Журнал обчислювальної та прикладної математики | 2021 № 3 (137) | Journal of Numerical & Applied Mathematics

UDC 519.6

THE WEIGHTED ERROR ESTIMATE OF THE FINITE-DIFFERENCE SCHEME FOR A SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION WITH A MIXED DERIVATIVE

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Резюме. У статті побудовано і досліджено скінченно-різницеву схему для розв'язування першої крайової задачі для еліптичного рівняння 2-го порядку з мішаною похідною в прямокутнику. За допомогою різницевої функції Гріна та інтегрального зображення похибки апроксимації одержано вагову оцінку в рівномірній сітковій метриці для швидкості збіжності схеми на узагальнених розв'язках. З вагової оцінки випливає, що точність схеми вища відповідно на пів порядку та порядок (щодо кроку) поблизу сторін і вершин прямокутника порівняно з 2-м порядком у внутрішніх вузлах сітки.

Abstract. We construct and investigate the finite-difference scheme for a second-order elliptic differential equation with a mixed derivative in a rectangle under the Dirichlet boundary condition. With the help of discrete Green's function and the integral representation of the approximation error, we obtain the weighted estimate for the convergence rate of the scheme in the uniform discrete norm and on generalized solutions. The estimate indicates that the accuracy order of the scheme is higher near the sides of the rectangle than in the inner nodes of the grid set.

1. Introduction

The convergence rate of any discrete method for solving boundary value problems is traditionally characterized by a priori estimates with an appropriate discretization parameter (or parameters). For example, in the case of the finite-difference method such parameter is a grid step h. However, error estimates do not usually reflect the influence of some other important factors. (One such factor is Dirichlet's boundary condition. Indeed, since an approximate solution satisfies it exactly, it is natural for the accuracy of the scheme to be higher near the boundary of the domain than inside of it.) Moreover, taking this impact into consideration is not only of theoretical but also of practical significance since near the boundary of the domain it allows to choose a coarser grid step.

The influence of the boundary condition was first named a *boundary effect* in [1] and is generally characterized by some weighted estimates containing, in addition to discretization parameters, the distance of a point to the boundary of the domain. A systematic study of the boundary effect dates back to the pioneering papers [1, 2]. Currently, there are a number of publications (although

Key words. Elliptic equation; mixed derivative; boundary value problem; Dirichlet's boundary condition; finite-difference scheme; weighted error estimate; boundary effect.
MSC 2020: 65N06, 65N12, 65N15, 65N80.

not very many) devoted to the weighted error estimates of the finite-difference schemes. For example, the boundary effect for the elliptic equations is considered in [2–5] and the initial and boundary effects for the evolution equations are investigated in [6–10]. To prove the weighted estimates in the above-mentioned papers, two different approaches are used: the first one is based on the comparison theorem from [11] (see, e.g, [4, 5, 10]) and the second one makes use of discrete Green's function and the main lemma from [12] (see, e.g., [6,7,9,13,14]).

The present paper is ideologically close to papers [13, 14]. Its main aim is to obtain the weighted error estimates of the finite-difference scheme for the inhomogeneous second-order elliptic equation with a mixed derivative and constant coefficients under Dilichlet's boundary condition in a rectangle.

We consider the problem

$$\begin{aligned} Lu \equiv L_1 u + L_2 u + 2L_{12}u = -f(x), \quad x \in D,\\ u(x) = 0, \quad x \in \Gamma, \end{aligned} \tag{1}$$

where $x = (x_1, x_2)$, $D = \{(x_1, x_2) : 0 < x_\alpha < l_\alpha,\, \alpha = 1, 2\}$ is a rectangle with the boundary $\Gamma = \partial D$, $L_\alpha u = k_{\alpha\alpha}\dfrac{\partial^2 u(x)}{\partial x_\alpha^2}$, $\alpha = 1, 2$, $L_{12}u = k_{12}\dfrac{\partial^2 u(x)}{\partial x_1 \partial x_2}$, and the coefficients $k_{\alpha\beta}$ satisfy the following ellipticity condition:

$$k_{11}\xi_1^2 + k_{22}\xi_2^2 + 2k_{12}\xi_1\xi_2 \geqslant \gamma \sum_{\alpha=1}^{2} \xi_\alpha^2 \quad \forall\, \xi_1, \xi_2 \in \mathbb{R} \quad (\gamma = \mathrm{const} > 0). \tag{2}$$

For convenience, we remind the reader about some traditional notation for grid sets and difference derivatives [11]:

$$\begin{gathered} \omega_\alpha = \bigl\{x_\alpha = i_\alpha h_\alpha,\, i_\alpha = 1, \ldots, N_\alpha - 1,\, h_\alpha = l_\alpha / N_\alpha \quad (2 \leqslant N_\alpha \in \mathbb{N})\bigr\},\\ \bar\omega_\alpha = \omega_\alpha \cup \{0\} \cup \{l_\alpha\}, \quad \omega_\alpha^- = \omega_\alpha \cup \{0\}, \quad \omega_\alpha^+ = \omega_\alpha \cup \{l_\alpha\},\\ \omega = \omega_1 \times \omega_2 \text{ is a set of the inner nodes}, \quad \bar\omega = \bar\omega_1 \times \bar\omega_2, \quad \gamma = \bar\omega \backslash \omega;\\ \gamma_{-\alpha} = \bigl\{x \in \gamma : x_\alpha = 0,\, x_{3-\alpha} \in \omega_{3-\alpha}\bigr\},\\ \gamma_{+\alpha} = \bigl\{x \in \gamma : x_\alpha = 1,\, x_{3-\alpha} \in \omega_{3-\alpha}\bigr\},\\ \gamma_\alpha = \gamma_{-\alpha} \cup \gamma_{+\alpha}, \quad \alpha = 1, 2; \end{gathered}$$

$$\begin{aligned} u_{x_1}(x) &= \frac{u(x_1 + h_1, x_2) - u(x)}{h_1}, \quad x \in \omega_1^- \times \bar\omega_2,\\ u_{\bar x_1}(x) &= \frac{u(x) - u(x_1 - h_1, x_2)}{h_1}, \quad x \in \omega_1^+ \times \bar\omega_2,\\ u_{\bar x_1 x_1}(x) &= \frac{u(x_1 + h_1, x_2) - 2u(x) + u(x_1 - h_1, x_2)}{h_1^2}, \quad x \in \omega_1 \times \bar\omega_2; \end{aligned}$$

the difference derivatives u_{x_2}, $u_{\bar x_2}$, $u_{\bar x_2 x_2}$ are defined in a similar way.

Now we introduce the Steklov averaging operators S_α^-, S_α^+ and the operators of the exact finite-difference schemes T_α which are developed in [12]. For example, with regard to the variable x_1, they are defined as follows:

$$S_1^+u(x) = \frac{1}{h_1}\int\limits_{x_1}^{x_1+h_1} u(\xi_1, x_2)\, d\xi_1, \quad S_1^-u(x) = \frac{1}{h_1}\int\limits_{x_1-h_1}^{x_1} u(\xi_1, x_2)\, d\xi_1,$$

$$T_1u(x) = \frac{1}{h_1^2}\int\limits_{x_1-h_1}^{x_1+h_1} \big(h_1 - |x_1 - \xi_1|\big)u(\xi_1, x_2)\, d\xi_1.$$

The operators S_2^-, S_2^+, T_2 are introduced similarly. Next we note some of their useful properties and their connection with the difference derivatives:

$$T_\alpha = S_\alpha^+ S_\alpha^- = S_\alpha^- S_\alpha^+, \quad T = T_1T_2 = T_2T_1,$$

$$S_\alpha^+\frac{\partial u}{\partial x_\alpha}(x) = u_{x_\alpha}(x), \quad S_\alpha^-\frac{\partial u}{\partial x_\alpha}(x) = u_{\bar{x}_\alpha}(x),$$

$$T_\alpha\frac{\partial^2 u}{\partial x_\alpha^2}(x) = u_{\bar{x}_\alpha x_\alpha}(x), \quad \alpha = 1, 2.$$

2. The finite-difference scheme and the properties of the discrete operators

Applying the operator $T = T_1T_2$ to the differential equation (1), we get the relation (the so called generalized balanced equation)

$$k_{11}(T_2u)_{\bar{x}_1x_1} + k_{22}(T_1u)_{\bar{x}_2x_2} + 2k_{12}\frac{(S_1^+S_2^-u)_{\bar{x}_1x_2} + (S_1^-S_2^+u)_{x_1\bar{x}_2}}{2} =$$
$$= -Tf(x), \quad x \in \omega.$$

Then we approximate problem (1) with the following finite-difference scheme:

$$\begin{gathered} \Lambda y \equiv \Lambda_1 y + \Lambda_2 y + 2\Lambda_{12} y = -Tf(x), \quad x \in \omega, \\ y(x) = 0, \quad x \in \gamma, \end{gathered} \tag{3}$$

with $\Lambda_\alpha y = k_{\alpha\alpha} y_{\bar{x}_\alpha x_\alpha}$, $\alpha = 1, 2$, $\Lambda_{12}y = 0{,}5k_{12}(y_{\bar{x}_1x_2} + y_{x_1\bar{x}_2})$.

Note (see, e.g., [11]) that the difference expression $\Lambda_{12}u$ approximates the differential expression $L_{12}u$ on the seven-point template

$$(x_1, x_2), \quad (x_1 \pm h_1, x_2), \quad (x_1, x_2 \pm h_2), \quad (x_1 + h_1, x_2 - h_2), \quad (x_1 - h_1, x_2 + h_2)$$

with the second order in $h = (h_1, h_2)$ on the smooth solutions, namely:

$$\Lambda_{12}u = L_{12}u + O(|h|^2), \quad |h|^2 = h_1^2 + h_2^2.$$

Next we introduce the space $\overset{0}{H}$ of functions defined on $\bar{\omega}$ and vanishing on γ with the inner product and the induced norm:

$$(y, v) = \sum_{x\in\omega} h_1h_2y(x)v(x), \quad \|v\| = \|v\|_{L_2(\omega)} = \sqrt{(v, v)} = \left\{\sum_{x\in\omega} h_1h_2v^2(x)\right\}^{1/2}.$$

We also employ some standard notation from [12]:

$$(y,v]_{1,2} = \sum_{x\in\omega_1^+\times\omega_2^+} h_1 h_2 y(x)v(x),$$

$$(y,v]_\alpha = \sum_{x\in\omega\cup\gamma_{+\alpha}} h_1 h_2 y(x)v(x)\,,\quad \|v]|_\alpha = \sqrt{(v,v]_\alpha}\,,\quad \alpha = 1,2,$$

$$|v|^2_{1,\omega} = |v|^2_{W_2^1(\omega)} = \sum_{\alpha=1}^{2} \|v_{\bar{x}_\alpha}]|^2_\alpha\,,\quad \|v\|^2_{1,\omega} = \|v\|^2_{W_2^1(\omega)} = |v|^2_{1,\omega} + \|v\|^2.$$

Note that the grid function $\varphi(x) = Tf(x)$ is defined in the inner nodes $x\in\omega$. Putting $\varphi(x) = 0$ for $x\in\gamma$, we get $\varphi\in\overset{0}{H}$. Then the discrete problem [3] can be rewritten in the form of the following operator equation:

$$Ay \equiv A_1 y + A_2 y + 2A_{12} y = \varphi,\quad y\in\overset{0}{H},\ \varphi\in\overset{0}{H}, \tag{4}$$

with A_1, A_2, A_{12}, $A:\overset{0}{H}\to\overset{0}{H}$, $A_\alpha y = -\Lambda_\alpha y$, $\alpha = 1,2$, $A_{12}y = -\Lambda_{12}y$.

Next we investigate the properties of the operator A.

Lemma 1. *The operator A is self-adjoint and positive definite in $\overset{0}{H}$.*

Proof. The self-adjointness of A follows from the relation

$$(Ay,v) = (A_1 y + A_2 y + 2A_{12}y, v) =$$

$$= -\big(k_{11}y_{\bar{x}_1x_1} + k_{22}y_{\bar{x}_2x_2} + k_{12}(y_{\bar{x}_1x_2} + y_{x_1\bar{x}_2}),\, v\big) =$$

$$= \sum_{\alpha=1}^{2} k_{\alpha\alpha}(y_{\bar{x}_\alpha}, v_{\bar{x}_\alpha}]_\alpha + k_{12}\big[(y_{\bar{x}_1}, v_{\bar{x}_2}) + (y_{\bar{x}_2}, v_{\bar{x}_1})\big] = (y, Av)\quad \forall\, y,v\in\overset{0}{H},$$

where the summation by parts formula is applied, for example:

$$\sum_{x_1\in\omega_1} h_1 y_{x_1}(x)v(x) = -\sum_{x_1\in\omega_1^+} h_1 y(x)v_{\bar{x}_1}(x) +$$

$$+\, y(l_1,x_2)v(l_1,x_2) - y(h_1,x_2)v(0,x_2),\quad x_2\in\omega_2,$$

where the relations $y_{\bar{x}_1} = 0$ for $x_2 = l_2$ and $u_{\bar{x}_2}$ for $x_1 = l_1$ are taken into consideration.

To prove the positive definiteness of A, we make use of the ellipticity condition (2) and the following inequality from [11]:

$$|v|^2_{1,\omega} \equiv \|v_{\bar{x}_1}]|^2_1 + \|v_{\bar{x}_2}]|^2_1 \geqslant \big(8/l_1^2 + 8/l_2^2\big)\|v\|^2\quad \forall\, v\in\overset{0}{H}. \tag{5}$$

We have

$$(Av,v) = k_{11}(v_{\bar{x}_1}, v_{\bar{x}_1}]_1 + k_{22}(v_{\bar{x}_2}, v_{\bar{x}_2}]_2 + 2k_{12}(v_{\bar{x}_1}, v_{\bar{x}_2}) =$$

$$= (k_{11}v^2_{\bar{x}_1} + k_{22}v^2_{\bar{x}_2} + 2k_{12}v_{\bar{x}_1}v_{\bar{x}_2},\, 1]_{1,2} \geqslant \gamma\big(\|v_{\bar{x}_1}]|^2_1 + \|v_{\bar{x}_2}]|^2_2\big),$$

which yields the useful inequality

$$(Av, v) \geqslant \gamma\, |v|^2_{1,\omega} \quad \forall\, v \in \overset{0}{H} \tag{6}$$

and the positive definiteness of A: $(Av, v) \geqslant \gamma(8/l_1^2 + 8/l_2^2)\|v\|^2 \;\; \forall\, v \in \overset{0}{H}$. □

Now we can move on to the following statement.

Theorem 1. *The discrete problem (4) is uniquely solvable for any right-hand side $\varphi(x)$, and for the discrete solution $y(x)$ the following estimate holds true:*

$$|y|_{1,\omega} \leqslant \frac{l_1 l_2}{2\sqrt{2}\,\gamma\sqrt{l_1^2 + l_2^2}}\, \|\varphi\|. \tag{7}$$

Proof. Lemma 1 implies the existence of the inverse operator $A^{-1} : \overset{0}{H} \to \overset{0}{H}$ and therefore the existence of the the unique solution $y \in \overset{0}{H}$ for each $\varphi \in \overset{0}{H}$. To obtain estimate (7), we multiply by y both sides of equation (4) scalarly in $\overset{0}{H}$ and then apply estimate (6) and the Cauchy–Bunyakovsky inequality to the left-hand side and the right-hand side respectively:

$$\gamma\, |y|^2_{1,\omega} \leqslant (Ay, y) = (\varphi, y) \leqslant \|\varphi\|\, \|y\| \leqslant \|\varphi\|(8/l_1^2 + 8/l_2^2)^{-1/2} |y|_{1,\omega},$$

which gives estimate (7). The theorem is proven. □

Next we introduce the following operators:

1) $B_1 : H_1 \to \overset{0}{H}$, $B_1 y = -y_{x_1 x_2}$, where H_1 is a space of the grid functions defined on the set $\widetilde{\omega} = \omega_1^+ \times \omega_2^+$;

2) $B_2 : H_2 \to \overset{0}{H}$, $B_2 y = -y_{\bar{x}_1 \bar{x}_2}$, where H_2 is a space of the grid functions defined on the set $\widetilde{\omega} = \omega_1^- \times \omega_2^-$;

3) $B_3 : H_3 \to \overset{0}{H}$, $B_3 y = -y_{x_1 \bar{x}_2}$, where H_3 is a space of the grid functions defined on the set $\widetilde{\omega} = \omega_1^+ \times \omega_2^-$;

4) $B_4 : H_4 \to \overset{0}{H}$, $B_4 y = -y_{\bar{x}_1 x_2}$, where H_4 is a space of the grid functions defined on the set $\widetilde{\omega} = \omega_1^- \times \omega_2^+$.

The inner product and the corresponding norm in the space H_k, $k = \overline{1,4}$, are defined as follows:

$$(y, v)_k = \sum_{x \in \widetilde{\omega}} h_1 h_2 y(x) v(x), \quad \|y\|_k = \sqrt{(y, y)_k} = \left\{ \sum_{x \in \widetilde{\omega}} h_1 h_2 y^2(x) \right\}^{1/2}.$$

Now we prove the inequalities that we need further.

Lemma 2. *The following estimates hold true:*

$$\|A^{-1} B_k y\| \leqslant \frac{1}{2(\sqrt{k_{11} k_{22}} - |k_{12}|)} \|y\|_k \quad \forall\, y \in H_k \quad (k = \overline{1,4}) \tag{8}$$

with $\sqrt{k_{11} k_{22}} - |k_{12}| > 0$ due to the ellipticity condition (2).

Proof. We denote by $B_k : H_k \to \overset{0}{H}$ the operator that is conjugate to the operator $B_k^* : \overset{0}{H} \to H_k$, namely:

$$(B_k y, v) = (y, B_k^* v)_k \quad \forall\, y \in H_k \ \forall\, v \in \overset{0}{H}.$$

From here we get $B_1^* y = -y_{\bar{x}_1\bar{x}_2}$, $B_2^* y = -y_{x_1x_2}$, $B_3^* y = -y_{\bar{x}_1x_2}$, $B_4^* y = -y_{x_1\bar{x}_2}$. Next we prove the inequality

$$\|Ay\| \geqslant 2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)\|B_k^* y\|_k \quad \forall y \in \overset{0}{H} \quad (k = \overline{1,4}).$$

For example, in the case $k = 1$ we have

$$\|Ay\| \geqslant \|A_1y + A_2y + 2A_{12}y\| \geqslant \|A_1y + A_2y\| - \|2A_{12}y\|,$$

where

$$\|A_1y + A_2y\|^2 = \|A_1y\|^2 + \|A_2y\|^2 + 2(A_1y, A_2y) \geqslant 4(A_1y, A_2y) =$$

$$= 4k_{11}k_{22}\sum_{x\in\omega} h_1h_2 y_{\bar{x}_1x_1} y_{\bar{x}_2x_2} = 4k_{11}k_{22}\sum_{x\in\omega_1^+\times\omega_2^+} h_1h_2 y^2_{\bar{x}_1\bar{x}_2} = 4k_{11}k_{22}\|B_1^*y\|_1^2,$$

$$\|A_{12}y\|^2 = (2A_{12}y, 2A_{12}y) = k_{12}^2\sum_{x\in\omega} h_1h_2(y^2_{\bar{x}_1x_2} + y^2_{x_1\bar{x}_2} + 2y_{\bar{x}_1x_2}y_{x_1\bar{x}_2}) \leqslant$$

$$\leqslant 2k_{12}^2\sum_{x\in\omega} h_1h_2(y^2_{\bar{x}_1x_2} + y^2_{x_1\bar{x}_2}) \leqslant 4k_{12}^2\sum_{x\in\omega_1^+\times\omega_2^+} h_1h_2 y^2_{\bar{x}_1\bar{x}_2} = 4k_{12}^2\|B_1^*y\|_1^2,$$

which consequently leads to the inequality

$$\|Ay\| \geqslant 2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)\, \|B_1^*y\|_1 \quad \forall\, y \in \overset{0}{H}.$$

Bearing this in mind and applying the main lemma from [12] (see p. 54) to the operators $A : \overset{0}{H} \to \overset{0}{H}$, $B_k : H_k \to \overset{0}{H}$, $B_k^* : \overset{0}{H} \to H_k$ $(k = \overline{1,4})$, we arrive at inequality (8). □

3. The estimate of discrete Green's function

We denote by $G(x,\xi)$ Green's function of the discrete boundary problem

$$\Lambda_\xi G(x,\xi) \equiv \Lambda_{1\xi}G(x,\xi) + \Lambda_{2\xi}G(x,\xi) + 2\Lambda_{12\xi}G(x,\xi) =$$
$$= -\frac{\delta(x_1,\xi_1)\delta(x_2,\xi_2)}{h_1h_2}, \ \xi \in \omega, \quad G(x,\xi) = 0, \quad \xi \in \gamma, \tag{9}$$

where $\xi = (\xi_1, \xi_2)$ and $\delta(r,s)$ is the Kronecker symbol.

Note that in (9) and further throughout the paper the subscript ξ means a finite difference in the variable $\xi = (\xi_1,\xi_2)$, for example: $\Lambda_{1\xi}G(x,\xi) = k_{11}G_{\bar{\xi}_1\xi_1}(x,\xi)$.

Now we prove the auxiliary proposition.

Lemma 3. *For Green's function $G(x,\xi)$, the following estimate holds true:*

$$\| G(x,\cdot)\| \leqslant \frac{1}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}\rho(x), \quad x\in\omega, \tag{10}$$

with $\rho(x)=\min\left\{\sqrt{x_1x_2},\sqrt{x_1(l_2-x_2)},\sqrt{x_2(l_1-x_1)},\sqrt{(l_1-x_1)(l_2-x_2)}\,\right\}$.

Proof. Introducing the Heaviside step function $H(x)=\begin{cases}1, & s\geqslant 0,\\ 0, & s<0,\end{cases}$ we rewrite problem (9) in the form

$$\Lambda_\xi G(x,\xi)=-\big(H(x_1-\xi_1)H(x_2-\xi_2)\big)_{\xi_1\xi_2}, \quad \xi\in\omega,$$
$$G(x,\xi)=0, \quad \xi\in\gamma,$$

and then reduce it to the operator equation

$$A_\xi G(x,\xi)=-B_{1\xi}\big(H(x_1-\xi_1)H(x_2-\xi_2)\big).$$

Applying here Lemma 2, we get

$$\|G(x,\cdot)\|\leqslant\|-A_\xi^{-1}B_{1\xi}H(x_1-\cdot)H(x_2-\cdot)\|\leqslant\frac{\|H(x_1-\cdot)H(x_2-\cdot)\|_1}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}=$$
$$=\frac{1}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}\left\{\sum_{\xi\in\omega_1^+\times\omega_2^+}h_1h_2H^2(x_1-\xi_1)H^2(x_2-\xi_2)\right\}^{1/2}= \tag{11}$$
$$=\frac{1}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}\left\{\sum_{\xi_1=h_1}^{x_1}h_1\right\}^{1/2}\left\{\sum_{\xi_2=h_2}^{x_2}h_2\right\}^{1/2}=\frac{\sqrt{x_1x_2}}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}.$$

Now we put problem (9) in a different way:

$$\Lambda_\xi G(x,\xi)=-\big(H(\xi_1-x_1)H(\xi_2-x_2)\big)_{\bar\xi_1\bar\xi_2}, \quad \xi\in\omega,$$
$$G(x,\xi)=0, \quad \xi\in\gamma,$$

and then rewrite it as the operator equation

$$A_\xi G(x,\xi)=-B_{2\xi}\big(H(\xi_1-x_1)H(\xi_2-x_2)\big).$$

Employing here Lemma 2, we obtain

$$\|G(x,\cdot)\|\leqslant\|-A_\xi^{-1}B_{2\xi}H(\cdot-x_1)H(\cdot-x_2)\|\leqslant\frac{\|H(\cdot-x_1)H(\cdot-x_2)\|_2}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}=$$
$$=\frac{1}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}\left\{\sum_{\xi\in\omega_1^-\times\omega_2^-}h_1h_2H^2(\xi_1-x_1)H^2(\xi_2-x_2)\right\}^{1/2}= \tag{12}$$
$$=\frac{1}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}\left\{\sum_{\xi_1=x_1}^{l_1-h_1}h_1\right\}^{1/2}\left\{\sum_{\xi_2=x_2}^{l_2-h_2}h_2\right\}^{1/2}=\frac{\sqrt{(l_1-x_1)(l_2-x_2)}}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}.$$

Next we express problem (9) as follows:

$$\Lambda_\xi G(x,\xi)=\big(H(x_1-\xi_1)H(\xi_2-x_2)\big)_{\xi_1\bar\xi_2}, \quad \xi\in\omega,$$
$$G(x,\xi)=0, \quad \xi\in\gamma,$$

which means the operator formulation

$$A_\xi G(x,\xi) = B_{3\xi}\big(H(\xi_1 - x_1)H(\xi_2 - x_2)\big).$$

Making use of Lemma 2, we have

$$\|G(x,\cdot)\| \leqslant \| - A_\xi^{-1} B_{3\xi} H(x_1 - \cdot)H(\cdot - x_2)\| \leqslant \frac{\|H(x_1 - \cdot)H(\cdot - x_2)\|_3}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} =$$

$$= \frac{1}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} \Bigg\{ \sum_{\xi \in \omega_1^+ \times \omega_2^-} h_1 h_2 H^2(x_1 - \xi_1) H^2(\xi_2 - x_2) \Bigg\}^{1/2} = \qquad (13)$$

$$= \frac{1}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} \Bigg\{ \sum_{\xi_1 = h_1}^{x_1} h_1 \Bigg\}^{1/2} \Bigg\{ \sum_{\xi_2 = x_2}^{l_2 - h_2} h_2 \Bigg\}^{1/2} = \frac{\sqrt{x_1(l_2 - x_2)}}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)}.$$

Finally, we can formulate problem (9) as that:

$$\Lambda_\xi G(x,\xi) = \big(H(\xi_1 - x_1)H(x_2 - \xi_2)\big)_{\bar\xi_1 \xi_2}, \quad \xi \in \omega,$$
$$G(x,\xi) = 0, \quad \xi \in \gamma,$$

which yields the operator equation

$$A_\xi G(x,\xi) = B_{4\xi}\big(H(\xi_1 - x_1)H(x_2 - \xi_2)\big).$$

Due to Lemma 2, we get

$$\|G(x,\cdot)\| \leqslant \| - A_\xi^{-1} B_{4\xi} H(x_1 - \cdot)H(\cdot - x_2)\| \leqslant \frac{\|H(\cdot - x_1)H(x_2 - \cdot)\|_4}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} =$$

$$= \frac{1}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} \Bigg\{ \sum_{\xi \in \omega_1^- \times \omega_2^+} h_1 h_2 H^2(\xi_1 - x_1) H^2(x_2 - \xi_2) \Bigg\}^{1/2} = \qquad (14)$$

$$= \frac{1}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} \Bigg\{ \sum_{\xi_1 \in \omega_1^-} h_1 H^2(\xi_1 - x_1) \Bigg\}^{1/2} \Bigg\{ \sum_{\xi_2 \in \omega_2^+} h_2 H^2(x_2 - \xi_2) \Bigg\}^{1/2} =$$

$$= \frac{1}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)} \Bigg\{ \sum_{\xi_1 = x_1}^{l_1 - h_1} h_1 \Bigg\}^{1/2} \Bigg\{ \sum_{\xi_2 = h_2}^{x_2} h_2 \Bigg\}^{1/2} = \frac{\sqrt{(l_1 - x_1)x_2}}{2\big(\sqrt{k_{11}k_{22}} - |k_{12}|\big)}.$$

Combining estimates (11)–(14), we eventually come to the conclusion of the lemma. □

4. The weighted error estimate

For the error $z(x) = y(x) - u(x)$ we have the problem

$$\Lambda z \equiv \Lambda_1 z + \Lambda_2 + 2\Lambda_{12} z = \psi(x), \quad x \in \omega,$$
$$z(x) = 0, \quad x \in \gamma, \qquad (15)$$

where $\psi(x)$ is the approximation error:

$$\psi(x) = Tf(x) + \Lambda_1 u(x) + \Lambda_2 u(x) + 2\Lambda_{12} u(x) =$$
$$= \eta_{1\bar x_1 x_1} + \eta_{2\bar x_2 x_2} + 2\,\eta_{12\bar x_1 x_2}, \qquad (16)$$

$$\eta_\alpha(x) = k_{\alpha\alpha}\big(u(x) - T_{3-\alpha}u(x)\big), \quad \alpha = 1,2,$$

$$\eta_{12}(x) = 0{,}5k_{12}\big(u(x) + u(x_1+h_1, x_2-h_2) - 2S_1^+S_2^-u(x)\big).$$

Further we use the relation $2\,\eta_{12\bar{x}_1x_2} = k_{12}\left(u_{\bar{x}_1x_2} + u_{x_1\bar{x}_2} - 2T\frac{\partial^2 u}{\partial x_1\partial x_2}\right)$ and a conventional notation $|u|_{W_2^4(D)}$ for a seminorm in $W_2^4(D)$:

$$|u|_{W_2^4(D)} = \left\{\sum_{\substack{k_1+k_2=4\\(k_1\geqslant 0,\ k_2\geqslant 0)}} \iint\limits_D \left(\frac{\partial^{k_1+k_2}u(x_1,x_2)}{\partial x_1^{k_1}\partial x_2^{k_2}}\right)^2 dx_1dx_2\right\}^{1/2}.$$

Lemma 4. *Let the solution $u(x)$ of problem (1) satisfy the condition $u \in W_2^4(D)$. Then for the approximation error $\psi(x)$ the following estimate holds true:*

$$\|\psi\| \leqslant \widetilde{M}|h|^2\,|u|_{W_2^4(D)} \tag{17}$$

with a positive constant $\widetilde{M} = \dfrac{8(k_{11}+k_{22})}{\sqrt{3}} + \sqrt{1344}\,|k_{12}|$ independent of $u(x)$, h_1, h_2.

Proof. The representation (16) yields the inequality

$$\begin{aligned}\|\psi\| &= \|\eta_{1\bar{x}_1x_1} + \eta_{2\bar{x}_2x_2} + 2\eta_{12\bar{x}_1x_2}\| \leqslant \\ &\leqslant \|\eta_{1\bar{x}_1x_1}\| + \|\eta_{2\bar{x}_2x_2}\| + \|2\eta_{12\bar{x}_1x_2}\|.\end{aligned} \tag{18}$$

Next we consider each of the three summands in (18). For $\eta_1(x)$ we find the representation

$$\eta_1(x) = k_{11}\left(u(x) - T_2u(x)\right) = \frac{k_{11}}{h_2^2}\int\limits_{x_2-h_2}^{x_2-h_2}\big(h_2 - |x_2-\xi|\big)\big[u(x) - u(x_1,\xi)\big]\,d\xi =$$

$$= \frac{k_{11}}{h_2^2}\int\limits_{x_2-h_2}^{x_2-h_2}\big(h_2 - |x_2-\xi|\big)\,d\xi\int\limits_{\xi}^{x_2}\frac{\partial u(x_1,\xi_1)}{\partial\xi_1}\,d\xi_1 =$$

$$= \frac{k_{11}}{h_2^2}\int\limits_{x_2-h_2}^{x_2-h_2}\big(h_2 - |x_2-\xi|\big)\,d\xi\int\limits_{\xi}^{x_2}\left[\frac{\partial u(x_1,\xi_1)}{\partial\xi_1} - \frac{1}{2h_2}\int\limits_{x_2-h_2}^{x_2+h_2}\frac{\partial u(x_1,\xi_2)}{\partial\xi_2}d\xi_2\right]d\xi_1 =$$

$$= \frac{k_{11}}{2h_2^3}\int\limits_{x_2-h_2}^{x_2-h_2}\big(h_2 - |x_2-\xi|\big)d\xi\int\limits_{\xi}^{x_2}d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{\xi_2}^{\xi_1}\frac{\partial^2 u(x_1,\xi_3)}{\partial\xi_3^2}\,d\xi_3\,.$$

Bearing in mind the relation $T_1\dfrac{\partial^2 u}{\partial x_1^2} = u_{\bar{x}_1x_1}$, we have the equality

$$\eta_{1\bar{x}_1x_1}(x) = \frac{k_{11}}{2h_2^3h_1^2}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_4|\big)d\xi_4\int\limits_{x_2-h_2}^{x_2-h_2}\big(h_2-|x_2-\xi|\big)\,d\xi\times$$
$$\times\int\limits_{\xi}^{x_2}d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{\xi_2}^{\xi_1}\frac{\partial^4u(\xi_4,\xi_3)}{\partial\xi_4^2\partial\xi_3^2}\,d\xi_3,$$

which gives the estimate

$$\big|\,\eta_{\bar{x}_1x_1}(x)\big| \leqslant \frac{k_{11}}{2h_2^3h_1^2}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_4|\big)d\xi_4\int\limits_{x_2-h_2}^{x_2-h_2}\big(h_2-|x_2-\xi|\big)d\xi\times$$
$$\times\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_2\int\limits_{x_2-h_2}^{x_2+h_2}\left|\frac{\partial^4u(\xi_4,\xi_3)}{\partial\xi_4^2\partial\xi_3^2}\right|d\xi_3 \leqslant$$
$$\leqslant \frac{k_{11}h_2^2\cdot 2h_2\cdot 2h_2}{2h_2^3h_1^2}\left\{\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1-|x_1-\xi_4|\big)^2d\xi_4\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_3\right\}^{1/2}\times$$
$$\times\left\{\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_4\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(\xi_4,\xi_3)}{\partial\xi_4^2\partial\xi_3^2}\right)^2d\xi_3\right\}^{1/2} =$$
$$= \frac{4k_{11}}{\sqrt{3}}\sqrt{\frac{h_2^3}{h_1}}\left\{\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_4\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(\xi_4,\xi_3)}{\partial\xi_4^2\partial\xi_3^2}\right)^2d\xi_3\right\}^{1/2}.$$

Thus we get

$$\|\eta_{1\bar{x}_1x_1}\| = \left\{\sum_{x\in\omega}h_1h_2\,\eta^2_{1\bar{x}_1x_1}(x)\right\}^{1/2} \leqslant$$
$$\leqslant \frac{8k_{11}h_2^2}{\sqrt{3}}\left\{\iint\limits_D\left|\frac{\partial^4u(x_1,x_2)}{\partial x_1^2\partial x_2^2}\right|^2dx_1dx_2\right\}^{1/2}. \tag{19}$$

Similarly we can obtain the estimate

$$\|\eta_{2\bar{x}_2x_2}\| = \left\{\sum_{x\in\omega}h_1h_2\,\eta^2_{2\bar{x}_2x_2}(x)\right\}^{1/2} \leqslant$$
$$\leqslant \frac{8k_{22}h_1^2}{\sqrt{3}}\left\{\iint\limits_D\left|\frac{\partial^4u(x_1,x_2)}{\partial x_1^2\partial x_2^2}\right|^2dx_1dx_2\right\}^{1/2}. \tag{20}$$

Now we move on to the third summand in equation (18). Omitting some technical details, we arrive at the representation

$$2\,\eta_{12\bar{x}_1x_1} = k_{12}\left(u_{\bar{x}_1x_2} + u_{x_1\bar{x}_2} - 2T\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) =$$

$$= \frac{k_{12}}{h_1^2h_2^2}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1 - |x_1-\xi_1|\big)d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)\times$$

$$\times\left[u_{\bar{x}_1x_2}(x) + u_{x_1\bar{x}_2}(x) - 2\frac{\partial^2 u(\xi_1,\xi_2)}{\partial\xi_1\partial\xi_2}\right]d\xi_2 =$$

$$= \frac{k_{12}}{4h_1^4h_2^4}\int\limits_{x_1-h_1}^{x_1+h_1}\big(h_1 - |x_1-\xi_1|\big)\,d\xi_1\int\limits_{x_2-h_2}^{x_2+h_2}\big(h_2-|x_2-\xi_2|\big)\times$$

$$\times\left[\int\limits_{x_1}^{x_1+h_1}d\xi_3\int\limits_{x_2-h_2}^{x_2}d\xi_4\int\limits_{\xi_1}^{\xi_3}d\xi_7\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_{11}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_{12}\times\right.$$

$$\times\left\{\int\limits_{\xi_{11}}^{\xi_7}\frac{\partial^4 u(\xi_{19},\xi_4)}{\partial\xi_{19}^3\partial\xi_4}d\xi_{19} + \int\limits_{\xi_{12}}^{\xi_4}\frac{\partial^4 u(\xi_{11},\xi_{20})}{\partial\xi_{11}^2\partial\xi_{20}^2}d\xi_{20}\right\}+$$

$$+\int\limits_{x_1}^{x_1+h_1}d\xi_3\int\limits_{x_2-h_2}^{x_2}d\xi_4\int\limits_{\xi_2}^{\xi_4}d\xi_8\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_{13}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_{14}\times$$

$$\times\left\{\int\limits_{\xi_{13}}^{\xi_1}\frac{\partial^4 u(\xi_{21},\xi_8)}{\partial\xi_{21}^2\partial\xi_8^2}d\xi_{21} + \int\limits_{\xi_{14}}^{\xi_8}\frac{\partial^4 u(\xi_{13},\xi_{22})}{\partial\xi_{13}\partial\xi_{22}^3}d\xi_{22}\right\}+$$

$$+\int\limits_{x_1-h_1}^{x_1}d\xi_5\int\limits_{x_2}^{x_2+h_2}d\xi_6\int\limits_{\xi_1}^{\xi_5}d\xi_9\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_{15}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_{16}\times$$

$$\times\left\{\int\limits_{\xi_{15}}^{\xi_9}\frac{\partial^4 u(\xi_{23},\xi_6)}{\partial\xi_{23}^3\partial\xi_6}d\xi_{23} + \int\limits_{\xi_{16}}^{\xi_6}\frac{\partial^4 u(\xi_{15},\xi_{24})}{\partial\xi_{15}^2\partial\xi_{24}^2}d\xi_{24}\right\}+$$

$$+\int\limits_{x_1-h_1}^{x_1}d\xi_5\int\limits_{x_2}^{x_2+h_2}d\xi_6\int\limits_{\xi_2}^{\xi_6}d\xi_{10}\int\limits_{x_1-h_1}^{x_1+h_1}d\xi_{17}\int\limits_{x_2-h_2}^{x_2+h_2}d\xi_{18}\times$$

$$\left.\times\left\{\int\limits_{\xi_{17}}^{\xi_1}\frac{\partial^4 u(\xi_{25},\xi_{10})}{\partial\xi_{25}^2\partial\xi_{10}^2}d\xi_{25} + \int\limits_{\xi_{18}}^{\xi_{10}}\frac{\partial^4 u(\xi_{17},\xi_{26})}{\partial\xi_{17}\partial\xi_{26}^3}d\xi_{26}\right\}\right]d\xi_2,$$

where the following relations were used:

$$\int\limits_{x_1-h_1}^{x_1+h_1} \bigl(h_1-|x_1-\xi_1|\bigr)\left[\int\limits_{x_1}^{x_1+h_1}(\xi_3-\xi_1)\,d\xi_3+\int\limits_{x_1-h_1}^{x_1}(\xi_5-\xi_1)\,d\xi_5\right]d\xi_1=0,$$

$$\int\limits_{x_2-h_2}^{x_2+h_2} \bigl(h_2-|x_2-\xi_2|\bigr)\left[\int\limits_{x_2-h_2}^{x_2}(\xi_4-\xi_2)\,d\xi_4+\int\limits_{x_2}^{x_2+h_2}(\xi_6-\xi_2)\,d\xi_6\right]d\xi_2=0.$$

This yields the estimate

$$\bigl|2\,\eta_{12\bar{x}_1x_2}(x)\bigr|\leqslant\frac{4\sqrt{2}\,|k_{12}|\sqrt{h_1^3}}{\sqrt{h_2}}\left\{\int\limits_{x_2-h_2}^{x_2+h_2}\int\limits_{x_1-h_1}^{x_1+h_1}\left(\frac{\partial^4u(x_1,x_2)}{\partial x_1^3\partial x_2}\right)^2dx_1dx_2\right\}^{1/2}+$$

$$+\,8|k_{12}|\sqrt{h_1h_2}\left\{\int\limits_{x_1-h_1}^{x_1+h_1}\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(x_1,x_2)}{\partial x_1^2\partial x_2^2}\right)^2dx_2dx_1\right\}^{1/2}+$$

$$+\,\frac{4\,|k_{12}|\sqrt{h_2^3}}{\sqrt{h_1}}\left\{\int\limits_{x_1-h_1}^{x_1+h_1}\int\limits_{x_2-h_2}^{x_2+h_2}\left(\frac{\partial^4u(x_1,x_2)}{\partial x_1\partial x_2^3}\right)^2dx_2dx_1\right\}^{1/2},\quad x\in\omega.$$

Employing the inequality $(a+b+c)^2\leqslant 3(a^2+b^2+c^2)$, we find

$$\|2\,\eta_{12\bar{x}_1x_2}(x)\|^2=\sum_{x\in\omega}h_1h_2\bigl(2\,\eta_{12\bar{x}_1x_2}\bigr)^2\leqslant$$

$$\leqslant 3k_{12}^2\left[128h_1^4\iint\limits_D\left(\frac{\partial^4u(x_1,x_2)}{\partial x_1^3\partial x_2}\right)^2dx_1dx_2+\right.$$

$$\left.+\,256h_1^2h_2^2\iint\limits_D\left(\frac{\partial^4u(x_1,x_2)}{\partial x_1^2\partial x_2^2}\right)^2dx_1dx_2+64h_2^4\iint\limits_D\left(\frac{\partial^4u(x_1,x_2)}{\partial x_1\partial x_2^3}\right)^2dx_1dx_2\right],$$

that is

$$\|2\,\eta_{12\bar{x}_1x_2}(x)\|\leqslant|k_{12}|\sqrt{1344}\,|h|^2|u|_{x_2^4(D)}.\tag{21}$$

Combining inequalities (18)–(21), we come to estimate (17). The lemma is proven. □

We finally arrive at the main proposition.

Theorem 2. *Let the solution $u(x)$ of problem (1) satisfy the condition $u\in W_2^4(D)$. Then the accuracy of the difference scheme (3) is characterized by the weighted estimate*

$$\max_{x\in\omega}\left|\frac{z(x)}{\rho(x)}\right|\leqslant M|h|^2|u|_{W_2^4(D)},\quad |h|^2=h_1^2+h_2^2,\tag{22}$$

with the weight function $\rho(x)$ defined in (10) and a constant M independent of $u(x),h_1,h_2$.

Proof. The solution of problem (15) can be presented in the form

$$z(x) = \big(G(x,\cdot),\psi(\cdot)\big) = \sum_{\xi\in\omega} h_1h_2G(x,\xi)\psi(\xi), \quad x\in\omega.$$

Due to Lemma 3 and Lemma 4, we get

$$|z(x)| = \big|\big(G(x,\cdot),\psi(\cdot)\big)\big| \leqslant \big\|G(x,\cdot)\big\|\,\|\psi\| \leqslant \\ \leqslant \frac{\rho(x)}{2\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}\widetilde{M}|h|^2|u|_{W_2^4(D)},$$

which leads to (22) with the constant

$$M = \frac{\widetilde{M}}{2\big(\sqrt{k_{11}k_{22}}-|k_{12}|\big)} = \frac{4(k_{11}+k_{22})+12\sqrt{7}\,|k_{12}|}{\sqrt{3}\left(\sqrt{k_{11}k_{22}}-|k_{12}|\right)}.$$

The theorem is proven. □

Remark 2. *The functionals $\eta_{1\bar{x}_1x_1},\eta_{2\bar{x}_2x_2},2\eta_{12\bar{x}_1x_2}$ in Lemma 4 can be estimated by means of the Bramble – Hilbert lemma (see [12], p. 29), however, with the unknown constants $\widetilde{M}$ and M in (17) and (22) respectively.*

Remark 3. *The weighted estimate (22) shows the influence of Dirichle's boundary condition and clearly indicates that the accuracy order of the finite-difference scheme (17) in the uniform norm is $O\big(|h|^2h_1^{1/2}\big)$, $O\big(|h|^2h_2^{1/2}\big)$, and $O\big(|h|^2(h_1h_2)^{1/2}\big)$ near the vertical sides $x_1=0$, $x_1=l_1$, near the horizontal sides $x_2=0$, $x_2=l_2$, and near the vertices of the rectangle D respectively.*

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Received 18.08.2021; revised 16.09.2021.