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ON THE APPROXIMATION OF URYSOHN OPERATOR ON A SYMMETRICAL INTERVAL WITH BERNSTEIN-TYPE OPERATOR POLYNOMIALS

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Резюме. При застосуванні поліномів Бернштейна крім стандартного випадку $[0,1]$ природно виділити симетричний відрізок $[-1,1]$, на якому поліноми Бернштейна раніше майже не вивчалися. Ясно, що симетричний відрізок тісно пов'язаний з міркуваннями парності і непарності. У роботі для оператора Урисона (F), що задається на симетричному відрізку з невідомим ядром, про властивості якого ми можемо судити тільки аналізуючи результат його дії на будь-які функції з деякого класу, будується та вивчається операторний поліном типу С.Н. Бернштейна, який з ростом його степеня як завгодно точно наближає F.

Abstract. During the application of Bernstein-type polynomials, with the exception of the generalized case $[0,1]$, it is natural to define a symmetrical interval $[-1,1]$. Bernstein-type polynomials, defined on this interval, were not much studied in the literature. It is clear that a symmetrical interval is closely tied with the ideas on paired and unpaired functions. In this paper the Urysohn (F) operator is defined on a symmetrical interval with an unknown kernel. Properties of it we can identify only by analyzing its effect on any functions from a specific class. For such operator, a Bernstein-type polynomial approximates F with arbitrary high accuracy, depending on polynomial's degree, is built and studied.

1. Introduction

Functional approximation $F : L_1(0,1) \to \Re^1$ on a continual set of nodes

$$x^n(z,\xi^n) = x_0(z) + \sum_{i=1}^{n} H\left(z-\xi_i\right)\left[x_i(z) - x_{i-1}(z)\right], \tag{1}$$

$$\xi^n = (\xi_1, \xi_2, \ldots, \xi_n) \in \Omega_{\mathbf{z}^n} = \{\mathbf{z}^n : 0 \le z_1 \le \ldots \le z_n \le 1\}$$

is studied in multiple papers, e.g., see [1–12].

Let $x_i(z) \in Q\,[0,1]\,,\ i = 0,1,\ldots$ be arbitrary fixed elements of the space $Q[0,1]$ of piecewise continuous functions on a segment $[0,1]$ with a finite number of discontinuity points of the first kind. The set of such functions is called the interpolant framework and H(t) is a Heaviside function.

In the works [1-4] research the approximation of Urison operator by Bernstein-type and Stancu-type polynomials, the works [5-7] study the polynomial functional approximation. The works [8-12] are dedicated to integral continued fractions.

Key words. Bernstein-type polynomials; symmetrical interval; continual set of nodes; combinatorial correlations.

Let us consider the usage of a continual set of nodes during the approximation of Urysohn operator with Bernstein-type polynomials.

2. On Bernstein-type polynomials

For functions $f(x)$, continuous on the interval $[0,1] \subset R$, we get standard Bernstein-type polynomials

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) C_n^k x^k (1-x)^{n-k}, \quad n \in N$$

where x is a real variable, and C_n^k are binomial coefficients

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad n \in N, \quad k = 0,1,2\ldots,n, \quad C_0^0 = 1.$$

These polynomials are used usually for approximation of the function $f(x)$ (e.g., see [1-4]).

The work [1] considers the Urysohn operator

$$F(t, x(\cdot)) = \int_0^1 f(t,z,x(z))dz \tag{2}$$

with an unknown kernel $f(t,z,x)$, the properties of which we can identify only by analyzing its effect on any functions $x(z)$ of a particular class. A situation like this is sometimes called "grey box".

The task lied in building a relatively simple approximation to the operator F, in our case, Bernstein-type operator polynomials, which with any growth of its degree, would beyond doubt approximate F.

Let us assume that the following interpolation conditions are valid

$$F(t, x_i(\cdot)) = \int_0^1 f(t,z,x_i(z))dz \qquad i = \overline{1,n} \tag{3}$$

where

$$x_i(z) = \frac{i}{n} H(z-\xi), \quad \xi \in [0,1], \qquad i = \overline{1,n}. \tag{4}$$

Based on the known functions $F(t, x_i(\cdot))$, approximation to the Urysohn operator (2.1) with an unknown function $f(t,z,x(z))$ needed to be built. Bernstein-type operator polynomials was taken for this approximation

$$B_n(F, x(\cdot)) = F(t,0) - \int_0^1 \sum_{k=0}^{n} \frac{\partial F(t, \frac{k}{n} H(\cdot - z))}{\partial z} C_n^k x^k(z)(1-x(z))^{n-k} dz. \tag{5}$$

Let us introduce the following function (operator)

$$F_1(t,z,x(\cdot)) = \frac{\partial F(t, x(\cdot) H(\cdot - z))}{\partial z} = f(t,z,0) - f(t,z,x(\cdot)), \tag{6}$$

and assume that

$$F_1(t,z,x) \in C([0,1]\times[0,1]\times[0,1]). \tag{7}$$

This paper proves the theorem about a uniform convergence and the speed of convergence for constructed the approximations.

Afterwards, using (2.2) – (2.6) and the method of proof from [1-4], a number of Harun Karsli works emerged, dedicated to the Bernstein-type polynomials (see,e.g., [13-15]).

Apart from the standard case $[0, 1]$, it is natural to define a symmetrical interval $[-1, 1]$. On this interval the Bernstein-type polynomials have been insufficiently studied in the scientific literature. It is clear that the symmetrical interval is closely tied with the ideas on paired and unpaired functions which are crucial for the analysis but not very natural on a standard interval $[0, 1]$.

For the function $f(x)$, which is continuous on the interval $[a, b] \subset R$, Bernstein-type polynomials are introduced using the following formula

$$B_n(f,x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} f(\frac{(b-a)k}{n} + a) C_n^k (x-a)^k (b-x)^{n-k}, \quad n \in N.$$

At the moment, we are interested in the case of the symmetrical interval $[-1, 1]$ which has its particulars (see [16]). The case of the symmetrical interval is important for practical tasks. The structure of many functions manifests itself more naturally on $[-1, 1]$ than during their transfer on $[0, 1]$.

According to the general definition, Bernstein-type polynomials for the function $f \in C[-1, 1]$ are introduced using formula

$$B_n(f,x) = \frac{1}{2^n} \sum_{k=0}^{n} f(\frac{2k}{n} - 1) C_n^k (1+x)^k (1-x)^{n-k}, \quad n \in N. \tag{8}$$

Additionally to the preceding definition (2.7), it is convenient to use an equivalent formula

$$B_n(f,x) = \frac{1}{2^n} \sum_{k=0}^{n} f(1 - \frac{2k}{n}) C_n^k (1-x)^k (1+x)^{n-k}, \quad n \in N$$

with the summation from the right boundary of the interval $[-1, 1]$ instead of the left one, as used in formula (2.7).

The basic transition between formulas is done by using equation $C_n^{n-k} = C_n^k$.

3. Task statement

We obtained the following problem. Let us consider the following Urysohn operator

$$F(t, x(\cdot)) = \int_{-1}^{1} f(t, z, x(z)) dz \tag{9}$$

an unknown kernel $f(t, z, x)$, the properties of which we can identify only by analyzing its effect on any functions $x(z)$ of a particular class.

The task lied in building Bernstein-type operator polynomial which with any growth of its degree, would approximate F with arbitrary high accuracy.

Let us define interpolation conditions

$$F(t, x_i(\cdot)) = \int_{-1}^{1} f(t, z, x_i(z))dz \qquad i = \overline{0, n}$$

where continual set of nodes is defined by the formula

$$x_i(z) = (-1 + \frac{2i}{n})H(z - \xi), \quad \xi \in [-1, 1], \qquad i = \overline{0, n}.$$

4. Approximation selection

Let us use the Bernstein-type operator polynomial for such approximation

$$B_n(F, x(\cdot)) = -\frac{1}{2^n} \int_{-1}^{1} \sum_{k=0}^{n} f(t, z, \frac{2k}{n} - 1) C_n^k (1 + x(z))^k (1 - x(z))^{n-k} dz. \quad (10)$$

However, on of its items is an unknown function $f(t, z, \frac{2k}{n} - 1)$, $k = \overline{0, n}$. Let us define them in the same way as in work [17]. We get

$$f(t, z, \frac{2k}{n} - 1) = \frac{\partial F(t, (\frac{2k}{n} - 1)H(\cdot - z))}{\partial z} + f(t, z, 0),$$

which allows defining (3.1) as follows

$$B_n(F, x(\cdot)) = F(t, 0) - \frac{1}{2^n} \int_{-1}^{1} \sum_{k=0}^{n} \frac{\partial F(t, (\frac{2k}{n} - 1)H(\cdot - z))}{\partial z} \times$$
$$\times C_n^k (1 + x(z))^k (1 - x(z))^{n-k} dz.$$

Let us introduce the following function (operator)

$$F_1(t, z, x(\cdot)) = \frac{\partial F(t, x(\cdot)H(\cdot - z))}{\partial z} = f(t, z, 0) - f(t, z, x(z))$$

and assume that

$$F_1(t, z, x) \in C([-1, 1] \times [-1, 1] \times [-1, 1]) \quad (11)$$

is true.

Theorem 1. *Let the Urysohn operator (3.1) be such that the function-operator $F_1(t, z, x(z))$ built by it, satisfies condition (4.2), and let operator (3.1) be considered on compact $\Phi \subset [-1, 1]$. Then*

$$lim_{n \to \infty} \|B_n(F, x(\cdot)) - F(t, x(\cdot))\|_{C[-1,1]} = 0$$

uniformly relative to $x(\cdot) \in \Phi$, where $\Phi = \{x(z) \in C[-1, 1] : -1 \leq x(z) \leq 1\}$.

Proof is obtained in the same way as in [1].

5. Thoughts on even and odd functions

Let us further generalize the qualitative properties of Bernstein-type polynomials reflected in the paper [16] on Bernstein-type operator polynomials that approximate the Urysohn operator.

Lemma 1. *(see [16]). Let us take $f \in C[-1,1]$ with Bernstein-type polynomials $B_n(F,x)$, defined using formula (2.7). Then the following statements are true:*

1) If $f(-x) = f(x)$ for all $x \in [-1,1]$, then $B_n(f,-x) = B_n(f,x)$, $x \in R$, $n \in N$;

2) If $f(-x) = -f(x)$ for all $x \in [-1,1]$, then $B_n(f,-x) = -B_n(f,x)$, $x \in R$, $n \in N$.

The symmetrical interval $[-1,1]$ is suited naturally to the study of paired and unpaired functions. It is logical to assume that polynomials (4.1) inherit the properties of whether the function $\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}$ is paired or unpaired. This is truly so.

Property 1. *Assuming that $x(z) \in \Phi = \{x(z) \in C[-1,1]; -1 \le x(z) \le 1\}$, then, if $\left[-\frac{\partial F(t,(x(\cdot))H(\cdot-z))}{\partial z}\right]$ is paired, then $B_n(F,x(\cdot)) - F(t,0)$ is also paired, that is $B_n(F,-x(\cdot)) - F(t,0) = B_n(F,x(\cdot)) - F(t,0)$.*

Proof.

$$B_n(F,-x(\cdot)) - F(t,0) =$$

$$= -\frac{1}{2^n}\int_{-1}^{1}\sum_{k=0}^{n}\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}\times$$

$$\times\, C_n^k(1-x(z))^k(1+x(z))^{n-k}dz = \{k = n-m\} =$$

$$= -\frac{1}{2^n}\int_{-1}^{1}\sum_{k=0}^{n}\frac{\partial F(t,(\frac{2(n-m)}{n}-1)H(\cdot-z))}{\partial z}\times$$

$$\times\, C_n^{n-m}(1-x(z))^{n-m}(1+x(z))^m dz =$$

$$= -\frac{1}{2^n}\int_{-1}^{1}\sum_{k=0}^{n}\frac{\partial F(t,(1-\frac{2m}{n})H(\cdot-z))}{\partial z}C_n^m(1+x(z))^m(1-x(z))^{n-m}dz =$$

$$= B_n(F,x(\cdot)) - F(t,0).$$

The following property is proved in the same way.

Property 2. *Assuming that $x(z) \in \Phi = \{x(z) \in C[-1,1]; -1 \le x(z) \le 1\}$, then, if $\left[-\frac{\partial F(t,(x(\cdot))H(\cdot-z))}{\partial z}\right]$ is unpaired, then $B_n(F,x(\cdot)) - F(t,0)$ is also unpaired.*

6. Modified Temple formula

The needed formula first appeared in Temple's work [18] for standard Bernstein-type polynomials on $[0,1]$. Modification on $[-1,1]$ looks bulkier but will be useful during the systematic study of Bernstein-type polynomials on the symmetrical interval.

Property 3. *Let $x(z) \in \Phi = \{x(z) \in C[-1,1]; -1 \le x(z) \le 1\}$ and for $\left[-\frac{\partial F(t,(x(\cdot))H(\cdot-z))}{\partial z}\right]$ exists (4.1) then*

$$B_{n+1}(F,x(\cdot))-B_n(F,x(\cdot))=$$
$$=-\frac{1}{2^{n+1}}\int_{-1}^{1}\sum_{k=1}^{n}Q_{n,k}(F)(1+x(z))^k(1-x(z))^{n-k+1}dz$$

where

$$Q_{n,k}(F)=\frac{\partial F(t,(\frac{2k}{n+1}-1)H(\cdot-z))}{\partial z}C_{n+1}^k-$$
$$-\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}C_n^k-\frac{\partial F(t,(\frac{2(k-1)}{n}-1)H(\cdot-z))}{\partial z}C_n^{k-1}.$$

Proof.

$$B_n(F,x(\cdot))=F(t,0)-$$
$$-\frac{1}{2^n}\int_{-1}^{1}\sum_{k=0}^{n}\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}\times\frac{(1-x(z))+(1+x(z))}{2}\times$$
$$\times C_n^k(1+x(z))^k(1-x(z))^{n-k}dz=$$
$$=F(t,0)-\frac{1}{2^{n+1}}\int_{-1}^{1}\sum_{k=0}^{n}\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}C_n^k\times$$
$$\times\left[(1+x(z))^k(1-x(z))^{n-k+1}+(1+x(z))^{k+1}(1-x(z))^{n-k}\right]dz.$$

Let us split into two sums, and in the second one, transition to the numeration through k from 1 to $n+1$. We get:

$$B_n(F,x(\cdot))-F(t,0)=$$
$$=-\frac{1}{2^{n+1}}\int_{-1}^{1}\sum_{k=0}^{n}\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}C_n^k(1+x(z))^k(1-x(z))^{n-k+1}dz-$$
$$-\frac{1}{2^{n+1}}\int_{-1}^{1}\sum_{k=1}^{n+1}\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}C_n^{k-1}(1+x(z))^k(1-x(z))^{n-k+1}dz.$$

Let us take addend $k=0$ from the first sum, and addend $k=n+1$ from the second sum

$$B_n(F,x(\cdot))-F(t,0)=$$
$$=-\frac{1}{2^{n+1}}\int_{-1}^{1}\Big(\frac{\partial F(t,-H(\cdot-z))}{\partial z}(1-x(z))^{n+1}+$$
$$+\frac{\partial F(t,H(\cdot-z))}{\partial z}(1+x(z))^{n+1}\Big)dz-$$
$$-\frac{1}{2^{n+1}}\int_{-1}^{1}\sum_{k=1}^{n}\Big[\frac{\partial F(t,(\frac{2k}{n}-1)H(\cdot-z))}{\partial z}C_n^k+$$
$$+\frac{\partial F(t,(\frac{2(k-1)}{n}-1)H(\cdot-z))}{\partial z}C_n^{k-1}\Big]\times$$
$$\times(1+x(z))^k(1-x(z))^{n-k+1}dz.$$

Thus, for $B_{n+1}(F, x(\cdot)) - F(t,0)$ we get

$$B_{n+1}(F, x(\cdot)) - F(t,0) =$$

$$= -\frac{1}{2^{n+1}} \int_{-1}^{1} \sum_{k=0}^{n+1} \frac{\partial F(t, (\frac{2k}{n+1} - 1)H(\cdot - z))}{\partial z} C_{n+1}^{k} (1+x(z))^{k} (1-x(z))^{n-k+1} dz =$$

$$= -\frac{1}{2^{n+1}} \int_{-1}^{1} \Big(\frac{\partial F(t, -H(\cdot - z))}{\partial z} (1 - x(z))^{n+1} +$$

$$+ \frac{\partial F(t, H(\cdot - z))}{\partial z} (1 + x(z))^{n+1} \Big) dz -$$

$$-\frac{1}{2^{n+1}} \int_{-1}^{1} \sum_{k=1}^{n} \frac{\partial F(t, (\frac{2k}{n+1} - 1)H(\cdot - z))}{\partial z} C_{n+1}^{k} (1 + x(z))^{k} (1 - x(z))^{n-k+1} dz.$$

Let us take the difference $B_{n+1}(F, x(\cdot)) - B_n(F, x(\cdot))$, and we get the confirmation of property 3.

Next, let us examine coefficients $Q_{n,k}(F)$. By using identity

$$C_{n+1}^{k} = C_{n}^{k-1} + C_{n}^{k}$$

we get

$$Q_{n,k}(F) = C_{n}^{k-1} \left(\frac{\partial F(t, (\frac{2k}{n+1} - 1)H(\cdot - z))}{\partial z} - \right.$$

$$\left. - \frac{\partial F(t, (\frac{2(k-1)}{n+1} - 1)H(\cdot - z))}{\partial z} \right) -$$

$$-C_{n}^{k} \left(\frac{\partial F(t, (\frac{2k}{n} - 1)H(\cdot - z))}{\partial z} - \frac{\partial F(t, (\frac{2k}{n+1} - 1)H(\cdot - z))}{\partial z} \right).$$

Now, let's use designations for divided differences

$$[f; x_1, x_0] \equiv \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

It is easy to check that

$$Q_{n,k}(F) = -\frac{2}{n+1} C_{n-1}^{k-1} \left(\left[\frac{\partial F(t, x(\cdot)H(\cdot - z))}{\partial z}; \frac{2k}{n} - 1, \frac{2k}{n+1} - 1 \right] - \right.$$

$$\left. - \left[\frac{\partial F(t, x(\cdot)H(\cdot - z))}{\partial z}; \frac{2k}{n+1} - 1, \frac{2(k-1)}{n} - 1 \right] \right).$$

Let us move on to the divided differences of the second order

$$[f; x_2, x_1, x_0] \equiv \frac{[f; x_2, x_1] - [f; x_1, x_0]}{x_2 - x_0}.$$

We get

$$Q_{n,k}(F) = -\frac{4}{n(n+1)} C_{n-1}^{k-1} \Big[\frac{\partial F(t, x(\cdot) H(\cdot - z))}{\partial z};$$

$$\frac{2k}{n} - 1, \frac{2k}{n+1} - 1, \frac{2(k-1)}{n} - 1 \Big].$$

When working with formulas $Q_{n,k}(F)$ it is useful to take into account the positioning of the dots

$$-1 \leq \frac{2(k-1)}{n} - 1 < \frac{2k}{n+1} - 1 < \frac{2k}{n} - 1 \leq 1, \quad k = 1, \ldots, n$$

true for any $n \in N$.

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