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ON SOLUTION OF THE INITIAL-VALUE PROBLEM FOR HOMOGENEOUS WAVE EQUATION WITH DYNAMIC BOUNDARY CONDITION IN WEIGHTED LEBESGUE SPACES

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Резюме. Розглянуто початково-крайову задачу для однорідного хвильового рівняння з динамічною крайовою умовою у тривимірній за просторовими координатами області з ліпшицевою компактною межею. Для теоретичних досліджень введено потрібні функційні простори, зокрема вагові простори Лебега і Соболєва, та визначено оператори сліду і взяття нормальної похідної у цих простраx. Крім того, доведено деякі властивості даних операторів. Далі сформульовано означення сильного розв'язку початково-крайової задачі та відповідні теореми про існування і єдиність її розв'язку. Для доведення цих теорем використано перетворення Лапласа векторозначних функцій. З його допомогою еволюційна задача зведена до еквівалентної крайової задачі для еліптичного рівняння.

Обгрунтування основних результатів подано у вигляді декількох проміжних етапів. Спочатку доведено єдиність сильного розв'язку задачі, після чого розглянуто еквівалентну і досліджено її коректність. На завершальному етапі доведено існування розв'язку еволюційної задачі і за допомогою перетворення Лапласа знайдено його зображення у відповідних функційних просторах.

Abstract. Initial-value problem for homogeneous wave equation with dynamic boundary condition is considered in three-dimensional by spatial variables domain with Lipschitz compact surface. Required functional spaces are introduced for theoretical researches, in particular weighted Lebesgue and Sobolev spaces, and the trace and normal derivative operators are defined in these spaces. In addition, some properties of these operators are proved. Then the definition of a strong solution to the initial-value problem and corresponding theorems on the existence and uniqueness of the solution are formulated. To prove these theorems the Laplace transform of vector-valued functions is applied. By using it, evolutionary problem is reduced to equivalent boundary value problem for elliptic equation.

The justification of main results is demonstrated in the form of several intermediate stages. At first the uniqueness of the strong solution to the problem is proved, then the equivalent problem is considered and its correctness is explored. At the final stage the existence of the solution to the evolutionary problem is proved and its image is found in corresponding functional spaces by applying the Laplace transform.

Key words. initial-value problem for wave equation; dynamic boundary condition; generalized solution; weighted Lebesgue and Sobolev spaces; Laplace and Laguerre transforms; boundary integral equations; strong solution.

1. Introduction

Let Ω be a domain (bounded or unbounded) in $\mathbb{R}^n$ ($n \geq 2$) and let Γ be a boundary of Ω. We assume that Γ is a Lipschitz compact surface and $\boldsymbol{\nu}(x)$ is a unit vector of outer normal to this surface at point $x \in \Gamma$. We denote $\mathbb{R}_+ := (0, \infty)$, $Q := \Omega \times \mathbb{R}_+$ and $\Sigma := \Gamma \times \mathbb{R}_+$.

We consider the following initial-value problem: find a function $u(x,t)$, $(x,t) \in \overline{Q}$, which satisfies (in a certain sense) the wave equation

$$\partial_t^2 u - \Delta u = 0 \quad \text{in} \;\; Q, \tag{1}$$

homogeneous initial conditions

$$u(\cdot, 0) = 0, \quad \partial_t u(\cdot, 0) = 0 \quad \text{in} \;\; \Omega, \tag{2}$$

and dynamic boundary condition

$$\partial_{\boldsymbol{\nu}} u + b\, \partial_t u = g \quad \text{on} \;\; \Sigma, \tag{3}$$

where g is a function given on Σ, $b \geqslant 0$ is a function given on Γ, ∂_t and $\partial_{\boldsymbol{\nu}}$ denote time and normal derivatives respectively, Δ is the Laplace operator.

Hereinafter, we call this problem (**HD**).

The research of initial-value problems for the wave equation has a rich history (see, for example, monographs [8, 12, 15, 22, 24]). However, the development of effective methods for numerical solution of such problems still remains an actual problem, since they usually require significant amount of computational resources. This particularly concerns the problems in three-dimensional domains with complex geometry and problems with complicated boundary conditions.

The mentioned problems can be conveniently considered in functional spaces that consist of functions of a real variable (the time variable) with values in corresponding Hilbert spaces. The advantage of such an approach is a simple algorithm of integral transformations by time variable for the purpose of transition to the equivalent elliptic problems that are well explored from both theoretical and practical viewpoints. In particular, a new approach was founded in papers [1, 2] that combines the Laplace transform and integral representations of solutions of corresponding elliptic boundary value problems and leads to boundary integral equations (BIEs). Examples of the application of the Laplace transform for investigating the existence and uniqueness of the solution of different evolutionary problems can be found in [8, section XVI].

Unlike theoretical researches, the application of the Laplace transform in practice is a complicated process because of the resource-intensive inverse transform. Therefore, special approaches are used to set dependency of the solutions to evolutionary problems on time variable, in particular, based on functional convolutions. Here we note the so-called convolution quadrature method [18]. It was further developed in papers [3, 13, 24]. Now this approach is widely used in numerical modelling (with the respective theoretical justification) of different kinds of evolutionary processes (see, for example, [4, 11, 23, 24] and references there).

The Laguerre transform is closely coupled with the Laplace transform. In particular, they both have a common domain of definition. On the one hand, it allows us to use the Laplace transform for theoretical researches when solving evolutionary problems, e.g., for investigation of the existence and uniqueness of the solution. However, the Laguerre transform is more constructive for finding numerical solutions to mentioned problems because of the simple inverse transformation in terms of computational resources. As a result, it allows us to effectively use the advantages of BIEs method for problems in three-dimensional domains by spatial variables. Examples of the application of the method that combines the Laguerre transform and BIEs for such problems can be found in [5, 9, 16, 17, 20, 21] and references there.

The results of numerical experiments presented in [10] demonstrate an application of the aforementioned combined method for solving initial-value problems with dynamic boundary conditions. Notice that in this case, with the help of the Laguerre transform, it is possible to get rid of the time derivative of the trace of the solution on the boundary and obtain boundary conditions containing only the trace and the normal derivative operators.

The goal of this article is to investigate the existence and uniqueness of the strong solution of the initial-value problem (1) – (3) in the weighted Lebesgue spaces and show its continuous dependency on input data of the problem. One of the main research methods here is the Laplace transform. Due to the connection of this transform with the Laguerre transform [19] in specified functional spaces, obtained results form a basis for justification of the Laguerre transform and BIE combination for numerical solution of the problems mentioned above.

The main definitions and terms are introduced in section 2. The definition of a strong solution to the problem and the formulation of theorems about its existence and uniqueness are given in section 3. A direct proof of these theorems is proposed in section 5 after clarifying auxiliary facts about the Laplace transform of vector-valued functions in section 4.

2. Main definitions and terms

At first, we consider required functional spaces. Let X be a complex Hilbert space with the inner product $(\cdot,\cdot)_X$ and the induced norm $\|\cdot\|_X$. Elements of the space X are called vectors. We denote by $\mathcal{D}(\mathbb{R})$ a linear space that consists of infinitely differentiable finite functions $\varphi : \mathbb{R} \to \mathbb{C}$. We say that the sequence $\{\varphi_n\}$ converges to φ in $\mathcal{D}(\mathbb{R})$ if and only if bounded interval $I \subset \mathbb{R}$ such that $\operatorname{supp}\varphi_n \subset \overline{I}$ exists for every $n \in \mathbb{N}$ and $\varphi_n^{(m)} \rightrightarrows \varphi^{(m)}$ when $n \to \infty$ on $\overline{I}$ for arbitrary $m \in \mathbb{N} \cup \{0\}$. By $\mathcal{D}'(\mathbb{R}; X)$ we mean a linear space that consists of linear continuous mappings $F : \mathcal{D}(\mathbb{R}) \to X_w$, where X_w is a linear space X with a weak topology. We denote by $\langle F, \varphi\rangle_{\mathcal{D}(\mathbb{R})}$ the action $F \in \mathcal{D}'(\mathbb{R}; X)$ on $\varphi \in \mathcal{D}(\mathbb{R})$. Elements of the space $\mathcal{D}'(\mathbb{R}; X)$ are called generalized vector-valued functions. For every function $F \in \mathcal{D}'(\mathbb{R}; X)$ and an arbitrary natural m we define derivative $F^{(m)}$ according to the rule

$$\langle F^{(m)}, \varphi\rangle_{\mathcal{D}(\mathbb{R})} = (-1)^m \langle F, \varphi^{(m)}\rangle_{\mathcal{D}(\mathbb{R})}, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Obviously, generalized vector-valued functions are infinitely differentiable.

Let $L^1_{\text{loc}}(\mathbb{R}; X)$ be a linear space of measurable functions $f : \mathbb{R} \to X$ such that for arbitrary bounded interval $I \subset \mathbb{R}$ restriction of the function f on I

belongs to $L^1(I;X)$, i. e. $\int_I \|f(t)\|_X \, dt < \infty$. As we can see, for every function $f \in L^1_{\rm loc}(\mathbb{R};X)$ mapping $F_f : \mathcal{D}(\mathbb{R}) \to X$ defined according to the rule

$$\langle F_f, \varphi \rangle_{\mathcal{D}(\mathbb{R})} = \int_{\mathbb{R}} f(t)\, \varphi(t)\, dt, \quad \varphi \in \mathcal{D}(\mathbb{R}), \tag{4}$$

is an element of the space $\mathcal{D}'(\mathbb{R};X)$.

Mapping of the space $L^1_{\rm loc}(\mathbb{R};X)$ into $\mathcal{D}'(\mathbb{R};X)$ defined by the rule (4) is injective. That allows us to identify $L^1_{\rm loc}(\mathbb{R};X)$ with its image in $\mathcal{D}'(\mathbb{R};X)$ (this image is a subspace). Due to this, we consider that $L^1_{\rm loc}(\mathbb{R};X) \subset \mathcal{D}'(\mathbb{R};X)$.

Let's denote by $\mathcal{D}'(\mathbb{R}_+;X)$ a subspace of the space $\mathcal{D}'(\mathbb{R}_+;X)$ that consists of such elements F that $\operatorname{supp} F \subset [0, +\infty)$, i. e.

$$F \in \mathcal{D}'(\mathbb{R}_+;X) \Leftrightarrow F \in \mathcal{D}'(\mathbb{R};X) \text{ and } \langle F, \varphi \rangle_{\mathcal{D}(\mathbb{R})} = 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),\ \operatorname{supp} \varphi \subset (-\infty, 0).$$

As we can see, for a function $f \in L^1_{\rm loc}(\mathbb{R};X)$ we have $f \in \mathcal{D}'(\mathbb{R}_+;X)$ if and only if $f(t) = 0$ for $t \in (-\infty, 0)$. We denote by $L^1_{\rm loc}(\mathbb{R}_+;X)$ a space of functions $f \in L^1_{\rm loc}(\mathbb{R};X)$ that belong to $\mathcal{D}'(\mathbb{R}_+;X)$, i. e. $f(t) = 0$ for $t \in (-\infty, 0)$.

Let $\alpha > 0$ be an arbitrary fixed number. By $L^2_\alpha(\mathbb{R}_+;X)$ we mean a linear space that consists of functions $f \in L^1_{\rm loc}(\mathbb{R}_+;X)$ such that

$$\int_{\mathbb{R}_+} \|f(t)\|^2_X \, e^{-\alpha t}\, dt < \infty,$$

with the inner product

$$(f,g)_{L^2_\alpha(\mathbb{R}_+;X)} = \int\limits_{\mathbb{R}_+} \big(f(t), g(t)\big)_X \, e^{-\alpha t}\, dt,$$

and induced norm

$$\|f\|_{L^2_\alpha(\mathbb{R}_+;X)} = \big[(f,f)_{L^2_\alpha(\mathbb{R}_+;X)}\big]^{1/2} \equiv \Big[\int\limits_{\mathbb{R}_+} \|f(t)\|^2_X \, e^{-\alpha t}\, dt\Big]^{1/2}, \tag{5}$$

where $L^2_\alpha(\mathbb{R}_+;X)$ is a Hilbert space.

We assume that the space $L^2_\alpha(\mathbb{R}_+;X)$ as a subspace of the space $L^1_{\rm loc}(\mathbb{R}_+;X)$ is identified with corresponding subspace of the space $\mathcal{D}'(\mathbb{R}_+;X)$. As a result we can consider the derivative $f^{(k)}$ of any element f from the space $L^2_\alpha(\mathbb{R}_+;X)$ in terms of the space $\mathcal{D}'(\mathbb{R};X)$, where k is an arbitrary natural number.

Let $\mathbb{N}$ be a set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For arbitrary $m \in \mathbb{N}$ we define the weighted Sobolev space

$$H^m_\alpha(\mathbb{R}_+;X) := \{f \in L^2_\alpha(\mathbb{R}_+;X) \,|\, f^{(k)} \in L^2_\alpha(\mathbb{R}_+;X),\ k = \overline{1,m}\}, \tag{6}$$

with the inner product

$$(f,g)_{H^m_\alpha(\mathbb{R}_+;X)} = \sum_{k=0}^{m} \int\limits_{\mathbb{R}_+} \big(f^{(k)}(t), g^{(k)}(t)\big)_X \, e^{-\alpha t}\, dt,$$

and standard norm

$$\|f\|_{H^m_\alpha(\mathbb{R}_+;X)} = \left[\sum_{k=0}^{m} ||f^{(k)}||^2_{L^2_\alpha(\mathbb{R}_+;X)}\right]^{1/2}. \tag{7}$$

It is known that for an arbitrary function $f \in H^m_\alpha(\mathbb{R}_+;X)$ and any point $t_0 \in [0,+\infty)$ there exist traces $f(t_0) \in X, ..., f^{(m-1)}(t_0) \in X$, and $f(0) = 0, ..., f^{(m-1)}(0) = 0$.

Let's now consider spaces of functions defined on Ω. We denote by $\mathcal{D}(\Omega)$ a linear space of infinitely differentiable finite functions $\varphi : \Omega \to \mathbb{C}$. We say that the sequence $\{\varphi_n\}$ converges to φ in $\mathcal{D}(\Omega)$ if there is a compact $K \subset \Omega$ such that $\operatorname{supp}\varphi_n \subset K$ for all $n \in \mathbb{N}$, and $D^\beta\varphi_n \rightrightarrows D^\beta\varphi$ when $n \to \infty$ on K for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$. We denote by $\mathcal{D}'(\Omega)$ a linear space of linear continuous functionals $F : \mathcal{D}(\Omega) \to \mathbb{C}$. Elements of the space $\mathcal{D}'(\Omega)$ are called generalized functions given on Ω. Also we denote by $\langle F, \varphi\rangle_{\mathcal{D}(\Omega)}$ an action of element $F \in \mathcal{D}'(\Omega)$ on element $\varphi \in \mathcal{D}(\Omega)$.

For arbitrary $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}_0^n$ and generalized function $F \in \mathcal{D}'(\Omega)$ a derivative $D^\beta F \in \mathcal{D}'(\Omega)$ is defined according to the rule

$$\langle D^\beta F, \varphi\rangle_{\mathcal{D}(\Omega)} = (-1)^{|\beta|}\langle F, D^\beta\varphi\rangle_{\mathcal{D}(\Omega)}, \quad \varphi \in \mathcal{D}(\Omega). \tag{8}$$

Let $f \in L^1_{\mathrm{loc}}(\Omega)$, i.e., $f : \Omega \to \mathbb{C}$ be a measured function such that for any compact $K \subset \Omega$ the restriction $f|_K$ of the function f on K belongs to the space $L^1(K)$. Obviously that functional $F_f : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by the rule

$$\langle F_f, \varphi\rangle_{\mathcal{D}(\Omega)} := \int_\Omega f(x)\,\varphi(x)\,dx, \quad \varphi \in \mathcal{D}(\Omega), \tag{9}$$

belongs to the space $\mathcal{D}'(\Omega)$.

Based on the Dubois-Reymond lemma, mapping of the space $L^1_{\mathrm{loc}}(\Omega)$ into the space $\mathcal{D}'(\Omega)$ that is defined in (9) is injective and, therefore, the space $L^1_{\mathrm{loc}}(\Omega)$ can be identified with its image in this mapping (this image is a linear subspace of the space $\mathcal{D}'(\Omega)$). Thus, we consider that $L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$.

Let $L^2(\Omega)$ be a linear space that consists of the functions $v \in L^1_{\mathrm{loc}}(\Omega)$ such that $\int_\Omega |v(x)|^2\,dx < \infty$. In this space the inner product and the induced norm are defined as

$$(v, w)_0 := \int_\Omega v(x)\overline{w(x)}\,dx, \quad ||v||_0 := \big[(v, v)_0\big]^{1/2}.$$

Henceforth, by derivatives of a function from $L^2(\Omega)$ we mean the derivatives of this function as elements of the space $\mathcal{D}'(\Omega)$ according to the rule (8).

Let's denote a Sobolev space

$$H^1(\Omega) := \{v \in L^2(\Omega) \mid v_{x_i} \in L^2(\Omega),\ i = \overline{1, n}\}.$$

It is a Hilbert space with the inner product

$$(v, w)_1 := (v, w)_0 + [v, w] \equiv \int_\Omega \big(v(x)\overline{w(x)} + \nabla v(x)\overline{\nabla w(x)}\big)\,dx,$$

and corresponding norm

$$\|v\|_1 = \left[\|v\|_0^2 + [v,v]\right]^{1/2},$$

where

$$[v,w] := \int_\Omega \nabla v(x)\overline{\nabla w(x)}dx \equiv \sum_{j=1}^n \int_\Omega v_{x_j}(x)\overline{w_{x_j}(x)}\,dx.$$

Let's introduce one more space

$$H^1(\Omega,\Delta) := \{v \in H^1(\Omega) \mid \Delta v \in L^2(\Omega)\}.$$

This space is a Hilbert space with the inner product

$$(v,w)_2 := (v,w)_1 + (\Delta v, \Delta w)_0 \equiv$$

$$\equiv \int_\Omega \left[v(x)\overline{w(x)} + \nabla v(x)\overline{\nabla w(x)} + \Delta v(x)\overline{\Delta w(x)}\right] dx$$

and norm

$$\|v\|_2 := \left[\|v\|_1^2 + \|\Delta v\|_0^2\right]^{1/2}, \quad v \in H^1(\Omega,\Delta). \tag{10}$$

Now we consider operator $A : H^1(\Omega,\Delta) \to L^2(\Omega)$ defined by the rule: $Av = \Delta v,\ v \in H^1(\Omega,\Delta)$. This operator is linear and closed. Henceforth, we denote it by Δ.

Next we denote by $H^{1/2}(\Gamma)$ a Sobolev space which consists of functions that belong to the space $L^2(\Gamma)$ and can be approximated by the elements of the Sobolev spaces $H^{1/2}(\mathbb{R}^{n-1})$ using the local parametric representations of the smooth parts of the Lipschitz boundary (for details see, f.e. [14, §7.3 in Chapt. 1]). $H^{-1/2}(\Gamma)$ is a dual of this space. Also we denote by $\|\cdot\|_{1/2}$ and $\|\cdot\|_{-1/2}$ the norms in spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively. $\langle\cdot,\cdot\rangle_{1/2}$ is an action of the element of $H^{-1/2}(\Gamma)$ on element of the space $H^{1/2}(\Gamma)$.

It is known that there exists a linear continuous and surjective operator

$$\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma),$$

which is a continuous extension of the operator $\widetilde{\gamma}_0 : C^1(\overline{\Omega}) \to C(\Gamma)$, defined by the rule $\widetilde{\gamma}_0 v = v|_\Gamma,\ v \in C^1(\overline{\Omega})$. In particular, there exists such a constant $C_1 > 0$ that

$$||\gamma_0 v||_{1/2} \leqslant C_1 ||v||_1, \quad v \in H^1(\Omega). \tag{11}$$

Operator γ_0 is called a *trace operator*.

In addition we need a normal derivative operator (in the weak sense)

$$\gamma_1 : H^1(\Omega,\Delta) \to H^{-1/2}(\Gamma),$$

which is defined by the identity

$$\langle\gamma_1 v, \gamma_0 w\rangle_{1/2} = (\Delta v, w)_0 + [v,w], \quad v \in H^1(\Omega,\Delta),\ w \in H^1(\Omega).$$

The mapping $\gamma_1 : H^1(\Omega,\Delta) \to H^{-1/2}(\Gamma)$ is continuous and for smooth functions $v \in C^1(\overline{\Omega})$ we have $\gamma_1 v = \partial_\nu v|_\Gamma$ [6, Lemma 3.2]. Based on this we obtain the Green's formula

$$(\Delta v, w)_0 = \langle\gamma_1 v, \gamma_0 w\rangle_{1/2} - [v,w], \quad v \in H^1(\Omega,\Delta),\ w \in H^1(\Omega), \tag{12}$$

and inequality

$$\|\gamma_1 v\|_{-1/2} \leqslant C_2 \|v\|_2, \quad v \in H^1(\Omega, \Delta), \tag{13}$$

where $C_2 > 0$ is some constant.

In the case of vector-valued functions we consider new operators

$$\widetilde{\gamma_0} : L^2_\alpha(\mathbb{R}_+; H^1(\Omega)) \to L^2_\alpha(\mathbb{R}_+; H^{1/2}(\Gamma))$$

and

$$\widetilde{\gamma_1} : L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta)) \to L^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$$

defined by the rules $\widetilde{\gamma_0} u(t) := \gamma_0(u(t)),\ t \in \mathbb{R}_+$, and $\widetilde{\gamma_1} u(t) := \gamma_1(u(t)),\ t \in \mathbb{R}_+$, respectively. For simplicity, in the sequel we suppress the symbol 'tilde' and write γ_0 and γ_1 instead of $\widetilde{\gamma_0}$ and $\widetilde{\gamma_1}$.

Lemma 1. *Let $u \in L^2_\alpha(\mathbb{R}_+; H^1(\Omega))$. Then $\gamma_0 u \in L^2_\alpha(\mathbb{R}_+; H^{1/2}(\Gamma))$ and operator γ_0 is continuous on $L^2_\alpha(\mathbb{R}_+; H^1(\Omega))$. If $u \in L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta))$, then $\gamma_1 u \in L^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ and operator γ_1 is continuous on $L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta))$. In addition, if $u \in H^2_\alpha(\mathbb{R}_+; L^2(\Omega)) \cap H^1_\alpha(\mathbb{R}_+; H^1(\Omega))$, then $u \in C(\mathbb{R}; H^1(\Omega))$ and $u' \in C(\mathbb{R}; L^2(\Omega))$, furthermore $u(t) = 0$, $u'(t) = 0$ when $t \leqslant 0$.*

Proof. This statement easily follows from the definition of the space $L^2_\alpha(\mathbb{R}_+; X)$ where X is a Hilbert space, and inequalities (11) and (13). □

Lemma 2. *Let $u \in H^1_\alpha(\mathbb{R}_+; H^1(\Omega))$. Then*

$$\gamma_0 u \in H^1_\alpha(\mathbb{R}_+; H^{1/2}(\Gamma)) \quad \text{and} \quad (\gamma_0 u)' = \gamma_0 u'. \tag{14}$$

Proof. Let $\gamma_0^* : H^{-1/2}(\Gamma) \to \big(H^1(\Omega)\big)'$ be a dual of the trace operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\big(H^1(\Omega)\big)'$ be a dual space of $H^1(\Omega)$. Notice, then for any $u \in H^1_\alpha(\mathbb{R}_+; H^1(\Omega))$ and $\varphi \in \mathcal{D}(\mathbb{R})$ an integral $A(u) := \int\limits_{\mathbb{R}} \gamma_0 u(t)\, \varphi'(t)\, dt$ is a linear continuous mapping $A : H^1_\alpha(\mathbb{R}_+; H^1(\Omega)) \to H^{1/2}(\Gamma)$ and the product $\langle \cdot, \cdot \rangle_{1/2}$ is also continuous, so we can interchange the order of these operations. Then for any $v \in H^{-1/2}(\Gamma)$ and $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned}
\langle v, \int\limits_{\mathbb{R}} \gamma_0 u(t)\, \varphi'(t)\, dt \rangle_{1/2} &= \int\limits_{\mathbb{R}} \langle v, \gamma_0 u(t) \rangle_{1/2}\, \varphi'(t)\, dt = \\
&= \int\limits_{\mathbb{R}} \langle \gamma_0^* v,\, u(t) \rangle_{H^1(\Omega)}\, \varphi'(t)\, dt = \langle \gamma_0^* v, \int\limits_{\mathbb{R}} u(t)\, \varphi'(t)\, dt \rangle_{H^1(\Omega)} = \\
&= -\langle \gamma_0^* v, \int\limits_{\mathbb{R}} u'(t)\, \varphi(t)\, dt \rangle_{H^1(\Omega)} = -\int\limits_{\mathbb{R}} \langle \gamma_0^* v,\, u'(t) \rangle_{H^1(\Omega)}\, \varphi(t)\, dt = \\
&= -\int\limits_{\mathbb{R}} \langle v, \gamma_0 u'(t) \rangle_{1/2}\, \varphi(t)\, dt = -\langle v, \int\limits_{\mathbb{R}} \gamma_0 u'(t)\, \varphi(t)\, dt \rangle_{1/2}.
\end{aligned}$$

Hence (14) directly follows from this. □

3. Main results

In this section we state the main results of this paper.

Let's define a strong solution of the problem (**HD**) for the given $\alpha > 0$, $g \in L^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ and $b \in L^\infty(\Gamma)$.

Definition 2. By a strong solution of the problem (**HD**) we mean the function

$$u \in H^2_\alpha(\mathbb{R}_+; L^2(\Omega)) \cap H^1_\alpha(\mathbb{R}_+; H^1(\Omega)) \cap L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta)), \tag{15}$$

for which the following equalities hold

$$u''(t) - \Delta u(t) = 0 \quad \text{in} \quad L^2(\Omega), \quad t \in \mathbb{R}_+, \tag{16}$$

$$\gamma_1 u(t) + b(\gamma_0 u(t))' = g(t) \quad \text{in} \quad H^{-1/2}(\Gamma), \quad t \in \mathbb{R}_+. \tag{17}$$

Theorem 1. *Let $g \in H^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for $\alpha > 0$ and $b \in L^\infty(\Gamma)$, $b \geqslant 0$ on Γ. Then the problem (**HD**) has no more than one strong solution.*

Theorem 2. *Let $g \in H^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ for $\alpha > 0$ and $b \in L^\infty(\Gamma)$, $b \geqslant 0$ on Γ. Then there exists a strong solution to the problem (**HD**) (and only one). In addition, it satisfies the estimate*

$$\begin{aligned}\|u\|_{H^2_\alpha(\mathbb{R}_+;L^2(\Omega))} + \|u\|_{H^1_\alpha(\mathbb{R}_+;H^1(\Omega))} + \|u\|_{L^2_\alpha(\mathbb{R}_+;H^1(\Omega,\Delta))} \leq \\ \leq C_3 \|g\|_{H^2_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))},\end{aligned} \tag{18}$$

where $C_3 > 0$ is a constant dependent on the input data only.

The proofs of the aforementioned theorems are provided in section 5.

4. Laplace transform of vector-valued functions

Let $\mathcal{S}(\mathbb{R})$ be a linear space, composed of the infinitely differentiated functions $\psi : \mathbb{R} \to \mathbb{C}$ such that

$$\sup_{t\in\mathbb{R}} |t|^k |\psi^{(m)}(t)| < \infty \quad \text{for any} \quad k, m \in \mathbb{N}_0.$$

By definition, the sequence $\{\psi_n\}$ converges to ψ in $\mathcal{S}(\mathbb{R})$ when

$$\sup_{t\in\mathbb{R}} (1 + |t|)^k |\psi_n^{(m)}(t) - \psi^{(m)}(t)| \underset{n\to\infty}{\to} 0 \quad \text{for arbitrary} \quad k, m \in \mathbb{N}_0.$$

By $\mathcal{S}'(\mathbb{R}; X)$ we denote a linear space of linear and continuous mappings $G : \mathcal{S}(\mathbb{R}) \to X_w$.

The elements of the space $\mathcal{S}(\mathbb{R})$ are called rapidly decreasing functions, and the elements of the space $\mathcal{S}'(\mathbb{R}; X)$ are the slowly increasing generalized vector-valued functions. This definition, in particular, is due to the fact that the element $G_g \in \mathcal{S}'(\mathbb{R}; X)$ determined by the rule

$$\langle G_g, \psi \rangle_{\mathcal{S}(\mathbb{R})} := \int_{\mathbb{R}} g(t)\, \psi(t)\, dt, \quad \psi \in \mathcal{S}(\mathbb{R}), \tag{19}$$

corresponds to the function $g \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ such that

$$\int_{\mathbb{R}} \|g(t)\|_X (1 + |t|)^{-s}\, dt < \infty, \quad \text{where} \quad s = s(g) \geqslant 0 \ - \ \text{some number}, \tag{20}$$

By $\langle \cdot, \cdot \rangle_{\mathcal{S}(\mathbb{R})}$ we denote an action of an element of the space $\mathcal{S}'(\mathbb{R}; X)$ on an element of the space $\mathcal{S}(\mathbb{R})$. Notice that the functions $g \in L^1_{\text{loc}}(\mathbb{R}; X)$, that satisfy the condition (20), form a linear space that due to the mapping (19) can be identified with the subspace of the space $\mathcal{S}'(\mathbb{R}; X)$ (hereafter we consider that to be done).

Let's recall the definition of the *Fourier transform* of elements of the space $\mathcal{S}'(\mathbb{R}; X)$. Before this, we need to define the Fourier transform of functions of the space $\mathcal{S}(\mathbb{R})$. According to the commonly accepted definition, by Fourier transform of the arbitrary function $\psi \in \mathcal{S}(\mathbb{R})$ we mean the function $\widetilde{\psi} : \mathbb{R} \to \mathbb{C}$, determined by the rule

$$\widetilde{\psi}(\eta) = \mathfrak{F}[\psi](\eta) \equiv \mathfrak{F}_{t \to \eta}[\psi(t)](\eta) := \int_{\mathbb{R}} \psi(t)\, e^{-i\eta t}\, dt, \quad \eta \in \mathbb{R}. \tag{21}$$

It is known that the mapping $\mathfrak{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism.

For the arbitrary function $g \in L^1(\mathbb{R}; X)$ the Fourier transform $\mathfrak{F}[g] : \mathbb{R} \to X$ is also defined by the rule (21) (with replacement of ψ by g), namely

$$\widetilde{g}(\eta) = \mathfrak{F}[g](\eta) \equiv \mathfrak{F}_{t \to \eta}[g(t)](\eta) := \int_{\mathbb{R}} g(t)\, e^{-i\eta t}\, dt, \quad \eta \in \mathbb{R}. \tag{22}$$

For any element $G \in \mathcal{S}'(\mathbb{R}; X)$ its Fourier transform is defined as the element $\mathfrak{F}[G] \in \mathcal{S}'(\mathbb{R}; X)$ such that

$$\langle \mathfrak{F}[G], \psi \rangle_{\mathcal{S}(\mathbb{R})} := \langle G, \mathfrak{F}[\psi] \rangle_{\mathcal{S}(\mathbb{R})}, \quad \psi \in \mathcal{S}(\mathbb{R}). \tag{23}$$

It is easy to verify that (22) follows from (23) if $G = g \in L^1(\mathbb{R}; X)$ and vice versa.

Let $\omega \in \mathbb{R}$ be an arbitrary number. We define

$$\mathcal{D}'(\omega, \mathbb{R}_+; X) := \big\{ F \in \mathcal{D}'(\mathbb{R}_+; X) \mid e^{-\xi \cdot} F(\cdot) \in \mathcal{S}'(\mathbb{R}_+; X) \quad \forall\, \xi > \omega \big\},$$

where $\mathcal{S}'(\mathbb{R}_+; X) := \mathcal{D}'(\mathbb{R}_+; X) \cap \mathcal{S}'(\mathbb{R}; X)$. In particular, the functions $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ such that

$$e^{-\xi \cdot} f(\cdot) \in L^1(\mathbb{R}_+; X) \quad \forall\, \xi > \omega, \tag{24}$$

are elements of the space $\mathcal{D}'(\omega, \mathbb{R}_+; X)$.

We denote $\Pi_\omega := \{p = \xi + i\eta \in \mathbb{C} \mid \operatorname{Re} p = \xi > \omega\}$. By the Laplace transform of the function $G \in \mathcal{D}'(\omega, \mathbb{R}_+; X)$ we mean a function $\widehat{G} : \Pi_\omega \to X$ determined by the rule

$$\widehat{G}(\xi + i\eta) \equiv \widehat{G}(p) = \mathfrak{L}[G](p) := \mathfrak{F}_{t \to \eta}[e^{-\xi t} G(t)](\xi + i\eta). \tag{25}$$

Hence, it easily follows that

$$G(t) = e^{\xi t} \mathfrak{F}^{-1}_{\eta \to t}[\widehat{G}(\xi + i\eta)](t), \quad t \in \mathbb{R}_+, \tag{26}$$

is an inverse Laplace transform.

From the definition of the Laplace transform of the functions $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ satisfying the condition (24) we obtain

$$\widehat{f}(p) = \mathfrak{L}[f](p) := \int_{\mathbb{R}_+} f(t) e^{-p\, t}\, dt, \quad p \in \Pi_\omega. \tag{27}$$

Note that if the function $\eta \mapsto \widehat{f}(\xi + i\eta) : \mathbb{R} \to X$ belongs to $L^1(\mathbb{R}; X)$ for some $\xi > \omega$, then according to (26) we will obtain

$$f(t) = \frac{1}{2\pi} e^{\xi t} \int\limits_{\mathbb{R}} \widehat{f}(\xi + i\eta)\, e^{i\eta t}\, d\eta = \\ = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} \widehat{f}(\xi + i\eta)\, e^{(\xi + i\eta)t}\, d(i\eta) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} p = \xi} \widehat{f}(p)\, e^{pt}\, dp, \quad t \in \mathbb{R}_+. \tag{28}$$

This is the inverse Laplace transform in this case.

We introduce a space $\mathcal{H}(\omega; X)$ which is composed of the analytic functions $h : \Pi_\omega \to X$, that satisfy the condition:
(H): for any $\xi_0 > \omega$ there exist the constants $C = C(\xi_0) \geqslant 0$ and $s = s(\xi_0) \geqslant 0$ such that

$$\|h(p)\|_X \leqslant C(1 + |p|^s), \quad \operatorname{Re} p > \xi_0. \tag{29}$$

Proposition 1 ([25], §10.4; [8], section XVI, §2)**.** *The Laplace transform bijectively maps the space $\mathcal{D}'(\omega, \mathbb{R}_+; X)$ to the space $\mathcal{H}(\omega; X)$. Furthermore, if the function $\widehat{f} \in \mathcal{H}(\omega; X)$ satisfies the condition* (29) *(with the replacement of h by $\widehat{f}$), then there is (by applying the Laplace transform) an image of the function $f \in \mathcal{D}'(\omega, \mathbb{R}_+; X)$ which is defined by the formula*

$$f(t) = \frac{1}{2\pi i} \left(\frac{d}{dt} - b\right)^k \int\limits_{\operatorname{Re} p = \xi} \frac{\widehat{f}(p)}{(p - b)^k} e^{pt}\, dp, \quad t \in \mathbb{R}_+, \tag{30}$$

where $b, \xi \in \mathbb{R},\ k \in \mathbb{N}$ are arbitrary numbers such that $b \leqslant \omega,\ \xi > \xi_0,\ k > s + 1$.

The formula (30) defines the inverse Laplace transform which in the special case has the representation (28).

We note that

$$L^2_\alpha(\mathbb{R}_+; X) \subset \mathcal{D}'(\alpha/2, \mathbb{R}_+; X),$$

because for the arbitrary function $f \in L^2_\alpha(\mathbb{R}_+; X)$ we deduce $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+; X)$ and the condition (24) holds:

$$\int_{\mathbb{R}_+} e^{-\xi t} \|f(t)\|_X dt = \int_{\mathbb{R}_+} e^{\alpha/2t} \|f(t)\|_X e^{(\alpha/2 - \xi)t} dt \leqslant \\ \leqslant \left[\int_{\mathbb{R}_+} e^{-\alpha t} \|f(t)\|^2_X dt\right]^{1/2} \left[\int_{\mathbb{R}_+} e^{(\alpha - 2\xi)t} dt\right]^{1/2} = \\ = \frac{1}{\sqrt{2(\xi - \alpha/2)}} \|f\|_{L^2_\alpha(\mathbb{R}_+; X)} < \infty$$

for any $\xi > \alpha/2$. Herein we used the Cauchy-Schwarz inequality.

We denote by $\mathcal{L}(\Pi_{\alpha/2}; X)$ a linear subspace of the space $\mathcal{H}(\alpha/2; X)$ composed of such functions $p \mapsto h(p) : \overline{\Pi_{\alpha/2}} \to X$ that are analytical on the open half-plane $\Pi_{\alpha/2}$, continuous on its closure $\overline{\Pi_{\alpha/2}}$ and satisfy the condition **(H)** when $s = 0$, and for every value $\xi \geqslant \alpha/2$ the functions $\eta \mapsto h(\xi + i\eta) : \mathbb{R} \to X$ belong to the space $L^2(\mathbb{R}; X)$, i.e. $\int\limits_{\mathbb{R}} \|h(\xi + i\eta)\|^2_X\, d\eta < \infty$.

Corollary 1. *The Laplace transform $\mathfrak{L}[\cdot]$ bijectively maps the space $L^2_\alpha(\mathbb{R}_+; X)$ on the space $\mathcal{L}(\Pi_{\alpha/2}; X)$ and, moreover, the arbitrary function $f \in L^2_\alpha(\mathbb{R}_+; X)$ has as an image the function*

$$\widehat{f}(p) = \mathfrak{L}[f](p) := \int\limits_{\mathbb{R}_+} f(t)\, e^{-p\,t}\, dt, \quad p \in \overline{\Pi_\alpha/2}, \tag{31}$$

and for an arbitrary function $\widehat{f} \in \mathcal{L}(\Pi_{\alpha/2}; X)$ its inverse image $f \in L^2_\alpha(\mathbb{R}_+; X)$ is defined by the formula

$$f(t) = \mathfrak{L}^{-1}[\widehat{f}](t) := \frac{1}{2\pi i} \int\limits_{\mathrm{Re} p = \xi} \widehat{f}(p)\, e^{pt}\, dp, \quad t \in \mathbb{R}_+, \quad \xi \geqslant \alpha/2. \tag{32}$$

Also the Parseval equality holds

$$\begin{aligned} \int\limits_{\mathbb{R}_+} e^{-2\xi t} \|f(t)\|^2_X\, dt = \frac{1}{2\pi i} \int\limits_{\mathrm{Re} p = \xi} \|\widehat{f}(p)\|^2_X\, dp \equiv \\ \equiv \frac{1}{2\pi} \int\limits_{\mathbb{R}} \|\widehat{f}(\xi + i\eta)\|^2_X\, d\eta, \quad \xi \geqslant \alpha/2, \end{aligned} \tag{33}$$

in particular,

$$\begin{aligned} \|f\|_{L^2_\alpha(\mathbb{R}_+;X)} = \left(\frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, p = \alpha/2} \|\widehat{f}(p)\|^2_X\, dp \right)^{1/2} \equiv \\ \equiv \left(\frac{1}{2\pi} \int\limits_{\mathbb{R}} \|\widehat{f}(\alpha/2 + i\eta)\|^2_X\, d\eta \right)^{1/2}. \end{aligned} \tag{34}$$

Proof. Let $f \in L^2_\alpha(\mathbb{R}_+; X)$. Then using the Laplace transform definition we obtain

$$\begin{aligned} \widehat{f}(p) = \mathfrak{L}_{t\to p}[f(t)](p) \overset{p=\xi+i\eta}{=} \mathfrak{F}_{t\to\eta}[e^{-\xi t} f(t)](\xi + i\eta) = \\ = \int_{\mathbb{R}_+} e^{-\xi t} f(t)\, e^{-i\eta t}\, dt = \int_{\mathbb{R}_+} f(t)\, e^{-pt}\, dt, \quad p \in \overline{\Pi_{\alpha/2}}. \end{aligned} \tag{35}$$

Since the function $t \mapsto t^k f(t)\, e^{-pt} : \mathbb{R}_+ \to X$ is absolutely integrable for arbitrary $p \in \Pi_{\alpha/2}$ and for arbitrary point $p_0 \in \overline{\Pi_{\alpha/2}}$ this function has the absolutely integrable function as its upper bound in some neighborhood of p_0, then the function $\widehat{f}$ is analytical in $\Pi_{\alpha/2}$. Furthermore, it can be proven that $\widehat{f}$ is continuous on $\overline{\Pi_{\alpha/2}}$.

Now we show that $\widehat{f}$ satisfies the condition **(H)** with $s = 0$. Indeed, according to (35), for $p = \xi + i\eta \in \Pi_{\alpha/2}$ we obtain

$$\begin{aligned} \|\widehat{f}(p)\|_X &\leqslant \int_{\mathbb{R}_+} \|f(t)\|_X \, e^{-\xi t} \, dt = \int_{\mathbb{R}_+} e^{-\alpha/2t} \|f(t)\|_X \, e^{(\alpha/2-\xi)t} \, dt \leqslant \\ &\leqslant \left[\int_{\mathbb{R}_+} e^{-\alpha t} \|f(t)\|_X^2 \, dt \right]^{1/2} \left[\int_{\mathbb{R}_+} e^{(\alpha-2\xi)t} \, dt \right]^{1/2} = \\ &= \frac{1}{\sqrt{2(\xi-\alpha/2)}} \|f\|_{L^2_\alpha(\mathbb{R}_+;X)}. \end{aligned} \tag{36}$$

Since the function $t \mapsto e^{-\xi t} f(t)$ belongs to $L^2(\mathbb{R}_+; X)$ for arbitrary $\xi \geqslant \alpha/2$ and the Fourier transform maps $L^2(\mathbb{R}; X)$ on $L^2(\mathbb{R}; X)$, then the function $\eta \mapsto \widehat{f}(\xi + i\eta)$ belongs to the space $L^2(\mathbb{R}; X)$ and the Parseval equality holds

$$\int_{\mathbb{R}_+} e^{-2\xi t} \|f(t)\|_X^2 \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \|\widehat{f}(\xi + i\eta)\|_X^2 \, d\eta = \frac{1}{2\pi i} \int\limits_{\mathrm{Re} p = \xi} \|\widehat{f}(p)\|_X^2 \, dp.$$

Therefore, we proved that $\widehat{f}(p)$, $p \in \overline{\Pi_{\alpha/2}}$, belongs to the space $\mathcal{L}(\Pi_{\alpha/2}; X)$.

Now we assume that $\widehat{f} \in \mathcal{L}(\Pi_{\alpha/2}; X)$ and demonstrate that there exists $f \in L^2_\alpha(\mathbb{R}_+; X)$ such that $\widehat{f}$ is the image of f when applying the Laplace transform. Since $\widehat{f} \in \mathcal{H}(\alpha/2; X)$, there exists $f \in \mathcal{D}'(\alpha/2, \mathbb{R}_+; X)$ such that

$$f(t) = \mathfrak{L}^{-1}[\widehat{f}(p)](t) \overset{p=\xi+i\eta}{=} e^{\xi t} \, \mathfrak{F}^{-1}_{\eta \to t}[\widehat{f}(\xi + i\eta)](t), \quad \xi > \alpha/2.$$

Hence, since the function $\eta \mapsto \widehat{f}(\xi + i\eta) : \mathbb{R} \to X$ belongs to the space $L^2(\mathbb{R}; X)$ for every $\xi \geqslant \alpha/2$, we obtain

$$t \mapsto e^{-\xi t} \, f(t) = \mathfrak{F}^{-1}_{\eta \to t}[\widehat{f}(\xi + i\eta)](t) \in L^2(\mathbb{R}_+; X) \quad \text{for arbitrary } \xi \geqslant \alpha/2$$

and, in addition,

$$\begin{aligned} f(t) &= \frac{1}{2\pi} e^{\xi t} \int_{\mathbb{R}} \widehat{f}(\xi + i\eta) \, e^{i\eta t} \, d\eta = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi + i\eta) \, e^{(\xi+i\eta)t} \, d\eta = \\ &= \frac{1}{2\pi i} \int\limits_{\mathrm{Re} p = \xi} \widehat{f}(p) \, e^{pt} \, dp, \quad t \in \mathbb{R}_+. \end{aligned} \tag{37}$$

Thus, we proved that $f = \mathfrak{L}^{-1}[\widehat{f}]$ belongs to $L^2_\alpha(\mathbb{R}_+; X)$ and the formula (37) holds. □

Corollary 2. *If a function f belongs to the space $H^m_\alpha(\mathbb{R}_+; X)$ $(m \in \mathbb{N})$ then the functions $p \mapsto p^k \widehat{f}(p)$ belong to the space $\mathcal{L}(\Pi_{\alpha/2}; X)$ for every $k \in \{0, 1, ..., m\}$. And vice versa, if the functions $p \mapsto p^k \widehat{f}(p)$ belong to the space $\mathcal{L}(\Pi_{\alpha/2}; X)$ for every $k \in \{0, 1, ..., m\}$ then the function $f := \mathfrak{L}^{-1}[\widehat{f}]$ belongs to* *the space* $H^m_\alpha(\mathbb{R}_+;X)$ *and*

$$f^{(k)}(t) = \mathfrak{L}^{-1}[p^k \widehat{f}(p)](t) = \\ = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} p^k \widehat{f}(p)\, e^{pt}\, dp, \quad t \in \mathbb{R}_+, \quad k = \overline{1,m}, \tag{38}$$

where $\xi \geqslant \alpha/2$ *is an arbitrary number and values of the function* f *do not depend on it.*

Furthermore, the following equalities hold

$$\|f\|^2_{H^m_\alpha(\mathbb{R}_+;X)} = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\alpha/2} \Big[\sum_{k=0}^m |p|^{2k}\Big]\, \|\widehat{f}(p)\|^2_X\, dp = \\ = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \Big[\sum_{k=0}^m \big(\alpha^2/4+\eta^2\big)^k\Big] \|\widehat{f}(\alpha/2+i\eta)\|^2_X\, d\eta.$$

Proof. It is sufficient to consider the case of $m=1$.

Let $f \in H^1_\alpha(\mathbb{R}_+;X)$, i.e. $f \in L^2_\alpha(\mathbb{R}_+;X) \cap C(\mathbb{R};X)$, $f' \in L^2_\alpha(\mathbb{R}_+;X)$, $f(0) = 0$. According to corollary 1, we obtain that the functions $p \mapsto \widehat{f}(p)$ and $p \mapsto \widehat{f'}(p)$ belong to the space $\mathcal{L}(\Pi_{\alpha/2};X)$. By the definition of the Laplace transform and by using the formula of integration by parts, we obtain

$$\widehat{f'}(p) = \int\limits_{\mathbb{R}_+} f'(t)\, e^{-pt}\, dt = \begin{bmatrix} u = e^{-pt}; & du = -p\, e^{-pt}\, dt \\ dv = f'(t)\, dt; & v = f(t) \end{bmatrix} = \\ = f(t)\, e^{-pt}\Big|_{t=0}^{t=+\infty} + p \int\limits_{\mathbb{R}_+} f(t)\, e^{-pt}\, dt = p\widehat{f}(p), \quad p \in \overline{\Pi_{\alpha/2}}.$$

Hence, it directly follows that $p \mapsto p\widehat{f}(p) \in \mathcal{L}(\Pi_{\alpha/2};X)$.

Now we assume that $p \mapsto \widehat{f}(p)$ and $p \mapsto p\widehat{f}(p)$ belong to the space $\mathcal{L}(\Pi_{\alpha/2};X)$. According to corollary 1, we obtain that the function

$$f(t) := \begin{cases} \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} \widehat{f}(p)\, e^{pt}\, dp, & \text{if } t > 0, \\ 0, & \text{if } t \leqslant 0, \end{cases} \tag{39}$$

belongs to the space $L^2_\alpha(\mathbb{R}_+;X)$. We are now in the position to show that $f \in C(\mathbb{R};X)$, $f' \in L^2_\alpha(\mathbb{R}_+;X)$ and, in particular, $f(0) = 0$.

First of all we notice that by the Cauchy–Schwarz inequality we obtain

$$\|\widehat{f}(p)\|_X = \frac{1}{|p|} \cdot |p| \|\widehat{f}(p)\|_X \leq \frac{1}{4|p|^2} + |p|^2 \|\widehat{f}(p)\|^2_X, \quad p \in \overline{\Pi_{\alpha/2}}.$$

Hence, since $p \mapsto p\widehat{f}(p) \in \mathcal{L}(\Pi_{\alpha/2};X)$, it follows that

$$\eta \mapsto \|\widehat{f}(\xi+i\eta)\|_X \in L^1(\mathbb{R}) \quad \text{for arbitrary } \xi \geq \alpha/2.$$

It means that the function f defined by the formula (39) belongs to the space $C(\mathbb{R};X)$, and since $f(t) = 0$ at $t < 0$, then $f(0) = 0$.

Let's show that

$$f'(t) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} p\widehat{f}(p)\, e^{pt}\, dp, \quad t \in \mathbb{R}_+. \tag{40}$$

Indeed, since $p \mapsto p\widehat{f}(p) \in \mathcal{L}(\Pi_{\alpha/2}; X)$, the right part in the formula (40) defines the function from the space $L^2_\alpha(\mathbb{R}; X)$. It remains to prove that the indefinite integral of this function which is equal to 0 at $t = 0$ coincides with f.

Let

$$t \mapsto f_1(t) = \begin{cases} \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} p\widehat{f}(p)\, e^{pt}\, dp, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{41}$$

We find

$$\int_0^t f_1(s)\, ds = \frac{1}{2\pi i} \int_0^t \left[\int\limits_{\mathrm{Re}p=\xi} p\widehat{f}(p)\, e^{ps}\, dp \right] ds =$$

$$= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} p\widehat{f}(p) \left[\int_0^t e^{ps}\, ds \right] dp = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} \widehat{f}(p) \left[e^{pt} - 1 \right] dp =$$

$$= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} \widehat{f}(p) e^{pt}\, dp - \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} \widehat{f}(p)\, dp = f(t), \quad t \in \mathbb{R}_+.$$

Here we took into account that

$$f(0) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\xi} \widehat{f}(p)\, dp = 0.$$

Therefore, the proof of the corollary 2 is completed. □

Let $X_j,\ j = 0, 1, 2$, be the Hilbert spaces with the inner products $(\cdot, \cdot)_j,\ j = 0, 1, 2$, and induced norms $\| \cdot \|_j,\ j = 0, 1, 2$, respectively. We assume that

$$X_2 \subset X_1 \subset X_0, \tag{42}$$

and these inclusions are continuous. The examples of such spaces are $X_0 = L^2(\Omega),\ X_1 = H^1(\Omega),\ X_2 = H^1(\Omega, \Delta)$.

It is obvious that

$$L^2_\alpha(\mathbb{R}_+; X_2) \subset L^2_\alpha(\mathbb{R}_+; X_1) \subset L^2_\alpha(\mathbb{R}_+; X_0) \subset \mathcal{D}'(\alpha/2, \mathbb{R}_+; X_0) \tag{43}$$

and

$$\mathcal{L}(\Pi_{\alpha/2}; X_2) \subset \mathcal{L}(\Pi_{\alpha/2}; X_1) \subset \mathcal{L}(\Pi_{\alpha/2}; X_0) \subset \mathcal{H}(\alpha/2; X_0). \tag{44}$$

In addition, for every $j \in \{0, 1, 2\}$ the Laplace transform

$$\mathfrak{L}[\cdot] \,:\, \mathcal{D}'(\alpha/2, \mathbb{R}_+; X_0) \to \mathcal{H}(\alpha/2; X_0)$$

bijectively maps the space $L^2_\alpha(\mathbb{R}_+; X_j)$ on the space $\mathcal{L}(\Pi_{\alpha/2}; X_j)$, and the Parseval equalities hold

$$\|f\|^2_{L^2_\alpha(\mathbb{R}_+; X_j)} = \frac{1}{2\pi} \|\widehat{f}(\alpha/2 + i\cdot)\|^2_{L^2(\mathbb{R}; X_j)}, \quad j = 0, 1, 2. \tag{45}$$

Corollary 3. *Let a function f belong to the space $H^2_\alpha(\mathbb{R}_+;X_0)\cap H^1_\alpha(\mathbb{R}_+;X_1)\cap L^2_\alpha(\mathbb{R}_+;X_2)$. Then for the function $\widehat{f}(p):=\int\limits_{\mathbb{R}_+} f(t)\,e^{-p\,t}\,dt$, $p\in\Pi_{\alpha/2}$, the following inclusions hold*

$$\begin{aligned} &p\mapsto \widehat{f}(p)\in\mathcal{L}(\Pi_{\alpha/2};X_2),\\ &p\mapsto p\widehat{f}(p)\in\mathcal{L}(\Pi_{\alpha/2};X_1),\\ &p\mapsto p^2\widehat{f}(p)\in\mathcal{L}(\Pi_{\alpha/2};X_0). \end{aligned} \tag{46}$$

And vice versa, if for some function $\widehat{f}(p)$, $p\in\Pi_{\alpha/2}$, the inclusions (46) hold, then the function

$$f(t):=\begin{cases} \frac{1}{2\pi i}\int\limits_{\mathrm{Re}p=\,\xi}\widehat{f}(p)\,e^{pt}\,dp, & \text{if } t\in\mathbb{R}_+,\\ 0, & \text{if } t\in\mathbb{R}\setminus\mathbb{R}_+, \end{cases} \tag{47}$$

where $\xi\geqslant\alpha/2$ is an arbitrary number (the value of f does not depends on ξ), belongs to the space

$$H^2_\alpha(\mathbb{R}_+;X_0)\cap H^1_\alpha(\mathbb{R}_+;X_1)\cap L^2_\alpha(\mathbb{R}_+;X_2),$$

and, moreover, $f\in C^1(\mathbb{R};X_0)\cap C(\mathbb{R};X_1)$.

Furthermore,

$$f'(t)=\frac{1}{2\pi i}\int\limits_{\mathrm{Re}p=\,\xi} p\,\widehat{f}(p)\,e^{pt}\,dp,\quad f''(t)=\frac{1}{2\pi i}\int\limits_{\mathrm{Re}p=\,\xi} p^2\,\widehat{f}(p)\,e^{pt}\,dp,\quad t\in\mathbb{R}_+,$$

and

$$\begin{aligned} \|f\|^2_{L^2_\alpha(\mathbb{R}_+;X_2)} =&\frac{1}{2\pi i}\int\limits_{\mathrm{Re}p=\,\alpha/2}\|\widehat{f}(p)\|_2^2\,dp=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\|\widehat{f}(\alpha/2+i\eta)\|_2^2\,d\eta,\\ \|f\|^2_{H^1_\alpha(\mathbb{R}_+;X_1)} =&\frac{1}{2\pi i}\int\limits_{\mathrm{Re}p=\,\alpha/2}\left[1+|p|^2\right]\|\widehat{f}(p)\|_1^2\,dp=\\ =&\frac{1}{2\pi}\int\limits_{\mathbb{R}}\left[1+\alpha^2/4+\eta^2\right]\|\widehat{f}(\alpha/2+i\eta)\|_1^2\,d\eta,\\ \|f\|^2_{H^2_\alpha(\mathbb{R}_+;X_0)} =&\frac{1}{2\pi i}\int\limits_{\mathrm{Re}p=\,\alpha/2}[1+|p|^2+|p|^4]\,\|\widehat{f}(p)\|_0^2\,dp=\\ =&\frac{1}{2\pi}\int\limits_{\mathbb{R}}\left[1+\alpha^2/4+\eta^2+\left(\alpha^2/4+\eta^2\right)^2\right]\|\widehat{f}(\alpha/2+i\eta)\|_0^2\,d\eta. \end{aligned}$$

Proof. This statement easily follows from the statement of the proposition 2. □

Corollary 4. *Let X and Y be Hilbert spaces, and $A:X\to Y$ be a linear continuous operator. Then, if f belongs to the space $L^2_\alpha(\mathbb{R}_+;X)$ then Af belongs to the space $L^2_\alpha(\mathbb{R}_+;Y)$ and*

$$A\mathfrak{L}_X[f](p)=\mathfrak{L}_Y[Af](p),\quad p\in\overline{\Pi_{\alpha/2}}.$$

Furthermore, if $\widehat{f}$ *belongs to the space* $\mathcal{L}(\Pi_{\alpha/2};X)$ *then* $A\widehat{f}$ *belongs to the space* $\mathcal{L}(\Pi_{\alpha/2};Y)$ *and*

$$A\mathfrak{L}_X^{-1}[\widehat{f}](t)=\mathfrak{L}_Y^{-1}[A\widehat{f}](t),\quad t\in\mathbb{R}_+,$$

where $\mathfrak{L}_X : L^2_\alpha(\mathbb{R}_+;X)\to\mathcal{L}(\Pi_{\alpha/2};X)$, $\mathfrak{L}_Y : L^2_\alpha(\mathbb{R}_+;Y)\to\mathcal{L}(\Pi_{\alpha/2};Y)$ *are the Laplace transform of the corresponding spaces.*

Proof. This statement easily follows from the definition of the direct and inverse Laplace transforms and properties of linear continuous operators in Banach spaces. □

5. The proofs of the main results

Proof of the Theorem 1. Let's prove by contradiction. So, we assume that statement of the theorem is incorrect, and let u_1 and u_2 be two arbitrary strong solutions of the given problem. We substitute them alternately in the equation (16) and boundary condition (17) and subtract the corresponding equalities. As a result for $v:=u_1-u_2$ we obtain equalities

$$v''(t)-\Delta v(t)=0\quad\text{in}\quad L^2(\Omega),\quad t\in\mathbb{R}_+,\tag{48}$$

$$\gamma_1 v(t)+b\gamma_0 v'(t)=0\quad\text{in}\quad H^{-1/2}(\Gamma),\quad t\in\mathbb{R}_+.\tag{49}$$

$$v(0)=0,\quad v'(0)=0.\tag{50}$$

Next we multiply (in a scalar way in $L^2(\Omega)$) the equality (48) by $v'(t)$ for almost every $t\in\mathbb{R}_+$:

$$\big(v''(t),v'(t)\big)_0-\big(\Delta v(t),v'(t)\big)_0=0,\quad t\in\mathbb{R}_+.\tag{51}$$

It is easy to see that

$$\big(v''(t),v'(t)\big)_0=\frac{1}{2}\left(\|v'(t)\|_0^2\right)',\quad t\in\mathbb{R}_+.\tag{52}$$

By using Green's formula and equality (49), we obtain

$$\begin{aligned}\big(\Delta v(t),v'(t)\big)_0&=\langle\gamma_1 v(t),\gamma_0 v'(t)\rangle_{1/2}-\big[v(t),v'(t)\big]=\\&=-\langle b\,\gamma_0 v'(t),\gamma_0 v'(t)\rangle_{1/2}-\frac{1}{2}\left(\|\nabla v(t)\|_0^2\right)',\quad t\in\mathbb{R}_+.\end{aligned}\tag{53}$$

Taking into account the inequality

$$\langle b\,\gamma_0 v'(t),\gamma_0 v'(t)\rangle_{1/2}=\big(b\,\gamma_0 v'(t),\gamma_0 v'(t)\big)_{L^2(\Gamma)}\geq 0\quad\text{for } b\geq 0\quad a.e.\ t\in\mathbb{R}_+,$$

and using (52) and (53) we obtain from (51)

$$\left(\|v'(t)\|_0^2\right)'+\left(\|\nabla v(t)\|_0^2\right)'\leq 0\quad a.e.\ t\in\mathbb{R}_+.\tag{54}$$

Then we substitute t by s in inequality (54) and integrate over s from 0 to $t>0$

$$\|v'(t)\|_0^2+\|\nabla v(t)\|_0^2\leq\|v'(0)\|_0^2+\|\nabla v(0)\|_0^2\quad a.e.\ t\in\mathbb{R}_+.\tag{55}$$

Hence, by taking into account (50), we have $v(t)=v_0,\ t\in\mathbb{R}_+$, where $v_0\in L^2(\Omega)$ is some element. From here and from the first condition of (50) we obtain that $v(t)=0,\ t\in\mathbb{R}_+$. This contradiction proves our statement. □

Proof of the Theorem 2. Let's prove the theorem 2 in several stages.

First stage. Let's denote by $\mathfrak{L}_0[\cdot]$ and $\mathfrak{L}_0^{-1}[\cdot]$ a direct and an inverse Laplace transform of the spaces $L^2_\alpha(\mathbb{R}_+; L^2(\Omega))$ and $\mathcal{L}(\Pi_{\alpha/2}; L^2(\Omega))$ respectively, by $\mathfrak{L}_1[\cdot]$ and $\mathfrak{L}_1^{-1}[\cdot]$ a direct and an inverse Laplace transform of the spaces $L^2_\alpha(\mathbb{R}_+; H^1(\Omega))$ and $\mathcal{L}(\Pi_{\alpha/2}; H^1(\Omega))$ respectively, by $\mathfrak{L}_2[\cdot]$ and $\mathfrak{L}_2^{-1}[\cdot]$ a direct and an inverse Laplace transform of the spaces $L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta))$ and $\mathcal{L}(\Pi_{\alpha/2}; H^1(\Omega, \Delta))$ respectively, by $\mathfrak{L}_{1/2}[\cdot]$ and $\mathfrak{L}_{1/2}^{-1}[\cdot]$ a direct and an inverse Laplace transform of the space $L^2_\alpha(\mathbb{R}_+; H^{1/2}(\Gamma))$ and $\mathcal{L}(\Pi_{\alpha/2}; H^{1/2}(\Gamma))$ respectively, by $\mathfrak{L}_{-1/2}[\cdot]$ and $\mathfrak{L}_{-1/2}^{-1}[\cdot]$ a direct and an inverse Laplace transform of the spaces $L^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$ and $\mathcal{L}(\Pi_{\alpha/2}; H^{-1/2}(\Gamma))$ respectively.

Let $u(\cdot, t)$, $t \in \mathbb{R}_+$, be a strong solution of the problem (**HD**), i.e, $u \in H^2_\alpha(\mathbb{R}_+; L^2(\Omega)) \cap H^1_\alpha(\mathbb{R}_+; H^1(\Omega)) \cap L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta))$ and the equalities (16) and (17) hold. Then we apply the Laplace transforms $\mathfrak{L}_0[\cdot]$ and $\mathfrak{L}_{1/2}^{-1}[\cdot]$ to this equalities respectively:

$$\mathfrak{L}_0[u''](p) - \mathfrak{L}_0[\Delta u](p) = 0, \tag{56}$$

$$\mathfrak{L}_{-1/2}[\gamma_1 u](p) + \mathfrak{L}_{-1/2}[b(\gamma_0 u)'](p) = \mathfrak{L}_{-1/2}[g](p), \quad p \in \overline{\Pi_{\alpha/2}}. \tag{57}$$

Notice that the operators $\Delta : H^1(\Omega, \Delta)) \to L^2(\Omega)$, $\gamma_0 : H^1(\Omega)) \to H^{1/2}(\Gamma)$ and $\gamma_1 : H^1(\Omega, \Delta)) \to H^{-1/2}(\Gamma)$ are linear and continuous. So, using the corollaries 1 – 4, we obtain that for every $p \in \overline{\Pi_{\alpha/2}}$ the Laplace transform $\widehat{u}(\cdot, p) := \mathfrak{L}_2[u](\cdot, p)$ as a function of the variable $x \in \Omega$ belongs to the space $H^1(\Omega, \Delta)$ and satisfies equalities

$$-\Delta \widehat{u}(\cdot, p) + p^2 \widehat{u}(\cdot, p) = 0, \tag{58}$$

$$\gamma_1 \widehat{u}(\cdot, p) + p\, b\, \gamma_0 \widehat{u}(\cdot, p) = \widehat{g}(\cdot, p), \quad p \in \overline{\Pi_{\alpha/2}}, \tag{59}$$

where $\widehat{g} := \mathfrak{L}_{-1/2}[g]$ is the Laplace transform of the function g.

Second stage. Let's consider the following problem: for every $p \in \overline{\Pi_{\alpha/2}}$ find a function $w(\cdot, p) \in H^1(\Omega, \Delta)$ which is a strong solution of the problem

$$-\Delta w + p^2 w = 0, \tag{60}$$

$$\gamma_1 w + p\, b\, \gamma_0 w = h(p), \tag{61}$$

where $h : \overline{\Pi_{\alpha/2}} \to H^{-1/2}(\Gamma)$ is a given function.

Lemma 3. *The problem* (60),(61) *has one and only one solution for every* $p \in \overline{\Pi_{\alpha/2}}$. *Moreover, it satisfies the following estimates*

$$\|w\|_0 \leqslant C_4 \, \|h\|_{-1/2}, \tag{62}$$

$$\|w\|_1 \leqslant C_5 \, |p| \, \|h\|_{-1/2}, \tag{63}$$

$$\|w\|_2 \leqslant C_6 \, |p|^2 \, \|h\|_{-1/2}, \tag{64}$$

where C_4, C_5, C_6 *are some constants.*

Furthermore, if the function $h : \Pi_{\alpha/2} \to H^{-1/2}(\Gamma)$ *is analytic then the function* $p \mapsto w(\cdot, p) : \Pi_{\alpha/2} \to H^1(\Omega, \Delta)$ *is also analytic.*

Proof of the lemma 3. As well as it was proved in [20, теорема 2.2] for the case $p \in \mathbb{R}$ the problem (60), (61) can be reduced to the variational identity

$$\widetilde{a}_p(w,v) = \langle h, \gamma_0 v\rangle_{1/2}, \quad v \in H^1(\Omega), \tag{65}$$

where

$$\widetilde{a}_p(w,v) := [w,v] + p^2(w,v)_0 + p\int_\Gamma b(x)\gamma_0 w(x)\overline{\gamma_0 v(x)}\, d\Gamma, \quad w,v \in H^1(\Omega). \tag{66}$$

Hereinafter, the argument p in the function representation $w(p,\cdot)$ is omitted for simplicity. Taking into account the Cauchy–Schwarz inequality and the continuity of the trace operator, for the continuous anti-linear form in the right part of the equation (65) we obtain the estimate

$$|\langle h, \gamma_0 v\rangle_{1/2}| \leqslant \|h\|_{-1/2}\|\gamma_0 v\|_{1/2} \leqslant C_1\|h\|_{-1/2}\|v\|_1, \quad v \in H^1(\Omega). \tag{67}$$

where C_1 is a constant from inequality (11).

The sesquilinear form $\widetilde{a}_p(\cdot,\cdot)$ is continuous on $H^1(\Omega)\times H^1(\Omega)$

$$|\widetilde{a}_p(w,v)| \leqslant C_7\|w\|_1\|v\|_1, \quad w,v \in H^1(\Omega), \tag{68}$$

where C_7 is a constant depended on p. It is known that for continuous sesquilinear form

$$a_p(w,v) := [w,v] + p^2(w,v)_0, \quad (w,v) \in H^1(\Omega)\times H^1(\Omega),$$

the following equality holds

$$\mathrm{Re}(e^{-i\mathrm{Arg}\,p}\, a_p(v,v)) = \frac{\mathrm{Re}\,p}{|p|}\|v\|^2_{|p|,\Omega}, \quad v \in H^1(\Omega), \tag{69}$$

where

$$\|v\|_{|p|,\Omega} := \left(\|\nabla v\|_0^2 + |p|^2\,\|v\|_0^2\right)^{1/2}, \quad v \in H^1(\Omega). \tag{70}$$

It is easy to see (see also [13]) that for arbitrary fixed p, $\mathrm{Re}\,p \geqslant \alpha/2$ the following inequalities hold

$$\kappa\|v\|_1 \leqslant \|v\|_{|p|,\Omega} \leqslant \frac{|p|}{\kappa}\|v\|_1, \quad v \in H^1(\Omega), \tag{71}$$

where $\kappa := \min\{1, \alpha/2\}$.

By taking into account this and equality $p\,e^{-i\mathrm{Arg}\,p} = |p|$, we obtain an estimate

$$\begin{aligned}\mathrm{Re}(e^{-i\mathrm{Arg}\,p}\,\widetilde{a}_p(v,v)) =& \frac{\mathrm{Re}\,p}{|p|}\|v\|^2_{|p|,\Omega} + |p|\int_\Gamma b(x)|\gamma_0 v(x)|^2\, d\Gamma \geqslant \\ \geqslant& \frac{\mathrm{Re}\,p}{|p|}\|v\|^2_{|p|,\Omega}, \quad v \in H^1(\Omega),\end{aligned} \tag{72}$$

i.e., the form $e^{-i\mathrm{Arg}\,p}\,\widetilde{a}_p(\cdot,\cdot)$ is coercive in the space $H^1(\Omega)$.

Therefore, for arbitrary $p \in \overline{\Pi_{\alpha/2}}$ all conditions of the Lax-Milgram theorem are true (see., for example, [7, section VII, §1, Theorem 1]) regarding the variational equality (65). Hence, the equality has a solution $w \in H^1(\Omega)$ and it is unique. Since the function w satisfies equation (60) in terms of distributions,

$$\Delta w = p^2 w \in L^2(\Omega), \tag{73}$$

i.e., $w \in H^1(\Omega, \Delta)$.

Now we obtain estimates of the solution to the problem (60),(61). By taking into account (67) and (71), we have

$$\begin{aligned} \operatorname{Re}(e^{-i\operatorname{Arg} p}\, \widetilde{a}_p(w,w)) &\leqslant |\widetilde{a}_p(w,w)| = |\langle h, \gamma_0 w\rangle_{1/2}| \leqslant \\ &\leqslant C_1\|h\|_{-1/2}\|w\|_1 \leqslant \frac{C_1}{\kappa}\|h\|_{-1/2}\|w\|_{|p|,\Omega}. \end{aligned} \tag{74}$$

From this and (72), we arrive at the inequalities

$$\|w\|_{|p|,\Omega} \leqslant \frac{C_1}{\kappa \operatorname{Re} p}|p|\|h\|_{-1/2} \leqslant \frac{2C_1}{\kappa\alpha}|p|\|h\|_{-1/2} = C_8|p|\|h\|_{-1/2},$$

where $C_8 := \frac{2C_1}{\kappa\alpha}$. Hence, considering (70), we have

$$\|\nabla w\|_0^2 + |p|^2\|w\|_0^2 \leqslant C_8^2|p|^2\|h\|_{-1/2}^2. \tag{75}$$

Then the following estimates follow from obtained inequality:

$$\|w\|_0^2 \leqslant C_8^2\,\|h\|_{-1/2}^2, \tag{76}$$

$$\|\nabla w\|_0^2 \leqslant C_8^2\,|p|^2\|h\|_{-1/2}^2. \tag{77}$$

Inequality (62) directly follows from (76) and from (77) we have:

$$\begin{aligned} \|w\|_1^2 = \|w\|_0^2 + \|\nabla w\|_0^2 &\leqslant C_8^2\|h\|_{-1/2}^2 + C_8^2|p|^2\|h\|_{-1/2}^2 = \\ &= C_8^2\left(\frac{1}{|p|^2}+1\right)|p|^2\|h\|_{-1/2}^2 \leqslant C_8^2\left(\frac{1}{\operatorname{Re}^2 p}+1\right)|p|^2\|h\|_{-1/2}^2 \leqslant \\ &\leqslant C_8^2\left(\frac{2}{\alpha}+1\right)^2|p|^2\|h\|_{-1/2}^2, \end{aligned} \tag{78}$$

hence, we obtain (63) when $C_5 := C_8\left(\frac{2}{\alpha}+1\right)$.

Taking into account equality (73) we have

$$\|\Delta w\|_0^2 = |p|^4\|w\|_0^2 \leqslant C_8^2|p|^4\|h\|_{-1/2}^2. \tag{79}$$

Therefore, we get

$$\begin{aligned} \|w\|_2^2 = \|w\|_1^2 + \|\Delta w\|_0^2 &\leqslant C_5^2|p|^2\|h\|_{-1/2}^2 + C_8^2|p|^4\|h\|_{-1/2}^2 = \\ &= \left(\frac{C_5^2}{|p|^2}+C_8^2\right)|p|^4\|h\|_{-1/2}^2 \leqslant \left(\frac{2C_5}{\alpha}+C_8\right)^2|p|^4\|h\|_{-1/2}^2, \end{aligned} \tag{80}$$

whence after denoting $C_6 := C_8 + 2C_5\alpha^{-1}$ we arrive at equality (64). □

Third stage. Let's prove the existence of a strong solution to the problem (**HD**) and find its image.

Let function $w(x,p)$, $x \in \Omega$, be a strong solution to the problem (60),(61) with $h = \widehat{g}(p)$ for every $p \in \overline{\Pi_{\alpha/2}}$. Now we show that function $p \mapsto w(\cdot,p)$ satisfies conditions of the corollary 3 with $X_0 = L^2(\Omega)$, $X_1 = H^1(\Omega)$, $X_2 = H^1(\Omega,\Delta)$. At first we set an equality

$$\widehat{g}(p) = p^{-k}\widehat{g^{(k)}}(p), \quad k \in \{1,2\}, \quad p \in \overline{\Pi_{\alpha/2}}. \tag{81}$$

In fact, for arbitrary $p \in \overline{\Pi_{\alpha/2}}$ after integration by parts we get

$$\widehat{g}(p) = \int\limits_{\mathbb{R}_+} g(t)\, e^{-p\,t}\, dt = \left[\begin{array}{ll} u = g(t); & du = g'(t)\, dt \\ dv = e^{-p\,t}\, dt; & v = -\frac{1}{p} e^{-p\,t} \end{array} \right] =$$

$$= -\frac{1}{p} g(t)\, e^{-p\,t} \Big|_{t=0}^{t=+\infty} + \frac{1}{p} \int\limits_{\mathbb{R}_+} g'(t)\, e^{-p\,t}\, dt = \frac{1}{p^2} \int\limits_{\mathbb{R}_+} g''(t)\, e^{-p\,t}\, dt = \frac{1}{p^2} \widehat{g''}(p),$$

whence we obtain (81).

From (81) an equality follows

$$\|\widehat{g}(p)\|_{-1/2} = |p|^{-k} \|\widehat{g^{(k)}}(p)\|_{-1/2}, \quad k \in \{1, 2\}, \quad p \in \overline{\Pi_{\alpha/2}}. \tag{82}$$

Based on the lemma (see estimates (62) – (64)) and the equality (82), we obtain

$$|p|^2 \|w(p)\|_0 \leqslant C_4 |p|^2 \|\widehat{g}(p)\|_{-1/2} = C_4 \|\widehat{g''}(p)\|_{-1/2}, \tag{83}$$

$$|p| \|w(p)\|_1 \leqslant C_5 |p|^2\, \|\widehat{g}(p)\|_{-1/2} = C_5 \|\widehat{g''}(p)\|_{-1/2}, \tag{84}$$

$$\|w(p)\|_2 \leqslant C_6 |p|^2\, \|\widehat{g}(p)\|_{-1/2} = C_6 \|\widehat{g''}(p)\|_{-1/2}, \quad p \in \overline{\Pi_{\alpha/2}}. \tag{85}$$

Since function $p \mapsto w(p)$ is an analytic in $\Pi_{\alpha/2}$ and continuous on $\overline{\Pi_{\alpha/2}}$, and g'' belongs to the space $L^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))$, based on the estimates (83) – (85), all conditions of the corollary 3 regarding function $p \mapsto w(p)$ hold, i.e.

$$p \mapsto w(p) \in \mathcal{L}(\Pi_{\alpha/2}; H^1(\Omega, \Delta)), \quad p \mapsto p\, w(p) \in \mathcal{L}(\Pi_{\alpha/2}; H^1(\Omega)),$$

$$p \mapsto p^2 w(p) \in \mathcal{L}(\Pi_{\alpha/2}; L^2(\Omega)).$$

Therefore, function

$$u(x,t) = \begin{cases} \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p = \,\xi} w(x,p)\, e^{pt}\, dp, & \text{for } x \in \overline{\Omega},\ t \in \mathbb{R}_+, \\ 0, & \text{for } x \in \overline{\Omega},\ t \in \mathbb{R} \setminus \mathbb{R}_+, \end{cases} \tag{86}$$

where $\xi \geqslant \alpha/2$ is arbitrary number and value of u does not depend on ξ, belongs to the space

$$H^2_\alpha(\mathbb{R}_+; L^2(\Omega)) \cap H^1_\alpha(\mathbb{R}_+; H^1(\Omega)) \cap L^2_\alpha(\mathbb{R}_+; H^1(\Omega, \Delta)),$$

and, moreover,

$$\begin{aligned} u_t(x,t) &= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p = \,\xi} p\, w(x,p)\, e^{pt}\, dp, \ u_{tt}(x,t) = \\ &= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p = \,\xi} p^2 w(x,p)\, e^{pt}\, dp, \ x \in \Omega, \ t \in \mathbb{R}_+, \end{aligned} \tag{87}$$

and

$$\begin{aligned} \|u\|^2_{L^2_\alpha(\mathbb{R}_+; H^1(\Omega,\Delta))} &= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p = \,\alpha/2} \|w(p)\|_2^2\, dp \leqslant \\ \leqslant \frac{C_6^2}{2\pi i} \int\limits_{\mathrm{Re}p = \,\alpha/2} &\|\widehat{g''}(p)\|^2_{-1/2}\, dp = C_6^2 \|g''\|^2_{L^2_\alpha(\mathbb{R}_+; H^{-1/2}(\Gamma))}, \end{aligned} \tag{88}$$

$$\begin{aligned} \|u\|^2_{H^1_\alpha(\mathbb{R}_+;H^1(\Omega))} &= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\alpha/2} (1+|p|^2)\|w(p)\|_1^2\, dp \leqslant \\ &\leqslant \frac{C_5^2}{2\pi i} \int\limits_{\mathrm{Re}p=\alpha/2} \left(|p|^2+|p|^4\right) \|\widehat{g}(p)\|_1^2\, dp \leqslant \\ &\leqslant \frac{C_5^2}{2\pi i} \int\limits_{\mathrm{Re}p=\alpha/2} \left(\|\widehat{g'}(p)\|^2_{-1/2} + \|\widehat{g''}(p)\|^2_{-1/2}\right) dp = \\ &= C_5^2 \left(\|g'\|^2_{L^2_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))} + \|g''\|^2_{L^2_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))}\right), \end{aligned} \tag{89}$$

$$\begin{aligned} \|u\|^2_{H^2_\alpha(\mathbb{R}_+;L^2(\Omega))} &= \frac{1}{2\pi i} \int\limits_{\mathrm{Re}p=\alpha/2} \left(1+|p|^2+|p|^4\right) \|w(p)\|_0^2\, dp \leqslant \\ \leqslant \frac{C_4^2}{2\pi i} \int_{\mathrm{Re}p=\alpha/2} &\left(1+|p|^2+|p|^4\right) \|\widehat{g}(p)\|^2_{-1/2}\, dp = C_4^2 \|g\|^2_{H^2_\alpha(\mathbb{R}_+;H^{-1/2}(\Gamma))}. \end{aligned} \tag{90}$$

Now we demonstrate that the function u is a strong solution of the problem (**HD**). Since the operators $\Delta : H^1(\Omega,\Delta)) \to L^2(\Omega)$, $\gamma_0 : H^1(\Omega)) \to H^{1/2}(\Gamma)$ and $\gamma_1 : H^1(\Omega,\Delta)) \to H^{-1/2}(\Gamma)$ are linear and continuous, from the aforementioned properties and the corollaries 1 – 4 we obtain

$$\begin{aligned} \mathfrak{L}_0^{-1}[\Delta w(p)](t) &= \Delta \mathfrak{L}_2^{-1}[w(p)](t) = \Delta u(t), \quad t \in \mathbb{R}_+, \\ \mathfrak{L}_0^{-1}[p^2 w(p)](t) &= u''(t), \quad t \in \mathbb{R}_+, \\ \mathfrak{L}_{-1/2}^{-1}[\gamma_1 w(p)](t) &= \gamma_1 \mathfrak{L}_2^{-1}[w(p)](t) = \gamma_1 u(t), \quad t \in \mathbb{R}_+, \\ \mathfrak{L}_{1/2}^{-1}[b\, p\, \gamma_0 w(p)](t) &= b\, \mathfrak{L}_{1/2}^{-1}[p\, \gamma_0 w(p)](t) = b\, \gamma_0 \mathfrak{L}_1^{-1}[p\, w(p)](t) = \\ &= b\, \gamma_0 u'(t), \quad t \in \mathbb{R}_+. \end{aligned}$$

Next we apply mapping $\mathfrak{L}_0^{-1}[\cdot]$ to the equality

$$-\Delta w + p^2 w = 0, \quad p \in \overline{\Pi_{\alpha/2}}, \tag{91}$$

and mapping $\mathfrak{L}_{-1/2}^{-1}[\cdot]$ to the quality

$$\gamma_1 w(p) + b\, p\, \gamma_0 w(p) = \widehat{g}(p), \quad p \in \overline{\Pi_{\alpha/2}}. \tag{92}$$

As a result, by taking into account the aforementioned, we obtain what is needed.

□

6. Conclusion

The results on the existence and the uniqueness of the solution to the problem (**HD**) with estimates in corresponding functional spaces obtained in this research are the basis for the development of methods for finding numerical solution of such problem. Since the Laguerre transform is also applied in mentioned spaces, we can use it to reduce the problem (**HD**) to an infinite sequence of boundary integral equations. One of the advantages of this approach is that each of these equations has the same integral operator in their left parts, and their right parts are recursively dependent [10]. Note that the numerical results obtained in [10] demonstrate the efficiency of the combined approach using the Laguerre transform and the boundary element method for modeling evolutionary processes described by the problem (**HD**).

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