laplace transformation in time, finite difference approximations, Laguerre transformation with a combination of method of the boundary integral equations or other methods, see [7,8,12]. Also, there is a studies when the MFS is directly applied (for heat equation), without time transformation, see [16].

The non-stationary problems govern many important physical problems and are widely used in different applications. The Dirichlet problem is one of the classical well–posed problem which has owns application and commonly used for generation of the input data for ill-posed problems, see [4, 7]. Important to have an efficient way of numerical solution for that problem. We build the algorithm on the study [7].

Key words. Dirichlet problem; heat equation; wave equation; Laguerre transformation; Houbolt method; method fundamental solutions.

Let us consider the Dirichlet problem for the wave equation

$$\begin{cases} \dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial t^2} = \Delta u & \text{in } D \times (0,\infty), \\ u = f_\ell & \text{on } \Gamma_\ell \times (0,\infty),\ \ell = 1,2, \\ \dfrac{\partial u}{\partial t}(x,0) = u(x,0) = 0 & \text{for } x \in D, \end{cases} \tag{1}$$

where $a > 0$ is the given constant speed of sound, D is an annular domain in $\mathbb{R}^d, d = 2,3$, bounded by two simple closed non-intersecting boundaries Γ_1 (inner) and Γ_2 (outer) with ν the outward unit normal to these boundaries, f_1 and f_2 are given smooth functions.

Similarly let's consider Dirichlet problem for the heat equation

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial u}{\partial t} = \Delta u & \text{in } D \times (0,\infty), \\ u = f_\ell & \text{on } \Gamma_\ell \times (0,\infty),\ \ell = 1,2, \\ u(x,0) = 0 & \text{for } x \in D, \end{cases} \tag{2}$$

where $c > 0$ a given constant specifying the heat diffusivity and f_1, f_2 and domain D defined as for the wave problem.

We reduce the non-stationary Dirichlet problem to a sequence of boundary value problems for elliptic equations, using either the Laguerre transformation [9] or the Houbolt method [14]. The set of fundamental solutions is known, thus it is possible to use the MFS without the need to transform the inhomogeneous right-hand side to a homogeneous one, like it has been done in previous studies. The solution of the elliptic problems is approximated by the linear combination of fundamental solutions with unknown coefficients, which in turn are obtained by the collocation method. In the end, we obtain the recursive system of linear equations for the unknown coefficients.

An outline of the work is: in section 2 we apply the Laguerre transformation and Houbolt method for time discretisation of the Dirichlet problem for the heat and wave equations. MFS for the obtained sequence of elliptic problems is developed in section 3. The definition of fundamental solutions and main steps of the algorithm is provided in that section. In section 4 we show algorithm of distribution of the source and collocation points for the MFS for specific domains together with the some numerical results, that show the applicability of the proposed approaches both for the wave and heat equations.

2. Time discretization

In the section we use two approaches to reduce (1) and (2) to a sequence of stationary elliptic Dirichlet problems. Let us start from the Laguerre transformation, see [7, 9].

2.1. **Laguerre transformation.**

Recall the definition and the main result of the Laguerre transform for our problems.

Definition 1. The Laguerre transformation with respect to the time-variable of an element $u(x,t)$ has the following representation:

$$u(x,t) = \kappa \sum_{p=0}^{\infty} u_p(x) L_p(\kappa t), \tag{3}$$

where $L_p(t) = \sum_{k=0}^{p} \binom{p}{k} \frac{(-t)^k}{k!}$ is the Laguerre polynomial of order p ([1, Chapt. 22]), $\kappa > 0$ is a given constant and the Fourier-Laguerre coefficients u_p are defined as:

$$u_p(x) = \int_0^{\infty} e^{-\kappa t} L_p(\kappa t) u(x,t)\, dt, \quad p = 0,1,2,\dots. \tag{4}$$

Theorem 1. *The function u defined in* (3) *is a solution of the Dirichlet problem for the wave equation* (1) *respectively the heat equation* (2) *provided that the Fourier-Laguerre coefficients u_p, $p = 0,1,2,\dots$, are the solution of the following sequence of elliptic Dirichlet problems:*

$$\begin{cases} \Delta u_p - \gamma^2 u_p = \displaystyle\sum_{m=0}^{p-1} \beta_{p-m} u_m & \text{in } D, \\ u_p = f_{\ell,p} & \text{on } \Gamma_\ell,\ \ell = 1,2, \end{cases} \tag{5}$$

where

$$f_{\ell,p}(x) = \int_0^{\infty} e^{-\kappa t} L_p(\kappa t) f_\ell(x,t)\, dt, \quad \ell = 1,2,\ p = 0,1,2,\dots,$$

with $\gamma^2 = \beta_0$ and the coefficients β_p being: in the case of the wave equation: $\beta_p = \dfrac{\kappa^2}{a^2}(p+1)$, $p = 0,1,2,\dots$, and in the case of the heat equation: $\beta_p = \dfrac{\kappa}{c}$, $p = 0,1,2,\dots$.

The approximation with respect to the time variable of the exact solution of the Dirichlet problems is obtained as a partial sum of the representation (3), that is limiting the value of $p = 0,1,2,\dots,N > 0$.

2.2. **Houbolt method.** In another hand, we can use the finite difference methods for time discretization. A commonly used method is the Rothe method [7,8]. But in our work we approximate time derivatives using Houbolt method [14], which is an unconditionally stable and second-order accurate linear scheme for second-order equations; properties and comparison with various second-order schemes are given in [11, 15, 17, 18]. Thus, start developing from the wave equation (1). The generic form of the Houbolt scheme is

$$\ddot{X}_{t+\delta t} = \frac{1}{\delta t^2}\Big(2X_{t+\delta t} - 5X_t + 4X_{t-\delta t} - X_{t-2\delta t}\Big),$$

where X is a twice continuously differentiable function, $\ddot{X}$ note the second derivative of the function, δt – time interval, $X_{t+k\delta t} = X(t+k\delta t), k \in \mathbb{N}$. To start this method, knowledge of the initial condition X_0 is required together with $X_{-\delta t}$ and $X_{-2\delta t}$. Therefore, a separate procedure is needed for the starting values; for an investigation of various initial strategies and consequences, see [6].

To apply the above discretisation to (1), we use the equidistant grid

$$t_p = (p+3)h_t, \quad \text{for } p = -3, -2, \dots, N > 0, \qquad \text{with } h_t = \frac{T}{N+3}, \text{ and } N \in \mathbb{N}, \tag{6}$$

where $T > 0$ is the given final time. We approximate the solution u by the sequence

$$u_p \approx u(\cdot, t_p), \quad p = -3, \dots, N. \tag{7}$$

Using the above Houbolt scheme, the elements $\{u_p\}$ satisfy

$$\Delta u_p = \frac{1}{a^2 h_t^2}(2u_p - 5u_{p-1} + 4u_{p-2} - u_{p-3}). \tag{8}$$

Applying the standard Euler scheme and initial conditions from (1), we obtain an approximation of the first three elements of the sequence:

$$u_{-3} = u(\cdot, 0) = 0,$$
$$u_{-2} \approx u_{-3} + h_t \frac{\partial u}{\partial t}(\cdot, 0) = 0,$$
$$u_{-1} \approx u_{-3} + 2h_t \frac{\partial u}{\partial t}(\cdot, 0) = 0.$$

We note that (8) can be written as

$$\Delta u_p - \gamma^2 u_p = \beta_1 u_{p-1} + \beta_2 u_{p-2} + \beta_3 u_{p-3}, \tag{9}$$

where

$$\gamma^2 = \beta_0 = \frac{2}{a^2 h_t^2}, \ \beta_1 = -\frac{5}{a^2 h_t^2}, \ \beta_2 = \frac{4}{a^2 h_t^2}, \text{ and } \beta_3 = -\frac{1}{a^2 h_t^2}. \tag{10}$$

Let us consider the application of the Houbolt method for the Dirichlet problem for the heat equation (2). For generating of the starting values we have to assume that $\frac{\partial u}{\partial t}(x, 0) = 0$, for $x \in D$. Then, based on [13] coefficients β_p will be as following:

$$\gamma^2 = \beta_0 = \frac{11}{6a^2 h_t}, \ \beta_1 = -\frac{3}{a^2 h_t}, \ \beta_2 = \frac{3}{2a^2 h_t}, \text{ and } \beta_3 = -\frac{1}{3a^2 h_t}. \tag{11}$$

In total, after applying the above time-discretisation to (1) or to (2), we get the following sequence of elliptic Dirichlet problems.

Theorem 2. *The function u approximated in (7) is a solution of the Dirichlet problem for the wave equation (1) when the coefficients u_p, $p = 0, 1, 2, \dots$, are the solution of the following sequence of elliptic Dirichlet problems:*

$$\begin{cases} \Delta u_p - \gamma^2 u_p = \sum_{m=0}^{p-1} \beta_{p-m} u_m & \text{in } D, \\ u_p = f_{\ell,p} & \text{on } \Gamma_\ell, \end{cases} \tag{12}$$

with $f_{\ell,p} = f_\ell(\cdot, t_p)$ *for* $\ell = 1, 2$, $p = 0, 1, \dots, N$. *The coefficients* γ, β_1, β_2, β_3 *for wave equation are defined in (10) and for heat equation – in (11), and* $\beta_4, \dots, \beta_{N-1} = 0$.

3. Method of fundamental solutions

Elliptic Dirichlet problems, obtained by the Laguerre transformation (5) or by the Houbolt method (12) has the same definition, the only difference is in the calculation of the coefficients β_p and $f_{\ell,p}$, $p = 0, 1, ..., N, \ell = 1, 2$. In that section we will focus on (5). To build the MFS for approximating a solution to (5), explicit expressions are needed for what is known as a fundamental sequence. We recall such expressions in $\mathbb{R}^d$, $d = 2, 3$.

3.1. **2-dimensional case.** The results are recalled from [7]. The functions Φ_p with

$$\Phi_p(x, y) = K_0(\gamma|x - y|)v_p(|x - y|) + K_1(\gamma|x - y|)w_p(|x - y|), \quad x \neq y \tag{13}$$

for $p = 0, 1, 2, \dots, N$, are a fundamental sequence of the elliptic equations (5) in the case of planar domains.

The elements K_0 and K_1 are what is known as modified Bessel functions [1]. The polynomials v_p and w_p for $p = 0, 1, \dots$, are given by

$$v_p(r) = \sum_{m=0}^{\left[\frac{p}{2}\right]} a_{p,2m} r^{2m} \quad \text{and} \quad w_p(r) = \sum_{m=0}^{\left[\frac{p-1}{2}\right]} a_{p,2m+1} r^{2m+1}, \; w_0 = 0,$$

with $[q]$ the largest integer not greater than q. The coefficients a_p for $p = 0, 1, \dots$, are obtained from the recurrence relations

$$a_{p,0} = 1;$$

$$a_{p,p} = -\frac{1}{2\gamma p}\beta_1 a_{p-1,p-1};$$

$$a_{p,k} = \frac{1}{2\gamma k}\left\{4\left[\frac{k+1}{2}\right]^2 a_{p,k+1} - \sum_{m=k-1}^{p-1} \beta_{p-m} a_{m,k-1}\right\}, \quad k = p - 1, \dots, 1.$$

3.2. **3-dimensional case.** In [8] shown that the functions Φ_p with

$$\Phi_p(x, y) = \frac{e^{-\gamma|x-y|}}{|x - y|}\widetilde{v}_p(|x - y|), \quad x \neq y \tag{14}$$

for $p = 0, 1, 2, \dots$, are a fundamental sequence of the elliptic equations (5) in the case of 3-dimensional domains.

The polynomials $\widetilde{v}_p$ for $p = 0, 1, \dots$, are given by

$$\widetilde{v}_p(r) = \sum_{m=0}^{p} \widetilde{a}_{p,m} r^m,$$

where the coefficients $\widetilde{a}_p$ for $p = 0, 1, \dots$, are obtained from the recurrence relations

$$\widetilde{a}_{p,0} = 1;$$

$$\widetilde{a}_{p,p} = -\frac{1}{2\gamma p}\beta_1 \widetilde{a}_{p-1,p-1};$$

$$\widetilde{a}_{p,k} = \frac{1}{2\gamma k}\left\{k(k+1)\widetilde{a}_{p,k+1} - \sum_{m=k-1}^{p-1} \beta_{p-m}\widetilde{a}_{m,k-1}\right\}, \quad k = p-1, \ldots, 1.$$

In [4] for 2-dimensional domains and in [5] for 3-dimensional domains shown that the sequence Φ_p, $p = 0, 1, ..., N$ are linear independent and dense.

3.3. **MFS for the sequence of Dirichlet problems (5).** The unknown solution of the (5) is approximated by the linear combination of the fundamental solutions, defined in (13) and in (14) for 2-dimensional and 3-dimensional domains respectively. Thus

$$u_p(x) \approx u_{p,n}(x) = \sum_{m=0}^{p}\sum_{k=1}^{n} \alpha_{mk}\Phi_{p-m}(x, y_k), \quad x \in D \tag{15}$$

for $n > 0$ with Φ_p given by (13) or by (14), and with coefficients $\alpha_{mk} \in \mathbb{R}$, $k = 1, 2, \ldots, n$, $m = 0, 1, \ldots, p$, to be determined. The so-called source points y_k, $k = 1, 2, \ldots, n$, are located outside of the domain D.

The coefficients α_{mk} in (15) is determined by collocating on the boundary of the solution domain D using a set of so-called collocation points, namely from a recurrent system of linear equations for $p = 0, 1, ..., N$:

$$\sum_{k=1}^{n} \alpha_{pk}\Phi_0(x_{\ell j}, y_k) = f_{\ell,p}(x_{\ell j}) - \sum_{m=0}^{p-1}\sum_{k=1}^{n} \alpha_{mk}\Phi_{p-m}(x_{\ell j}, y_k), \tag{16}$$

where $n > 0$, $j = 1, ..., n/2$, $\ell = 1, 2, x_{\ell j} \in \Gamma_\ell$ – selected collocation points.

There is no one way of selecting the source points. Since the domain D is doubly-connected, source points have to be placed, according to [2], both in the unbounded exterior region of D as well as in the bounded region enclosed by Γ_1. The collocation points are evenly distributed on boundaries $\Gamma_\ell, \ell = 1, 2$. More about distribution of the source and collocation points will be noted in numerical results for known representation of boundaries of D.

Thus the final solution of (1) and (2) based on Laguerre transformation (3) and (15) is approximated by

$$\begin{aligned} u(x,t) &\approx u_{N,n}(x,t) = \\ &= \kappa \sum_{p=0}^{N} L_p(\kappa t) \sum_{m=0}^{p}\sum_{k=1}^{n} \alpha_{mk}\Phi_{p-m}(x, y_k),\ (x,t) \in D \times (0, \infty) \end{aligned} \tag{17}$$

for $N, n > 0$ and α_{mk} – solution of the (16).

And in case of Houbolt method (9) for wave equation (1) we have:

$$u(x,t) \approx u_{N,n}(x,t_p) = \sum_{m=0}^{p}\sum_{k=1}^{n} \alpha_{mk}\Phi_{p-m}(x, y_k),\ (x,t) \in D \times (0, \infty). \tag{18}$$

3.4. **The algorithm of the MFS.** The summarization of the main steps of the numerical procedures for solving the Dirichlet problem for the wave (1) or heat (2) equations is as follows:

– Initialization:
  1). Select discretization parameters $N > 0$ – truncation of the sequence (5) or (12).
  2). Select discretization parameters $n > 0$ – number of the collocation and source point in MFS (15).
  3). For the Laguerre transformation approach:
    3.1) select scaling parameter κ in (3);
    3.2) calculate the constants β_p, $p = 0, 1, ..., N$ specified in the theorem 1, depends on type of the equation;
    3.3) calculate the Laguerre transformation $f_{\ell,p}$, $\ell = 1, 2$, $p = 0$, 1, ..., N, given in the theorem 1.
  For the Houbolt scheme approach:
    3.1) calculate the constants β_p, $p = 0, 1, ..., N$ specified in the theorem 2, depends on type of the equation;
    3.2) calculate the functions $f_{\ell,p}$, $\ell = 1, 2, p = 0, 1, ..., N$, given in the theorem 2.
  4). Generate the source points y_k and collocation points $x_{\ell j}$ for the MFS (15).
  5). Calculate the matrix in the linear system (16), where the fundamental solutions Φ_0 for 2-dimensional domains are given in (13) and for the 3-dimensional domains are given in (14).
– Iterative procedure ($p = 0, 1, ..., N$):
  1). Calculate the right-side vector in the linear system (16), where the sequence of fundamental solutions Φ_m, $m = 0, 1, ..., p$ are given in (13) and (14).
  2). Solve the (16) and obtain coefficients α_{pk}, $k = 1, ..., n$.
– Build the solution:
  1). Using obtained coefficients α_{pk}, $p = 0, 1, ..., N$, $k = 1, ..., n$ build the solution of the Dirichlet problem (1) or (2) by (17) for the Laguerre transformation approach or by (18) for the Houbolt method.

4. Numerical examples

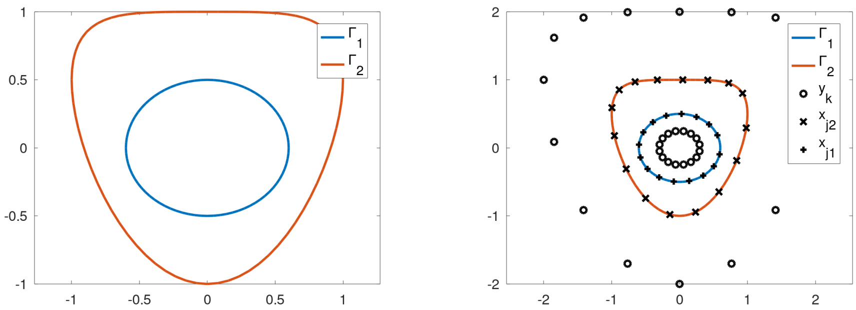
In this section we consider a few results of numerical experiments of finding the solution of Dirichlet problem for the wave (1) and heat (2) equations in 2- and 3-dimensional domains.

The concrete definition of the domain D is provided together with the algorithm of the selection of the source and collocation points for the MFS.

4.1. **Example 1 (Laguerre transformation for 2D case).** Lets consider the application of the Laguerre transformation approach for the Dirichlet problem for the heat equation (2) with $c = 1$ in 2-dimensional domain. The boundaries of the domain D has following representation (see Fig. 1a.):

$$\Gamma_1 = \{x_1(t) = (0.6\cos t, 0.5\sin t),\ t \in [0, 2\pi]\},$$
$$\Gamma_2 = \left\{x_2(t) = (\cos t, \sin t - 0.5\sin^2 t + 0.5),\ t \in [0, 2\pi]\right\}.$$

For generating of the source points we generate an artificial boundary in the



a) The domain D in Ex. 1

b) The distribution of the $y_k, x_{\ell,j}$ in Ex. 1

FIG. 1. Domain, source and collocation points for $n = 32$, used in the Ex. 1

unbounded exterior region of the D and in the bounded region enclosed by Γ_1 and place evenly distributed source points y_k on generated boundaries by the next rule:

$$y_k = \begin{cases} 2x_2(s_k), & \text{for even } k, \\ 0.5x_1(s_k), & \text{for odd } k, \end{cases} \tag{19}$$

where $s_k = \dfrac{2\pi}{n}k$, for $k = 1, \ldots, n$. Note that n should be an even integer.

The collocation points are generated similarly:

$$x_{\ell j} = x_\ell(\widetilde{s}_j),\ \widetilde{s}_j = \frac{4\pi}{n+1}j,\ \ell = 1, 2, j = 1, ..., n/2. \tag{20}$$

The distribution of the source and collocation points are given in Fig. 1b.

As the exact solution, we use the fundamental solution of the heat equation:

$$u_{ex}(x, t) = \frac{100}{4\pi t}e^{-\frac{|x-x^*|^2}{4t}}, \quad (x, t) \in D \times (0, \infty),\ x^* = (0, 4).$$

Then the Dirichlet data on the boundaries Γ_ℓ is

$$f_\ell(x, t) = u_{ex}(x, t), \quad (x, t) \in \Gamma_\ell \times (0, \infty),\ l = 1, 2.$$

The Laguerre transformation of f_ℓ, $\ell = 1, 2$ is calculated exactly and is:

$$f_{\ell p} = \frac{50}{\pi}\Phi_p(x, x^*), \quad p = 0, 1, ..., N.$$

Scaling parameter κ is equal to 1. The absolute errors $|u_{ex}(x, t) - u_{N,n}(x, t)|$ of the approximation of the solution (2) in the 2-dimensional domain D for test points are given in Table 1.

TABL. 1. Errors for the approximated solution in the domain D in Ex. 1

	$x=(0,0.7)^\top$, $t=1$			$x=(0.6,0)^\top$, $t=2$		
N/n	8	16	32	8	16	32
0	$1.3e-01$	$1.3e-01$	$1.3e-01$	$3.5e-01$	$3.5e-01$	$3.5e-01$
10	$2.8e-02$	$2.4e-02$	$2.3e-02$	$3.4e-02$	$1.0e-02$	$9.9e-03$
20	$2.0e-02$	$1.6e-02$	$1.5e-02$	$9.0e-03$	$2.4e-03$	$2.3e-03$
30	$8.7e-03$	$3.8e-03$	$3.1e-03$	$8.0e-03$	$1.6e-03$	$1.5e-03$
40	$7.7e-03$	$2.8e-03$	$2.2e-03$	$5.9e-03$	$5.4e-04$	$2.8e-04$
50	$5.2e-03$	$5.3e-04$	$4.0e-04$	$6.0e-03$	$5.0e-04$	$2.4e-04$

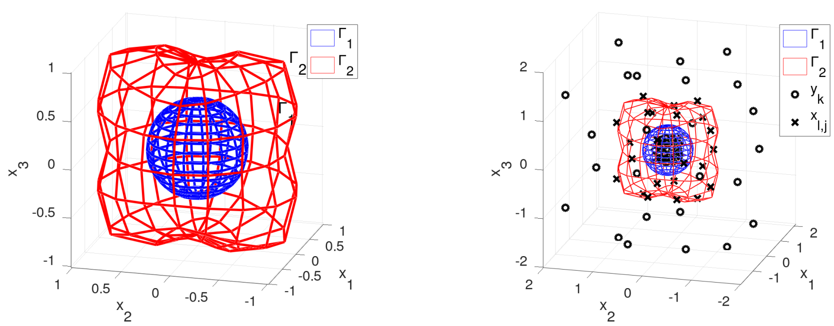
4.2. **Example 2 (Houbolt scheme for 3D case).** Let us consider the application of the Houbolt scheme for the Dirichlet problem for the wave equation (1) with wave speed $a=10$ in 3-dimensional domain. The boundaries of the domain D has following representation (see Fig. 2a.):

$$\Gamma_1=\{x_1(\theta,\phi)=0.5(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\ \theta\in[0,\pi],\phi\in[0,2\pi]\}$$

and

$$\Gamma_2=\{x_2(\theta,\phi)=\rho(\theta,\phi)(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\ \theta\in[0,\pi],\phi\in[0,2\pi]\},$$

where $\rho(\theta,\phi)=\sqrt{0.8+0.2(\cos 2\phi-1)(\cos 4\theta-1)}$.



a) The domain D in Ex. 2

b) The distribution of the $y_k, x_{\ell,j}$ in Ex. 2

FIG. 2. Domain, source and collocation points for $n=32$, used in the Ex. 2

As in previous example, for generating of the source points we generate an artificial boundaries and place evenly distributed points y_k by the next rule:

$$y_k=\begin{cases}2x_2(\theta_k,\phi_k), & \text{for even } k,\\ 0.5x_1(\theta_k,\phi_k), & \text{for odd } k,\end{cases} \qquad (21)$$

where

$$\theta_k = \pi \frac{\left\{\frac{k-1}{\widetilde{n}}\right\}+1}{\widetilde{n}+1}, \quad \phi_k = \pi \frac{\left[\frac{k-1}{\widetilde{n}}\right]+1}{\widetilde{n}+1},$$

$$\widetilde{n} = \sqrt{\frac{n}{2}}, \text{ with } \{q\} = q - [q] \text{ for } k = 1, \dots, n.$$

The collocation points are generated similarly:

$$\begin{aligned} x_{\ell j} =& x_\ell(\widetilde{\theta}_j, \widetilde{\phi}_j), \quad \widetilde{\theta}_j = \pi \frac{\left\{\frac{2j-1}{\widetilde{n}}\right\}+1}{\widetilde{n}+1}, \\ \widetilde{\phi}_j =& \pi \frac{\left[\frac{2j-1}{\widetilde{n}}\right]+1}{\widetilde{n}+1}, \text{ for } \ell = 1,2, \ j = 1, \dots, n/2. \end{aligned} \tag{22}$$

The distribution of the source and collocation points are given in Fig. 2b. We assume that $n = 2\xi^2$, where $\xi \in \mathbb{N}$.

The Dirichlet data on the boundaries Γ_ℓ, $\ell = 1, 2$ is

$$f_\ell(x,t) = t \sin t(x_1 + x_2 + x_3), \quad \text{on } \Gamma_\ell \times (0,T), \ \ \ell = 1,2.$$

The absolute errors $|u_{40,128}(x,t) - u_{N,n}(x,t)|$ of the approximation of the solution (1) in the 3-dimensional domain D for test points are given in Table 2. Note that for the current example exact solution is unknown, thus to test our approach we use as the exact solution the numerical one for $N = 40, n = 128$ and the final time T is equal to 5.

TABL. 2. Errors for the approximated solution in the domain D in Ex. 2

	$x = (0, 0.6, 0.5)^\top$, $t = 2$			$x = (0, 0, 7)^\top$, $t = 5$		
N/n	18	50	98	18	50	98
0	$2.3e-01$	$1.9e-01$	$1.8e-01$	$2.3e-01$	$1.4e-01$	$1.3e-01$
10	$8.7e-02$	$1.6e-02$	$2.4e-04$	$9.4e-02$	$6.1e-02$	$2.2e-03$
20	$6.5e-02$	$7.1e-03$	$1.2e-04$	$8.5e-02$	$9.3e-03$	$1.1e-03$
30	$6.5e-02$	$7.0e-03$	$3.7e-05$	$8.2e-02$	$8.1e-03$	$3.8e-04$

Numerical results from both examples confirm the applicability of the proposed approaches for the numerical solution of the time-dependent Dirichlet problems.

5. Conclusion

MFS has been proposed for the numerical solution of the Dirichlet problem for the wave and heat equations in 2- and 3-dimensional planar bounded domains. The original problem is reduced by the Laguerre transform in time or by the Houbolt method to a sequence of elliptic Dirichlet problems. The solution of the elliptic problems is approximated by the linear combination of the given fundamental sequence with source points evenly distributed outside the solution domain. By collocating on the boundary of the solution domain, recurrent linear systems are obtained for finding the unknown coefficients in the MFS approximation. Numerical results are provided for both heat and wave equations, which confirms the applicability of the proposed approaches.

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