Журнал обчислювальної та прикладної математики | **2021** **№ 3 (137)** | Journal of Numerical & Applied Mathematics

UDC 519.6

SOLVING SYSTEMS OF NONLINEAR EQUATIONS WITH MATRIX CONTINUED FRACTIONS

A. S. Chernikova, A. M. Nedashkovska

Резюме. Актуальність дослідження та використання ланцюгових дробів в чисельних методах є очевидною. Так ланцюгові дроби застосовували при розробці сонячного календаря, для доведення ірраціональності чисел (наприклад, за допомогою ланцюгових дробів доведена ірраціональність дзета-функції Рімана ζ числа π), в алгоритмі Ланцьоша, який використовує ланцюгові дроби для обчислення власних значень великих розріджених матриць, в деяких алгоритмах факторизації і т.д. Важко переоцінити роль ланцюгових дробів і при розв'язуванні систем рівнянь. В даній роботі будуть розглянуті системи нелінійних рівнянь. Для їх розв'язування буде використана відносно проста схема, запропонована раніше у роботі [1]. Розв'язок системи нелінійних рівнянь подано у вигляді матричного ланцюгового дробу та сформульовано достатню умову його збіжності. Результати обчислення тестових задач підтвердили збіжність та ефективність застосування запропонованої схеми.

Abstract. The relevance of the research and the use of continued fractions in numerical methods is obvious. Thus, continued fractions were used in the development of the solar calendar, to prove the irrationality of numbers (for example, with the help of continued fractions proved the irrationality of the Riemann zeta function ζ number π), in the Lanczos algorithm, which uses continued fractions to calculate eigenvalues of large sparse matrices, in some factorization algorithms, etc. It is difficult to overestimate the role of continued fractions in solving systems of equations. In this paper, systems of nonlinear equations are considered. To solve them, a relatively simple scheme proposed earlier in [1] will be used. The solution of the problem is given in the form of a matrix continued fraction and a sufficient condition of its convergence is formulated. The calculation's results of test problems confirmed the convergence and effectiveness of the proposed scheme.

1. Introduction

The importance of developing methods for solving nonlinear systems is due to the wide range of their application in practice. Bright examples of the usage of such schemes are the problems of the ballistics of rocket engines and their utilization, as well as the problems of virology and immunology, which are more important than ever today.

Key words. Systems of nonlinear equations; matrix equations; continued fractions.

Consider a system of n nonlinear equations with n unknowns:

$$\begin{cases} f_1\left(x_1, x_2, \ldots, x_n\right) = 0; \\ f_2\left(x_1, x_2, \ldots, x_n\right) = 0; \\ \vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) = 0, \end{cases} \tag{1}$$

where f_i $(i = \overline{1, n})$ are functions of real variables $x_1, x_2, \ldots, x_n$.

Let the functions of the n variables $y_i = f_i\left(x_1, x_2 \ldots, x_n\right), (i = \overline{1, n})$ be definite and continuous together with all their partial derivatives up to and including the second order in some δ-neighborhood of the point $x^{(0)} = \big(x_1^{(0)}, x_2^{(0)} \ldots, x_n^{(0)}\big)$.

Suppose that $x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}$, $(k = 0, 1, \ldots)$ are the approximate roots of the system of equations (1). Suppose that for step k, that is, for approximations $x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}$, the system equation (1) is not executed. Then we will look for corrections $h_j^{(k)}, (j = \overline{1, n})$ at which as a result of substitution $x_j^{(k+1)} = x_j^{(k)} + h_j^{(k)}$ the equations should be contented:

$$\begin{cases} f_1\left(x_1^{(k)} + h_1^{(k)}, x_2^{(k)} + h_2^{(k)}, \ldots, x_n^{(k)} + h_n^{(k)}\right) = 0; \\ f_2\left(x_1^{(k)} + h_1^{(k)}, x_2^{(k)} + h_2^{(k)}, \ldots, x_n^{(k)} + h_n^{(k)}\right) = 0; \\ \vdots \\ f_n\left(x_1^{(k)} + h_1^{(k)}, x_2^{(k)} + h_2^{(k)}, \ldots, x_n^{(k)} + h_n^{(k)}\right) = 0. \end{cases} \tag{2}$$

Let's write Taylor's formula for the function of n variables like this:

$$\begin{aligned} & f_i\left(x_1^{(k)} + h_1^{(k)}, x_2^{(k)} + h_2^{(k)}, \ldots, x_n^{(k)} + h_n^{(k)}\right) = f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right) + \\ & + \sum_{j=1}^{n} \frac{\partial f_i\left(x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}\right)}{\partial x_j^{(k)}} h_j^{(k)} + \\ & + \frac{1}{2!}\left(\sum_{j=1}^{n}\sum_{q=1}^{n} \frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}\right)}{\partial x_j^{(k)} \partial x_q^{(k)}} h_j^{(k)} h_q^{(k)}\right) + \\ & + o(h^2), \;\; h = \sqrt{\left(h_1^{(k)}\right)^2 + \left(h_2^{(k)}\right)^2 + \ldots + \left(h_n^{(k)}\right)^2}, \;\; i = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

Then based on the system (2) and the Taylor's formula (3) to determine the corrections $h_1^{(k)}, h_2^{(k)}, \ldots, h_n^{(k)}$ with an accuracy of $o(h^2)$ we can write the following system of equations:

$$
\begin{gathered}
f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right) + \sum_{j=1}^{n} \frac{\partial f_i\left(x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}\right)}{\partial x_j^{(k)}} h_j^{(k)} + \\
+ \frac{1}{2!}\left(\sum_{j=1}^{n}\sum_{q=1}^{n} \frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}\right)}{\partial x_j^{(k)} \partial x_q^{(k)}} h_j^{(k)} h_q^{(k)}\right) + \\
+ o(h^2) = 0, \quad i = 1, 2, \ldots, n.
\end{gathered} \tag{4}
$$

Therefore, we obtain a method for solving systems of second-order algebraic equations with multiple unknowns.

2. Computational scheme of the method

The essence of the proposed method is to use periodic matrix continued fractions to calculate the solutions of the original system by certain elementary transformations that reduce it to some recurrent relation. Apply this approach to (4).

The system (4) can be represented as:

$$
\begin{gathered}
\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)^2}\left(h_1^{(k)}\right)^2 + \frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial x_1^{(k)} x_2^{(k)}} h_1^{(k)} h_2^{(k)} + \\
+ \ldots + \frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial x_1^{(k)} x_n^{(k)}} h_1^{(k)} h_n^{(k)} + \\
+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial x_2^{(k)} x_1^{(k)}} h_2^{(k)} h_1^{(k)} + \frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\left(\partial x_2^{(k)}\right)^2}\left(h_2^{(k)}\right)^2 + \\
+ \ldots + \frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial x_2^{(k)} x_n^{(k)}} h_2^{(k)} h_n^{(k)} + \ldots + \\
+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial x_n^{(k)} x_1^{(k)}} h_n^{(k)} h_1^{(k)} + \frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial x_n^{(k)} x_2^{(k)}} h_n^{(k)} h_2^{(k)} + \\
+ \ldots + \frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)^2}\left(h_n^{(k)}\right)^2 + \frac{\partial f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)} h_1^{(k)} + \\
+\frac{\partial f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial\left(x_2^{(k)}\right)} h_2^{(k)} + \ldots + \frac{\partial f_i\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)} h_n^{(k)} = \\
= -f_i\left(x_1^{(k)}, x_2^{(k)} \ldots, x_n^{(k)}\right), \quad i = \overline{1, n}
\end{gathered}
$$

or

$$\left(\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)^2}h_1^{(k)}+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_1^{(k)}x_2^{(k)}}h_2^{(k)}+\dots+\right.$$

$$\left.+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_1^{(k)}x_n^{(k)}}h_n^{(k)}+\frac{\partial f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)}\right)h_1^{(k)}+$$

$$\left(\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_2^{(k)}x_1^{(k)}}h_1^{(k)}+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\left(\partial x_2^{(k)}\right)^2}h_2^{(k)}+\right. \quad (5)$$

$$\left.+\dots+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_2^{(k)}x_n^{(k)}}h_n^{(k)}+\frac{\partial f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_2^{(k)}\right)}\right)h_2^{(k)}+$$

$$+\dots+\left(\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_n^{(k)}x_1^{(k)}}h_1^{(k)}+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_n^{(k)}x_2^{(k)}}h_2^{(k)}+\right.$$

$$\left.+\dots+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)^2}h_n^{(k)}+\frac{\partial f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)}\right)h_n^{(k)}=$$

$$=-f_i\left(x_1^{(k)},x_2^{(k)}\dots,x_n^{(k)}\right),\quad i=\overline{1,n}.$$

The system (5) in matrix form can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix}\begin{pmatrix} h_1^{(k)}\\ h_2^{(k)}\\ \vdots\\ h_n^{(k)}\end{pmatrix}=\begin{pmatrix} -f_1\left(x_1^{(k)},x_2^{(k)}\dots,x_n^{(k)}\right)\\ -f_2\left(x_1^{(k)},x_2^{(k)}\dots,x_n^{(k)}\right)\\ \vdots\\ -f_n\left(x_1^{(k)},x_2^{(k)}\dots,x_n^{(k)}\right)\end{pmatrix}, \quad (6)$$

where

$$a_{i1}=\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)^2}h_1^{(k)}+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_1^{(k)}x_2^{(k)}}h_2^{(k)}$$

$$+\dots+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_1^{(k)}x_n^{(k)}}h_n^{(k)}+\frac{\partial f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)};$$

$$a_{i2}=\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\partial x_2^{(k)}x_1^{(k)}}h_1^{(k)}+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)}{\left(\partial x_2^{(k)}\right)^2}h_2^{(k)}+$$

$$+\ldots+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}x_n^{(k)}}h_n^{(k)}+\frac{\partial f_i\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial\left(x_2^{(k)}\right)};$$

$$\vdots$$

$$a_{in}=\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}x_1^{(k)}}h_1^{(k)}+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}x_2^{(k)}}h_2^{(k)}+$$

$$+\ldots+\frac{1}{2}\frac{\partial^2 f_i\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)^2}h_n^{(k)}+\frac{\partial f_i\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)};\ i=\overline{1,n}.$$

We introduce the following notation:

$$A_1=\frac{1}{2}\begin{pmatrix}\frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)^2} & \frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}\partial x_2^{(k)}} & \cdots & \frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}\partial x_n^{(k)}}\\ \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)^2} & \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}\partial x_2^{(k)}} & \cdots & \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}\partial x_n^{(k)}}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial\left(x_1^{(k)}\right)^2} & \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}\partial x_2^{(k)}} & \cdots & \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}\partial x_n^{(k)}}\end{pmatrix},$$

$$A_2=\frac{1}{2}\begin{pmatrix}\frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}\partial x_1^{(k)}} & \frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial\left(x_2^{(k)}\right)^2} & \cdots & \frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}\partial x_n^{(k)}}\\ \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}\partial x_1^{(k)}} & \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial\left(x_2^{(k)}\right)^2} & \cdots & \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}\partial x_n^{(k)}}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}\partial x_1^{(k)}} & \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial\left(x_2^{(k)}\right)^2} & \cdots & \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}\partial x_n^{(k)}}\end{pmatrix},$$

$$\vdots$$

$$A_n=\frac{1}{2}\begin{pmatrix}\frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}\partial x_1^{(k)}} & \frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}\partial x_2^{(k)}} & \cdots & \frac{\partial^2 f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)^2}\\ \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}\partial x_1^{(k)}} & \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}\partial x_2^{(k)}} & \cdots & \frac{\partial^2 f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)^2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}\partial x_1^{(k)}} & \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}\partial x_2^{(k)}} & \cdots & \frac{\partial^2 f_n\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial\left(x_n^{(k)}\right)^2}\end{pmatrix},$$

$$A_{n+1}=\begin{pmatrix} \frac{\partial f_1\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}} & \frac{\partial f_1\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}} & \ldots & \frac{\partial f_1\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}} \\ \frac{\partial f_2\left(x_1^{(k)},x_2^{(k)}\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}} & \frac{\partial f_2\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}} & \ldots & \frac{\partial f_2\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_1^{(k)}} & \frac{\partial f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_2^{(k)}} & \ldots & \frac{\partial f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)}{\partial x_n^{(k)}} \end{pmatrix},$$

$$b=\begin{pmatrix} -f_1\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right) \\ -f_2\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right) \\ \vdots \\ -f_n\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right) \end{pmatrix}. \tag{7}$$

Then (6) can be written as follows:

$$\left(A_1h_1^{(k)}+A_2h_2^{(k)}\ldots+A_nh_n^{(k)}+A_{n+1}\right)\left(h_1^{(k)}\quad h_2^{(k)}\quad\ldots\quad h_n^{(k)}\right)^T=b. \tag{8}$$

The entry

$$\begin{pmatrix}A_1 & A_2 & \ldots & A_n\end{pmatrix}\circ\left(h_1^{(k)}\quad h_1^{(k)}\quad\ldots\quad h_n^{(k)}\right)^T$$

denotes the functionality $\sum\limits_{i=1}^{n}A_ih_i^{(k)}$, then (8) can be represented as:

$$\begin{gathered}\left(\begin{pmatrix}A_1 & A_2\ldots & A_n\end{pmatrix}\circ\left(h_1^{(k)}\ \ h_2^{(k)}\ \ \ldots\ \ h_n^{(k)}\right)^T+A_{n+1}\right)\times\\ \times\left(h_1^{(k)}\ \ h_1^{(k)}\ \ \ldots\ \ h_n^{(k)}\right)^T=b.\end{gathered} \tag{9}$$

Suppose now that $\left(\begin{pmatrix}A_1 & A_2 & \ldots & A_n\end{pmatrix}\circ\left(h_1^{(k)}\quad h_2^{(k)}\ldots\quad h_n^{(k)}\right)^T+A_{n+1}\right)$ is non degenerate matrix. From the relation (9) we obtain:

$$\begin{gathered}\left(h_1^{(k)}\ \ h_2^{(k)}\ \ \ldots\ \ h_n^{(k)}\right)^T=\\ =\left(\begin{pmatrix}A_1 & A_2 & \ldots & A_n\end{pmatrix}\circ\left(h_1^{(k)}\ \ h_2^{(k)}\ \ \ldots\ \ h_n^{(k)}\right)^T+A_{n+1}\right)^{-1}b.\end{gathered} \tag{10}$$

In addition, there is a representation (10) in the form of a matrix continued fraction

$$\left(h_1^{(k)}\quad\ldots\quad h_n^{(k)}\right)^T=\cfrac{b}{A_{n+1}+\begin{pmatrix}A_1 & \ldots & A_n\end{pmatrix}\circ\cfrac{b}{A_{n+1}+\begin{pmatrix}A_1 & \ldots & A_n\end{pmatrix}\circ\cfrac{b}{A_{n+1}+\ldots}}}.$$

or in the form of Pringsheim

$$\begin{aligned}\left(h_1^{(k)} \quad \dots \quad h_n^{(k)}\right)^T = \frac{b|}{|A_{n+1}} + \left(A_1 \, \dots \, A_n\right) \circ \frac{b|}{|A_{n+1}} + \\ + \left(A_1 \, \dots \, A_n\right) \circ \frac{b|}{|A_{n+1}} + \dots \, .\end{aligned} \tag{11}$$

So, we can write a recurrent formula for calculating the approximate solution of the system (1) using matrix continued fractions:

$$\begin{aligned}\begin{pmatrix} h_1^{(m+1)} \\ \vdots \\ h_n^{(m+1)} \end{pmatrix} = -\left(\begin{pmatrix} A_1 & \dots & A_n \end{pmatrix} \circ \begin{pmatrix} h_1^{(m)} \\ \vdots \\ h_n^{(m)} \end{pmatrix} + A_{n+1}\right)^{-1} \times \\ \times \begin{pmatrix} f_1\left(x_1^{(m)}, \dots, x_n^{(m)}\right) \\ \vdots \\ f_n\left(x_1^{(m)}, \dots, x_n^{(m)}\right) \end{pmatrix}, \quad m = 1, 2... \, .\end{aligned} \tag{12}$$

Thus, to find the solution of the system of equations (1) we can construct the following algorithm on the basis of (12):

1. Set error $\varepsilon > 0$;
2. Set the initial approximation, $x_1^{(0)}, \dots, x_n^{(0)} \in \mathbb{R}$, $h_1^{(0)}, \dots, h_n^{(0)} \in \mathbb{R}$;
3. Set counter $m = 1$;
4. Find the next correction using the formula (12):

$$\begin{aligned}\begin{pmatrix} h_1^{(m+1)} \\ \vdots \\ h_n^{(m+1)} \end{pmatrix} = -\left(\begin{pmatrix} A_1 & \dots & A_n \end{pmatrix} \circ \begin{pmatrix} h_1^{(m)} \\ \vdots \\ h_n^{(m)} \end{pmatrix} + A_{n+1}\right)^{-1} \times \\ \times \begin{pmatrix} f_1\left(x_1^{(m)}, \dots, x_n^{(m)}\right) \\ \vdots \\ f_n\left(x_1^{(m)}, \dots, x_n^{(m)}\right) \end{pmatrix}, \quad m = 1, 2...\end{aligned}$$

5. Calculate new approximations $x_1^{(m+1)} = x_1^{(m)} + h_1^{(m)}, \dots, x_n^{(m+1)} = x_n^{(m)} + h_n^{(m)}$;
6. Check the condition $\left\| x_i^{(k+1)} - x_i^{(k)} \right\| \le \varepsilon \ (i = 1, 2, \dots, n)$. If the condition is not satisfied, set the counter $m = m + 1$, assign values $x_i^{(m+1)} (i = 1, 2, \dots, n)$ to elements $x_i^{(m)}$ and go to step 4, otherwise return $x_i^{(m)} (i = 1, 2, \dots, n)$.

3. On the convergence of the computational scheme

Convergence of the matrix periodic continued fraction (11) is a necessary condition for the convergence of the iterative process (12).

In the work [2] a sufficient condition for the convergence of Vorpitsky was generalized and later it was proved in [3]. This feature can be used in the analysis of the convergence of the matrix continued fraction (11):

Theorem 1. *Matrix branched continued fraction*

$$\sum_{k_1=1}^{n} \frac{A_{k_1}|}{|E} + \sum_{k_2=1}^{n} \frac{A_{k_1k_2}|}{|E} + \ldots + \sum_{k_l=1}^{n} \frac{A_{k_1k_2\ldots k_l}|}{|E} + \ldots$$

is absolutely convergent if the condition

$$\|A_{k_1k_2\ldots k_l}\| \leq \frac{1}{4n} \quad (i = 1, 2, 3, \ldots;\ k_l = 1, 2, \ldots, n)$$

is valid.

Let us apply the theorem (1) to a continued fraction(11). Obviously, this continued fraction will be absolutely convergent if the condition

$$\left\| \begin{pmatrix} A_1 & A_2 & \ldots & A_n \end{pmatrix} \circ (A_{n+1})^{-1} b \right\| \leq \frac{1}{4} \tag{13}$$

is executed.

Substitute the values of $A_1, A_2, \ldots, A_{n+1}, b$ into the inequality (13) and obtain a sufficient condition for the convergence of the periodic chain fraction (11).

4. Computational experiments

The proposed scheme was implemented in the language *Python* by using of *Jupyter Notebook*, which is a handy open-source web application.To demonstrate the applicability of the computational scheme and its efficiency, consider the following examples:

Example 1. Consider the following system of polynomial equations

$$\begin{cases} x_1^2 - 2x_2^2 - x_1x_2 + 2x_1 - x_2 + 1 = 0; \\ 2x_1^2 - x_2^2 + x_1x_2 + 3x_2 - 5 = 0. \end{cases} \tag{14}$$

Let present the system (14) in the form (8):

$$\left(A_1 h_1^{(k)} + A_2 h_2^{(k)} + A_3\right) \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T = b, \tag{15}$$

where A_1, A_2, A_3 and b write down the formula (7):

$$A_1 = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix},$$

$$A_2 = \frac{1}{2} \begin{pmatrix} -4 & -1 \\ -2 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2x_1 - x_2 + 2 & -4x_2 - x_1 - 1 \\ 2x_1 + x_2 & -2x_2 + 3 + x_1 \end{pmatrix},$$

$$b = \begin{pmatrix} x_1^2 - 2x_2^2 - x_1x_2 + 2x_1 - x_2 + 1 \\ 2x_1^2 - x_2^2 + x_1x_2 + 3x_2 - 5 \end{pmatrix}.$$

Then (15) can be written other ways using the formula (9):

$$\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix} \circ \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T + A_3 \right) \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T = b$$

So, if the system matrix (14) is nondegenerate, then then to calculate the unknowns $h_1^{(k)}$ and $h_2^{(k)}$ we construct the following iterative process, using the formula (12):

$$\begin{pmatrix} h_1^{(m+1)} \\ h_2^{(m+1)} \end{pmatrix} = - \left(\begin{pmatrix} A_1 & A_2 \end{pmatrix} \circ \begin{pmatrix} h_1^{(m)} \\ h_2^{(m)} \end{pmatrix} + A_3 \right)^{-1} b, \; m = 1, 2...$$

Let n_1 denote the number of iterations obtained using the scheme (12), n_2 is the number of iterations obtained using Newton's method, and ε is specified accuracy. The initial approximations for the system (14) and in the case of the scheme (12), and in the case of Newton's method are given as follows:

– $x_1^{(0)} = 2.0$ and $x_2^{(0)} = 2.0$;
– $h_1^{(0)} = 0.0$ and $h_2^{(0)} = 0.0$.

Then we get the results shown in the table (1).

TABL. 1. The results of calculations for the system (14) by the method (12) and Newton's method

ε	n_1	n_2	Appr. solution for scheme (12)	Appr. solution for Newton's m.	Error for (12)	Error for Newton's method
0.1	5	5	$\widetilde{x}_1^{(n_1)} = 0.98047239$, $\widetilde{x}_2^{(n_1)} = 0.99107229$	$\widetilde{x}_1^{(n_2)} = 0.9750123$, $\widetilde{x}_2^{(n_2)} = 0.98752701$	0.38283455	0.4671275
0.01	8	8	$\widetilde{x}_1^{(n_1)} = 1.00251982$, $\widetilde{x}_2^{(n_1)} = 1.00126725$	$\widetilde{x}_1^{(n_2)} = 1.00316779$, $\widetilde{x}_2^{(n_2)} = 1.00158390$	0.04491348	0.56773749
0.001	11	12	$\widetilde{x}_1^{(n_1)} = 0.99968647$, $\widetilde{x}_2^{(n_1)} = 0.99984335$	$\widetilde{x}_1^{(n_2)} = 1.00019769$, $\widetilde{x}_2^{(n_2)} = 1.00009885$	0.00565064	0.00355751
0.0001	15	15	$\widetilde{x}_1^{(n_1)} = 0.99998039$, $\widetilde{x}_2^{(n_1)} = 0.99999019$	$\widetilde{x}_1^{(n_2)} = 0.99997529$, $\widetilde{x}_2^{(n_2)} = 0.99998765$	0.00035292	0.00044478
0.00001	18	18	$\widetilde{x}_1^{(n_1)} = 1.00000245$, $\widetilde{x}_2^{(n_1)} = 1.00000123$	$\widetilde{x}_1^{(n_2)} = 1.00000309$, $\widetilde{x}_2^{(n_2)} = 1.00000154$	0.00004411	0.00005559

Example 2. Consider a system of nonlinear equations of the form:

$$\begin{cases} e^{2x_2} + e^{5x_1} + 4x_1x_2^3 + 2x_1^4x_2 + x_1^4 - 2 = 0; \\ 2e^{2x_1} + 5e^{x_2} + 8x_1x_2 + 4x_2^2 + x_2^4 - 7 = 0. \end{cases} \tag{16}$$

Let's write this system in the form (8):

$$\left(A_1h_1^{(k)} + A_2h_2^{(k)} + A_3 \right) \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T = b, \tag{17}$$

where A_1, A_2, A_3 and b calculated as (7):

$$A_1 = \frac{1}{2}\begin{pmatrix} 25e^{5x_1} + 24x_1^2x_2 + 12x_1^2 & 12x_2^2 + 8x_1^3 \\ 2e^{x_1} & 8 \end{pmatrix},$$

$$A_2 = \frac{1}{2}\begin{pmatrix} 4e^{2x_2} + 24x_1x_2 & 12x_2^2 + 8x_1^3 \\ 5e^{x_2} + 8 + 12x_2^2 & 8 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 5e^{5x_1} + 4x_2^3 + 8x_1^3x_2 + 4x_1^3 & 2e^{2x_2} + 12x_1x_2^2 + 2x_1^4 \\ 2e^{x_1} + 8x_2 & 5e^{x_2} + 8x_1 + 8x_2 + 4x_2^3 \end{pmatrix},$$

$$b = \begin{pmatrix} e^{2x_2} + e^{5x_1} + 4x_1x_2^3 + 2x_1^4x_2 + x_1^4 - 2 \\ 2e^{2x_1} + 5e^{x_2} + 8x_1x_2 + 4x_2^2 + x_2^4 - 7 \end{pmatrix}.$$

Then using (9) formula (17) can be written otherwise:

$$\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix} \circ \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T + A_3\right)\begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T = b.$$

Next, if the system matrix (16) is nondegenerate,then to calculate the unknowns h_1 and h_2 build an iterative process using the formula (12):

$$\begin{pmatrix} h_1^{(m+1)} \\ h_2^{(m+1)} \end{pmatrix} = -\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix} \circ \begin{pmatrix} h_1^{(m)} \\ h_2^{(m)} \end{pmatrix} + A_3\right)^{-1} b, \ m = 1, 2...$$

Let n_1 be the number of iterations obtained using the scheme (12), n_2 is the number of iterations obtained using the Newton method, and ε is the specified accuracy. In the initial approach to the system (16)

- $x_1^{(0)} = 0.2$ and $x_2^{(0)} = 0.2$;
- $h_1^{(0)} = 0.0$ and $h_2^{(0)} = 0.0$

we obtain the results presented in the table (2).

TABL. 2. The results of calculations for the system (16) by the method (12) and Newton's method

ε	n_1	n_2	Appr. solution for scheme (12)	Appr. solution for Newton's m.	Error for (12)	Error for Newton's method
0.1	2	2	$\tilde{x}_1^{(n_1)} = -0.01528742$, $\tilde{x}_2^{(n_1)} = -0.01569091$	$\tilde{x}_1^{(n_2)} = 0.01239569$, $\tilde{x}_2^{(n_2)} = 0.00103799$	0.63213448	0.46163384
0.01	4	4	$\tilde{x}_1^{(n_1)} = 0.00020276$, $\tilde{x}_2^{(n_1)} = 0.00016086$	$\tilde{x}_1^{(n_2)} = 0.00000054$, $\tilde{x}_2^{(n_2)} = -0.00000024$	0.02604419	0.00187028
0.001	5	4	$\tilde{x}_1^{(n_1)} = -0.00000169$, $\tilde{x}_2^{(n_1)} = -0.00000112$	$\tilde{x}_1^{(n_2)} = 0.00000054$, $\tilde{x}_2^{(n_2)} = -0.00000024$	0.00134682	0.00187028
0.0001	6	5	$\tilde{x}_1^{(n_1)} = 0.00000000$, $\tilde{x}_2^{(n_1)} = 0.00000000$	$\tilde{x}_1^{(n_2)} = 0.00000000$, $\tilde{x}_2^{(n_2)} = -0.00000000$	0.00001073	0.00000001
0.00001	6	5	$\tilde{x}_1^{(n_1)} = 0.00000000$, $\tilde{x}_2^{(n_1)} = 0.00000000$	$\tilde{x}_1^{(n_2)} = 0.00000000$, $\tilde{x}_2^{(n_2)} = -0.00000000$	0.00001073	0.00000001

Example 3. Consider the following system of nonlinear equations:

$$\begin{cases} 2\cos x_1 + x_1 \sin x_2 + 3x_1^4 + 4x_2^2 + 7x_1x_2^2 + x_1^3 - 2 = 0; \\ 2\sin x_1 + 24x_1^3x_2 + 3\cos x_2 + 8x_1 + 4x_2 + x_2^3 - 3 = 0. \end{cases} \quad (18)$$

Let's write this system in the form (8):

$$\left(A_1 h_1^{(k)} + A_2 h_2^{(k)} + A_3\right)\begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T = b, \quad (19)$$

with A_1, A_2, A_3 and b in the form (7):

$$A_1 = \frac{1}{2}\begin{pmatrix} -2\cos x_1 + 36x_1^2 + 6x_1 & \cos x_2 + 14x_2 \\ -2\sin x_1 + 144x_1x_2 & 72x_1^2 \end{pmatrix},$$

$$A_2 = \frac{1}{2}\begin{pmatrix} -x_1\sin x_2 + 8 + 14x_1 & \cos x_2 + 14x_2 \\ -3\cos x_2 + 6x_2 & 72x_1^2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -2\sin x_1 + \sin x_2 + 12x_1^3 + 7x_2^2 + 3x_1^2 & x_1\cos x_2 + 8x_2 + 14x_1x_2 \\ 2\cos x_1 + 72x_1^2x_2 + 8 & 24x_1^3 - 3\sin x_2 + 4 + 3x_2^2 \end{pmatrix},$$

$$b = \begin{pmatrix} 2\cos x_1 + x_1\sin x_2 + 3x_1^4 + 4x_2^2 + 7x_1x_2^2 + x_1^3 - 2 \\ 2\sin x_1 + 24x_1^3x_2 + 3\cos x_2 + 8x_1 + 4x_2 + x_2^3 - 3 \end{pmatrix}.$$

Let's use (12) and write the system (18) like:

$$\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix} \circ \begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T + A_3\right)\begin{pmatrix} h_1^{(k)} & h_2^{(k)} \end{pmatrix}^T = b.$$

TABL. 3. The results of calculations for the system (18) by the method (12) and Newton's method

ε	n_1	n_2	Appr. solution for scheme (12)	Appr. solution for Newton's m.	Error for (12)	Error for Newton's method
0.1	2	2	$\tilde{x}_1^{(n_1)} = -0.01156658$, $\tilde{x}_2^{(n_1)} = 0.02855345$	$\tilde{x}_1^{(n_2)} = -0.01649241$, $\tilde{x}_2^{(n_2)} = 0.04168796$	0.00766547	0.00566992
0.01	3	4	$\tilde{x}_1^{(n_1)} = -0.00815887$, $\tilde{x}_2^{(n_1)} = 0.02048567$	$\tilde{x}_1^{(n_2)} = -0.0068928$, $\tilde{x}_2^{(n_2)} = 0.01731047$	0.00238516	0.00017725
0.001	9	10	$\tilde{x}_1^{(n_1)} = -0.00078319$, $\tilde{x}_2^{(n_1)} = 0.00195892$	$\tilde{x}_1^{(n_2)} = -0.00049904$, $\tilde{x}_2^{(n_2)} = 0.00124801$	0.00001099	0.00005312
0.0001	15	15	$\tilde{x}_1^{(n_1)} = -0.00007594$, $\tilde{x}_2^{(n_1)} = 0.00018986$	$\tilde{x}_1^{(n_2)} = -0.00005587$, $\tilde{x}_2^{(n_2)} = 0.0001397$	0.00000010	0.00000007
0.00001	15	15	$\tilde{x}_1^{(n_1)} = -0.00007594$, $\tilde{x}_2^{(n_1)} = 0.00018986$	$\tilde{x}_1^{(n_2)} = -0.00005587$, $\tilde{x}_2^{(n_2)} = 0.0001397$	0.00000010	0.00000007

Next, check whether the matrix of the system (18) is nondegenerate, and to calculate the unknowns h_1 and h_2 build an iterative process by the formula (12):

$$\begin{pmatrix} h_1^{(m+1)} \\ h_2^{(m+1)} \end{pmatrix} = -\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix} \circ \begin{pmatrix} h_1^{(m)} \\ h_2^{(m)} \end{pmatrix} + A_3\right)^{-1} b, \ m = 1, 2...$$

At initial approximations for the system (18)

– $x_1^{(0)} = 0,1$ та $x_2^{(0)} = 0,1$;
– $h_1^{(0)} = 0,0$ та $h_2^{(0)} = 0,0$,

Then we get the results shown in the table (3).

Here n_1 is the number of iterations obtained using the scheme (12), n_2 is the number of iterations obtained using the Newton method, and ε - specified accuracy.

5. Conclusions

In this paper, we have been considered systems of nonlinear equations over a field of real numbers. A scheme for solving these systems is proposed and recurrent relations are obtained to find their approximate solutions. The convergence of continued fractions used in the computational scheme is investigated. Numerous experiments have been performed to confirm the applicability and effectiveness of the proposed approach.

Bibliography

1. Nedashkovska A.M. Iteratsiinyi metod rozviazuvannia systemy polinomialnykh rivnian druhoho stepenia / A.M. Nedashkovska // Fizyko-matematychne modeliuvannia ta informatsiini tekhnolohii. – 2015. – Vol. 21. – P. 150-161.
2. Bodnar D.I. Vetvyashchiyesya tsepnyye drobi / D.I. Bodnar Kyiv: Naukova dumka, 1986. – 176 p.
3. Nedashkovskyi M.O. Oznaky zbizhnosti matrychnykh hiliastykh drobiv / M.O. Nedashkovskyi // Matematychni metody ta fizyko-mekhanichni polia. – 2003. – Vol. 46, № 4. – P. 50-56.

A. S. Chernikova, A. M. Nedashkovska,
Faculty of Applied Mathematics and Informatics,
Ivan Franko National University of Lviv,
1, Universytets'ka Str., Lviv, 79000, Ukraine.

Received 15.08.2021; revised 29.09.2021.