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ERROR BOUNDS FOR FOURIER-LEGENDRE TRUNCATION METHOD IN NUMERICAL DIFFERENTIATION

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Резюме. Розглядається задача чисельного диференціювання функцій $f(t,\tau)$ скінченної гладкості. Для відновлення похідної $f^{(1,1)}$ застосовано стандартний та модифікований варіанти методу зрізки, що використовують в якості вхідної інформації неточні значення коефіцієнтів Фур'є-Лежандра. Для обох варіантів методу зрізки досліджено апроксимаційні та інформаційні властивості, а також наведено їх порівняльний аналіз.

Abstract. The problem of numerical differentiation of functions $f(t,\tau)$ with finite smoothness is considered. For recovery of the derivative $f^{(1,1)}$ we apply standard and modified variants of the truncation method using perturbed values of Fourier-Legendre coefficients as input information. For both these variants some approximation and information properties are investigated, and their comparative analysis is given.

1. Introduction

The problem of numerical differentiation is an actual problem arising in many applied fields such as analytical chemistry, mathematical physics and pattern recognition. The problem of numerical differentiation of functions is a classic problem that is unstable to small perturbations and therefore requires application of regularization to ensure the stability of the approximation. It should be noted that intensive and effective research of the problem of stable differentiation began in the 60s of last century due to the development of the theory of ill-posed problems. The first paper on the differentiation of functions, which was written in terms of the theory of ill-posed problems, is [2]. Thus far, many researchers have proposed and substantiated different methods of numerical differentiation of univariate functions (see, for example, [14], [16], [3], [4], [1], [12], [17], [5], [15]). As to the functions of several (even two) variables, the problem is still studying. Only works can be mentioned here [11], [18], [7], [15]. Within this work the questions of numerical differentiation of bivariate functions with finite smoothness are carried out.

Let $C = C(Q)$ be the space of continuous on $Q = [-1,1]^2$ bivariate functions with uniform metric. By $\{\varphi_k(t)\}_{k=0}^{\infty}$ we denote the system of Legendre polynomials orthonormal on $[-1,1]$ as

$$\varphi_k(t) = \sqrt{k+1/2} \quad (2^k k!)^{-1} \frac{d^k}{dt^k}[(t^2-1)^k], \quad k = 0,1,2,\ldots.$$

Key words. Numerical differentiation; Legendre polynomials; truncation method.

By $L_2 = L_2(Q)$ we mean space of square-summable on Q functions $f(t,\tau)$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1}\int_{-1}^{1} f(t,\tau)g(t,\tau)d\tau dt$$

and standard norm

$$\|f\|_2^2 = \sum_{k,j=0}^{\infty} \langle f, \varphi_{k,j}\rangle^2 < \infty,$$

where

$$\langle f, \varphi_{k,j}\rangle = \int_{-1}^{1}\int_{-1}^{1} f(t,\tau)\varphi_k(t)\varphi_j(\tau)d\tau dt, \quad , k, j = 0, 1, 2, \ldots$$

are Fourier-Legendre coefficients of f.

For any $\mu > 0$ we introduce the space of functions

$$L_{2,2}^{\mu}(Q) = \{f \in L_2(Q): \quad \|f\|_{\mu}^2 = \sum_{k,j=0}^{\infty} (kj)^{2\mu} |\langle f, \varphi_{k,j}\rangle|^2 < \infty\}.$$

It should be noted that $L_{2,2}^{\mu}$ is generalization of the class of bivariate functions with dominating mixed partial derivative.

Suppose that instead of exact values of Fourier-Legendre coefficients $\langle f, \varphi_{k,j}\rangle$ we have only their perturbations, the error level δ of which is known in the metric of ℓ_2. More accurately, we assume that there is a sequence of numbers $\langle f_\delta, \varphi_{k,j}\rangle$ such that

$$\sum_{k,j=0}^{\infty} \xi_{k,j}^2 \le \delta^2, \tag{1}$$

where $\xi_{k,j} = \langle f - f_\delta, \varphi_{k,j}\rangle$, $k, j = 0, 1, 2, \ldots$. As the derivative of f we take the series

$$f^{(1,1)}(t,\tau) = \sum_{k,j=1}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi_k'(t)\varphi_j'(\tau). \tag{2}$$

In what follows, we need the following auxiliary relations (see Lemma 18 [10])

$$\max_{-1\le t\le 1} |\varphi_k(t)| = \sqrt{k + 1/2},\ k = 0, 1, 2, \ldots, \tag{3}$$

$$\varphi_k'(t) = 2\sqrt{k+1/2} \sum_{i=0}^{(k-q_k-1)/2} \sqrt{2i + q_k + 1/2}\varphi_{2i+q_k}(t), \tag{4}$$

where $q_k = \begin{cases} 1, & k \quad \text{is odd}, \\ 0, & k \quad \text{is even}. \end{cases}$

By replacing the variables $l = 2i + q_k$, with (4) we have

$$\begin{aligned} \varphi_k'(t) &= \sqrt{k+1/2} \sum_{l=q_k}^{k-1} \sqrt{l+1/2}\,\varphi_l(t) = \\ &= \sqrt{k+1/2} \sum_{l=0}^{k-1}{}^{*} \sqrt{l+1/2}\,\varphi_l(t), \quad k \in \mathbb{N}, \end{aligned} \tag{5}$$

where in aggregate $\sum_{l=0}^{k-1}{}^{*} \sqrt{l+1/2}\,\varphi_l(t)$ the summation is extended over only those terms for which $k+l$ is odd.

2. Standard approach

Our research is devoted to the application of the truncation method for the numerical differentiation of functions from $L^{\mu}_{2,2}(Q)$. The essence of this method is to replace the Fourier series (2) with a finite Fourier sum using perturbed data $\langle f_\delta, \varphi_{k,j}\rangle$. In the truncation method to ensure the stability of the approximation and achieve the required order accuracy, it is necessary to choose properly the discretization parameter, which here serves as a regularization parameter. So, the process of regularization in method under consideration consists in matching the discretization parameter with the perturbation level of the input data. Simplicity of implementation is the main advantage of this method.

By now, there are several approaches to selecting a domain of the coordinate plane for the indices of the involved Fourier-Legendre coefficients. Our research will consider the two most effective and popular approaches to select this domain, first, the standard variant, when the domain is a square, as well as a modified variant, which is to apply so-called hyperbolic cross (for more details on the use of hyperbolic cross in solving ill-posed problems see [13], [8], [9]).

At first we consider standard variant of such sum:

$$\mathcal{D}_n f_\delta(t,\tau) = \sum_{k,j=1}^{n} \langle f_\delta, \varphi_{k,j}\rangle \varphi'_k(t)\varphi'_j(\tau). \tag{6}$$

We give the approximation error in the form

$$f^{(1,1)}(t,\tau) - \mathcal{D}_n f_\delta(t,\tau) = (f^{(1,1)}(t,\tau) - \mathcal{D}_n f(t,\tau)) + (\mathcal{D}_n f(t,\tau) - \mathcal{D}_n f_\delta(t,\tau)). \tag{7}$$

Let's present the first term on the right-hand side of (7) as

$$f^{(1,1)}(t,\tau) - \mathcal{D}_n f(t,\tau) = \triangle_1(t,\tau) + \triangle_2(t,\tau),$$

where

$$\triangle_1(t,\tau) = \sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi'_k(t)\varphi'_j(\tau), \tag{8}$$

$$\triangle_2(t,\tau) = \sum_{k=1}^{n} \sum_{j=n+1}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi'_k(t)\varphi'_j(\tau).$$

Applying twice the formula (5) to (8) for the derivatives $\varphi'_k(t)$, $\varphi'_j(\tau)$, we get

$$\triangle_1(t,\tau) = \sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} \langle f, \varphi_{k,j}\rangle \sum_{l_1=0}^{k-1}{}^{*} 2\sqrt{k+1/2}\sqrt{l_1+1/2}\,\varphi_{l_1}(t) \times$$

$$\times \sum_{l_2=0}^{j-1}{}^{*} 2\sqrt{j+1/2}\sqrt{l_2+1/2}\,\varphi_{l_2}(\tau).$$

Here and below, the symbol $\sum^{*}$ means that the summation is extended over only those terms for which $k+l_1$ and $j+l_2$ are odd.

Whence, after changing the order of summation, we arrive at the representation

$$\triangle_1(t,\tau) = \triangle_{11}(t,\tau) + \triangle_{12}(t,\tau),$$

where

$$\begin{aligned}\triangle_{11}(t,\tau) = 4 {\sum_{l_1=0}^{n}}^{*} \sqrt{l_1+1/2}\varphi_{l_1}(t) {\sum_{l_2=0}^{\infty}}^{*} \sqrt{l_2+1/2}\varphi_{l_2}(\tau)\times \\ \times \sum_{k=n+1}^{\infty} \sqrt{k+1/2} \sum_{j=l_2+1}^{\infty} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle,\end{aligned} \tag{9}$$

$$\begin{aligned}\triangle_{12}(t,\tau) = 4 {\sum_{l_1=n+1}^{\infty}}^{*} \sqrt{l_1+1/2}\varphi_{l_1}(t) {\sum_{l_2=0}^{\infty}}^{*} \sqrt{l_2+1/2}\varphi_{l_2}(\tau)\times \\ \times \sum_{k=l_1+1}^{\infty} \sqrt{k+1/2} \sum_{j=l_2+1}^{\infty} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle.\end{aligned} \tag{10}$$

This can be written as follows

$$\triangle_1(t,\tau) = 4 {\sum_{l_1=0}^{\infty}}^{*} {\sum_{l_2=0}^{\infty}}^{*} \sqrt{l_1+1/2}\sqrt{l_2+1/2}\varphi_{l_1}(t)\varphi_{l_2}(\tau) a_{l_1,l_2},$$

where

$$a_{l_1,l_2} = \begin{cases} \sum\limits_{k=n+1}^{\infty} \sqrt{k+1/2} \sum\limits_{j=l_2+1}^{\infty} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle, & l_1 \le n \\ \sum\limits_{k=l_1+1}^{\infty} \sqrt{k+1/2} \sum\limits_{j=l_2+1}^{\infty} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle & l_1 > n. \end{cases}$$

Lemma 1. *For arbitrary function $f \in L^{\mu}_{2,2}(Q)$, $\mu > 2$, it holds true*

$$\|f^{1,1} - \mathcal{D}_n f\|_2 \le c\|f\|_\mu n^{-\mu+2}.$$

Proof. First of all we estimate the norm of $\triangle_1$ in the metric of $L_2(Q)$:

$$\|\triangle_1\|_2^2 \le 16 \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty} (l_1+1/2)(l_2+1/2)a^2_{l_1,l_2}.$$

Further we should majorize $a^2_{l_1,l_2}$.

1) Let $l_1 \le n$. Then

$$a^2_{l_1,l_2} = \left(\sum_{k=n+1}^{\infty}\sum_{j=l_2+1}^{\infty} \sqrt{k+1/2}\sqrt{j+1/2}\frac{(kj)^\mu}{(kj)^\mu}\langle f, \varphi_{k,j}\rangle \right)^2 \le$$

$$\le \sum_{k=n+1}^{\infty}\sum_{j=l_2+1}^{\infty} \frac{(k+1/2)(j+1/2)}{(kj)^{2\mu}} \sum_{k=n+1}^{\infty}\sum_{j=l_2+1}^{\infty} (kj)^{2\mu}|\langle f, \varphi_{k,j}\rangle|^2 \le$$

$$\le \|f\|^2_\mu \sum_{k=n+1}^{\infty}\sum_{j=l_2+1}^{\infty} \frac{1}{(kj)^{2\mu-1}} \le$$

$$\leq c\|f\|_\mu^2 \sum_{k=n+1}^{\infty} \frac{1}{(k)^{2\mu-1}}(1/2+l_2)^{-2\mu+2} \leq c\|f\|_\mu^2 [n(1/2+l_2)]^{-2\mu+2}.$$

2) Let $l_1 > n$. Then

$$a_{l_1,l_2}^2 = \left(\sum_{k=l_1+1}^{\infty} \sum_{j=l_2+1}^{\infty} \sqrt{k+1/2}\sqrt{j+1/2}\frac{(kj)^\mu}{(kj)^\mu}\langle f, \varphi_{k,j}\rangle \right)^2 \leq$$

$$\leq \sum_{k=l_1+1}^{\infty} \sum_{j=l_2+1}^{\infty} \frac{(k+1/2)(j+1/2)}{(kj)^{2\mu}} \sum_{k=l_1+1}^{\infty} \sum_{j=l_2+1}^{\infty} (kj)^{2\mu} |\langle f, \varphi_{k,j}\rangle|^2 \leq$$

$$\leq \|f\|_\mu^2 \sum_{k=l_1+1}^{\infty} \frac{k+1/2}{k^{2\mu}} \sum_{j=l_2+1}^{\infty} \frac{j+1/2}{j^{2\mu}} \leq c\|f\|_\mu^2 [(l_1+1/2)(l_2+1/2)]^{-2\mu+2}.$$

In order to obtain an estimate for the $\|\Delta_1\|_2$, we note that for $\mu > 2$

$$\sum_{l_2=0}^{\infty} \frac{l_2+1/2}{(1/2+l_2)^{2\mu-2}} = c,$$

and with the help of following relations

$$\sum_{l_1=0}^{n} (l_1+1/2) n^{-2\mu+2} = cn^{-2\mu+4},$$

$$\sum_{l_1=n+1}^{\infty} (l_1+1/2)^{-2\mu+2}(l_1+1/2) = cn^{-2\mu+4},$$

we get

$$\|\triangle_1\|_2 \leq c\|f\|_\mu n^{-\mu+2}. \tag{11}$$

Next, we need to bound $\triangle_2(t,\tau)$. Thus due to (4) we have

$$\triangle_2(t,\tau) = \sum_{k=1}^{n} \sum_{j=n+1}^{\infty} \langle f, \varphi_{k,j}\rangle \varphi_k'(t)\varphi_j'(\tau) = \sum_{k=1}^{n} \sum_{j=n+1}^{\infty} \langle f, \varphi_{k,j}\rangle \times$$

$$\times \sum_{i=0}^{(k-q_k-1)/2} 2\sqrt{k+1/2}\sqrt{2i+q_k+1/2}\varphi_{2i+q_k}(t) \times$$

$$\times \sum_{m=0}^{(j-q_j-1)/2} 2\sqrt{j+1/2}\sqrt{2m+q_j+1/2}\varphi_{2m+q_j}(\tau).$$

Let's replace variables $l_1 = 2i+q_k$ and $l_2 = 2m+q_j$ and change the summation order. Then we get

$$\triangle_2(t,\tau) = 4\sum_{k=1}^{n} \sum_{j=n+1}^{\infty} \langle f, \varphi_{k,j}\rangle \sum_{l_1=0}^{k-1}{}^{*} \sqrt{k+1/2}\sqrt{l_1+1/2}\varphi_{l_1}(t) \times$$

$$\times \sum_{l_2=0}^{j-1}{}^{*} \sqrt{j+1/2}\sqrt{l_2+1/2}\varphi_{l_2}(\tau) =$$

$$= \triangle_{21}(t,\tau) + \triangle_{22}(t,\tau),$$

where

$$\begin{aligned} \triangle_{21}(t,\tau) = 4 \sum_{l_1=0}^{n-1}{}^{*} \sqrt{l_1+1/2}\varphi_{l_1}(t) \sum_{l_2=0}^{n}{}^{*} \sqrt{l_2+1/2}\varphi_{l_2}(\tau) \times \\ \times \sum_{k=l_1+1}^{n} \sqrt{k+1/2} \sum_{j=n+1}^{\infty} \sqrt{j+1/2} \langle f, \varphi_{k,j}\rangle, \end{aligned} \tag{12}$$

$$\begin{aligned} \triangle_{22}(t,\tau) = 4 \sum_{l_1=0}^{n-1}{}^{*} \sqrt{l_1+1/2}\varphi_{l_1}(t) \sum_{l_2=n+1}^{\infty}{}^{*} \sqrt{l_2+1/2}\varphi_{l_2}(\tau) \times \\ \times \sum_{k=l_1+1}^{n} \sqrt{k+1/2} \sum_{j=l_2+1}^{\infty} \sqrt{j+1/2} \langle f, \varphi_{k,j}\rangle. \end{aligned} \tag{13}$$

It can be rewritten by the following way

$$\triangle_2(t,\tau) = 4 \sum_{l_1=0}^{n-1}{}^{*} \sum_{l_2=0}^{\infty}{}^{*} \sqrt{l_1+1/2}\sqrt{l_2+1/2}\varphi_{l_1}(t)\varphi_{l_2}(\tau)\hat{a}_{l_1,l_2},$$

where

$$\hat{a}_{l_1,l_2} = \begin{cases} \sum\limits_{k=l_1+1}^{n} \sqrt{k+1/2} \sum\limits_{j=n+1}^{\infty} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle, & l_2 \le n \\ \sum\limits_{k=l_1+1}^{n} \sqrt{k+1/2} \sum\limits_{j=l_2+1}^{\infty} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle & l_2 > n. \end{cases}$$

Further we estimate the norm of $\triangle_2$ in the metric of $L_2(Q)$:

$$\|\triangle_2\|_2^2 \le 16 \sum_{l_1=0}^{n-1} \sum_{l_2=0}^{\infty} (l_1+1/2)(l_2+1/2)\hat{a}^2_{l_1,l_2}.$$

So, we should bound $\hat{a}^2_{l_1,l_2}$.

1) Let $l_2 \le n$. Then

$$\hat{a}^2_{l_1,l_2} = \left(\sum_{k=l_1+1}^{n} \sum_{j=n+1}^{\infty} \sqrt{k+1/2}\sqrt{j+1/2}\frac{(kj)^\mu}{(kj)^\mu}\langle f, \varphi_{k,j}\rangle \right)^2 \le$$

$$\le \sum_{k=l_1+1}^{n} \sum_{j=n+1}^{\infty} \frac{(k+1/2)(j+1/2)}{(kj)^{2\mu}} \sum_{k=l_1+1}^{n} \sum_{j=n+1}^{\infty} (kj)^{2\mu}|\langle f, \varphi_{k,j}\rangle|^2 \le$$

$$\le c\|f\|_\mu^2 n^{-2\mu+2} \sum_{k=l_1+1}^{n} k^{-2\mu+1} \le$$

$$\le c\|f\|_\mu^2 [n(1/2+l_1)]^{-2\mu+2}.$$

2) Let $l_2 > n$. Then

$$\hat{a}^2_{l_1,l_2} = \left(\sum_{k=l_1+1}^{n} \sum_{j=l_2+1}^{\infty} \sqrt{k+1/2}\sqrt{j+1/2}\frac{(kj)^\mu}{(kj)^\mu}\langle f, \varphi_{k,j}\rangle \right)^2 \le$$

$$\le \sum_{k=l_1+1}^{n} \sum_{j=l_2+1}^{\infty} \frac{(k+1/2)(j+1/2)}{(kj)^{2\mu}} \sum_{k=l_1+1}^{n} \sum_{j=l_2+1}^{\infty} (kj)^{2\mu}\langle f, \varphi_{k,j}\rangle^2 \le$$

$$\le c\|f\|^2_\mu \sum_{k=l_1+1}^{n} k^{-2\mu+1} \sum_{j=l_2+1}^{\infty} j^{-2\mu+1} \le c\|f\|^2_\mu [(l_1+1/2)(l_2+1/2)]^{-2\mu+2}.$$

For $\mu > 2$ it is true

$$\sum_{l_1=0}^{n-1} \frac{l_1+1/2}{(1/2+l_1)^{2\mu-2}} = c,$$

and with the help of following relations

$$\sum_{l_2=0}^{n} (l_2+1/2)n^{-2\mu+2} = cn^{-2\mu+4},$$

$$\sum_{l_2=n+1}^{\infty} (l_2+1/2)^{-2\mu+3} = cn^{-2\mu+4},$$

we have

$$\|\triangle_2\|_2 \le c\|f\|_\mu n^{-\mu+2}. \tag{14}$$

Combining (11) and (14), we get the statement of Lemma. □

Lemma 2. *Let the condition (1) be satisfied. Then for arbitrary function $f \in L_2(Q)$ it holds true*

$$\|\mathcal{D}_n f - \mathcal{D}_n f_\delta\|_2 \le \frac{\delta}{6} n(3n^3 + 8n^2 + 6n + 1).$$

Proof.

So, it remains to estimate the norm of the second summand on right-hand side of (7):

$$\mathcal{D}_n f(t,\tau) - \mathcal{D}_n f_\delta(t,\tau) = \sum_{k,j=1}^{n} \langle f - f_\delta, \varphi_{k,j}\rangle \varphi'_k(t)\varphi'_j(\tau) = \sum_{k,j=1}^{n} \xi_{kj}\varphi'_k(t)\varphi'_j(\tau) =$$

$$= \sum_{k,j=1}^{n} \xi_{kj} \sum_{i=0}^{(k-q_k-1)/2} 2\sqrt{k+1/2}\sqrt{2i+q_k+1/2}\varphi_{2i+q_k}(t) \times$$

$$\times \sum_{m=0}^{(j-q_j-1)/2} 2\sqrt{j+1/2}\sqrt{2m+q_j+1/2}\varphi_{2m+q_j}(\tau).$$

Let's replace variables $l_1 = 2i + q_k$ and $l_2 = 2m + q_j$ and change the summation order. Then we have

$$\mathcal{D}_n f(t,\tau) - \mathcal{D}_n f_\delta(t,\tau) = 4 \sum_{k,j=1}^{n} \xi_{kj} \sum_{l_1=0}^{k-1}{}^{*} \sqrt{k+1/2}\sqrt{l_1+1/2}\varphi_{l_1}(t) \times$$

$$\times \sum_{l_2=0}^{j-1}{}^{*} \sqrt{j+1/2}\sqrt{l_2+1/2}\varphi_{l_2}(\tau) =$$

$$= 4\sum_{l_1=0}^{n-1}{}^{*} \sum_{l_2=0}^{n-1}{}^{*} \sqrt{l_1+1/2}\sqrt{l_2+1/2}\varphi_{l_1}(t)\varphi_{l_2}(\tau)\times$$
$$\times \sum_{k=l_1+1}^{n} \sum_{j=l_2+1}^{n} \sqrt{k+1/2}\sqrt{j+1/2}\xi_{k,j}. \quad (15)$$

Finally we get

$$\|\mathcal{D}_n f - \mathcal{D}_n f_\delta\|_2^2 \le$$
$$\le 16 \sum_{l_1,l_2=0}^{n-1} (l_1+1/2)(l_2+1/2) \left(\sum_{k=l_1+1}^{n} \sum_{j=l_2+1}^{n} \sqrt{k+1/2}\sqrt{j+1/2}\xi_{k,j} \right)^2 \le$$
$$\le 16\delta^2 \sum_{l_1,l_2=0}^{n-1} (l_1+1/2)(l_2+1/2) \sum_{k=l_1+1}^{n} (k+1/2) \sum_{j=l_2+1}^{n} (j+1/2) =$$
$$= \frac{\delta^2}{36} n^2 (3n^3+8n^2+6n+1)^2.$$

Thus, Lemma is proved. □

The combination of Lemmas 1 and 2 gives

Theorem 1. *Let $\mu > 2$. Then for arbitrary function $f \in L^{\mu}_{2,2}(Q)$, $\|f\|_\mu \le 1$, and $n \asymp \delta^{-\frac{1}{\mu+2}}$ it holds true*

$$\|f^{(1,1)} - \mathcal{D}_n f_\delta\|_2 \le c\delta^{\frac{\mu-2}{\mu+2}}.$$

Corollary 5. *Under the assumptions of Theorem 1, for achieving the accuracy*

$$O\left(\delta^{\frac{\mu-2}{\mu+2}}\right)$$

the method (6) *requires the following amount of perturbed Fourier-Legendre coefficients:*

$$card \asymp \mathrm{n}^2 \asymp \delta^{-\frac{2}{\mu+2}}$$

with indices from the square $\{(k,j):\ 1 \le k,j \le n; k,j \in \mathbb{N}\}$.

Let's estimate the error of the method (6) in uniform metric.

Lemma 3. *For arbitrary function $f \in L^{\mu}_{2,2}(Q)$, $\mu > 3$, it holds true*

$$\|f^{(1,1)} - \mathcal{D}_n f\|_C \le c\|f\|_\mu n^{-\mu+3}.$$

Proof. So, for $\mu > 3$ from (9) we have

$$\|\triangle_{11}\|_C \le 4 \sum_{l_1=0}^{n} \sqrt{l_1+1/2} \max_{-1\le t\le 1} |\varphi_{l_1}(t)| \sum_{l_2=0}^{\infty} \sqrt{l_2+1/2} \max_{-1\le t\le 1} |\varphi_{l_2}(t)| \times$$

$$\times \sum_{k=n+1}^{\infty} \sqrt{k+1/2} \sum_{j=l_2+1}^{\infty} \sqrt{j+1/2} |\langle f, \varphi_{k,j}\rangle| = 4 \sum_{l_1=0}^{n} (l_1+1/2) \sum_{l_2=0}^{\infty} (l_2+1/2) \times$$

$$\times \sum_{k=n+1}^{\infty} \frac{\sqrt{k+1/2}}{k^{\mu}} \sum_{j=l_2+1}^{\infty} \frac{\sqrt{j+1/2}}{j^{\mu}} (kj)^{\mu} |\langle f, \varphi_{k,j}\rangle| \le$$

$$\le 4\|f\|_{\mu} \sum_{l_1=0}^{n} (l_1+1/2) \sum_{l_2=0}^{\infty} (l_2+1/2) \left(\sum_{k=n+1}^{\infty} \frac{k+1/2}{k^{2\mu}} \sum_{j=l_2+1}^{\infty} \frac{j+1/2}{j^{2\mu}} \right)^{1/2} \le$$

$$\le c\|f\|_{\mu} n^{-\mu+1} \sum_{l_1=0}^{n} (l_1+1/2) \sum_{l_2=0}^{\infty} \frac{1}{(l_2+1)^{\mu-2}} \le c\|f\|_{\mu} n^{-\mu+3}.$$

Similarly, from (10) we find

$$\|\triangle_{12}\|_C \le 4 \sum_{l_1=n+1}^{\infty} \sqrt{l_1+1/2} \max_{-1\le t\le 1} |\varphi_{l_1}(t)| \sum_{l_2=0}^{\infty} \sqrt{l_2+1/2} \max_{-1\le t\le 1} |\varphi_{l_2}(\tau)| \times$$

$$\sum_{k=l_1+1}^{\infty} \sqrt{k+1/2} \sum_{j=l_2+1}^{\infty} \sqrt{j+1/2} |\langle f, \varphi_{k,j}\rangle| \le$$

$$\le 4\|f\|_{\mu} \sum_{l_1=n+1}^{\infty} (l_1+1/2) \sum_{l_2=0}^{\infty} (l_2+1/2) \left(\sum_{k=l_1+1}^{\infty} \frac{k+1/2}{k^{2\mu}} \sum_{j=l_2+1}^{\infty} \frac{j+1/2}{j^{2\mu}} \right)^{1/2} \le$$

$$c\|f\|_{\mu} \sum_{l_1=n+1}^{\infty} \frac{1}{(l_1+1)^{\mu-2}} \sum_{l_2=0}^{\infty} \frac{1}{(l_2+1)^{\mu-2}} \le c\|f\|_{\mu} n^{-\mu+3}.$$

So,

$$\|\triangle_1\|_C \le c\|f\|_{\mu} n^{-\mu+3}.$$

It remains to bound $\|\triangle_2\|_C$. Taking into account (3), due to (12) we have

$$\|\triangle_{21}\|_C \le 4 \sum_{l_1=0}^{n-1} (l_1+1/2) \sum_{l_2=0}^{n} (l_2+1/2) \times$$

$$\times \sum_{k=l_1+1}^{n} \frac{\sqrt{k+1/2}}{k^{\mu}} \sum_{j=n+1}^{\infty} \frac{\sqrt{j+1/2}}{j^{\mu}} (kj)^{\mu} |\langle f, \varphi_{k,j}\rangle| \le$$

$$\le 4\|f\|_{\mu} \sum_{l_1=0}^{n-1} (l_1+1/2) \sum_{l_2=0}^{n} (l_2+1/2) \left(\sum_{k=l_1+1}^{n} \frac{k+1/2}{k^{2\mu}} \sum_{j=n+1}^{\infty} \frac{j+1/2}{j^{2\mu}} \right)^{1/2} \le$$

$$\le c\|f\|_{\mu} n^{-\mu+1} \sum_{l_1=0}^{n-1} \frac{1}{(l_1+1)^{\mu-2}} \sum_{l_2=0}^{n} (l_2+1/2) \le c\|f\|_{\mu} n^{-\mu+3}.$$

Similarly, from (13) we get

$$\|\triangle_{22}\|_C \le 4\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=n+1}^{\infty}(l_2+1/2)\times$$

$$\times\sum_{k=l_1+1}^{n}\frac{\sqrt{k+1/2}}{k^\mu}\sum_{j=l_2+1}^{\infty}\frac{\sqrt{j+1/2}}{j^\mu}(kj)^\mu|\langle f,\varphi_{k,j}\rangle| \le$$

$$\le 4\|f\|_\mu\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=n+1}^{\infty}(l_2+1/2)\left(\sum_{k=l_1+1}^{n}\frac{k+1/2}{k^{2\mu}}\sum_{j=l_2+1}^{\infty}\frac{j+1/2}{j^{2\mu}}\right)^{1/2} \le$$

$$\le c\|f\|_\mu\sum_{l_1=0}^{n-1}\frac{1}{(l_1+1)^{\mu-2}}\sum_{l_2=n+1}^{\infty}\frac{1}{(l_2+1)^{\mu-2}} \le c\|f\|_\mu n^{-\mu+3}.$$

Combining the estimates obtained above, we have

$$\|f^{(1,1)}-\mathcal{D}_n f\|_C \le \|\triangle_1\|_C + \|\triangle_2\|_C \le c\|f\|_\mu n^{-\mu+3}.$$

□

Lemma 4. *Let the condition (1) be satisfied. Then for arbitrary function $f \in L_2(Q)$ it holds true*

$$\|\mathcal{D}_n f - \mathcal{D}_n f_\delta\|_C \le c\delta n^6.$$

Proof. So, from (1) and (15) it follows

$$\|\mathcal{D}_n f - \mathcal{D}_n f_\delta\|_C \le 4\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{n-1}(l_2+1/2)\times$$

$$\times\sum_{k=l_1+1}^{n}\sqrt{k+1/2}\sum_{j=l_2+1}^{n}\sqrt{j+1/2}\,\xi_{k,j} \le$$

$$\le 4\delta\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{n-1}(l_2+1/2)\left(\sum_{k=l_1+1}^{n}(k+1/2)\sum_{j=l_2+1}^{n}(j+1/2)\right)^{1/2} \le$$

$$\le c\delta n^2\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{n-1}(l_2+1/2) \le c\delta n^6.$$

□

Combining Lemmas 3 and 4 we obtain

Theorem 2. *Let $\mu > 3$. Then for arbitrary function $f \in L^\mu_{2,2}(Q)$, $\|f\|_\mu \le 1$, with $n = O(\delta^{-\frac{1}{\mu+3}})$ it holds true*

$$\|f^{(1,1)}-\mathcal{D}_n f_\delta\|_C \le c\delta^{\frac{\mu-3}{\mu+3}}.$$

Corollary 6. *Under the assumptions of Theorem 2, for achieving the accuracy*

$$O\left(\delta^{\frac{\mu-3}{\mu+3}}\right)$$

the method (6) *requires the following amount of perturbed Fourier-Legendre coefficients*

$$card \asymp \mathrm{n}^2 \asymp \delta^{-\frac{2}{\mu+3}}$$

with the indices from the square $\{(k,j) : 1 \le k, j \le n; k, j \in \mathbb{N}\}$.

3. Hyperbolic Cross

In this section we consider the second version of the truncation method for differentiation of bivariate functions. This approach uses an idea of the so-called hyperbolic cross and consists in approximating the derivative (2) by the aggregate

$$\overline{\mathcal{D}}_n f_\delta(t,\tau) = \sum_{1\le kj\le n} \langle f_\delta, \varphi_{k,j}\rangle \varphi'_k(t)\varphi'_j(\tau). \tag{16}$$

It is easy to see that

$$f^{(1,1)}(t,\tau) - \overline{\mathcal{D}}_n f_\delta(t,\tau) = [f^{(1,1)}(t,\tau) - \overline{\mathcal{D}}_n f(t,\tau)] + [\overline{\mathcal{D}}_n f(t,\tau) - \overline{\mathcal{D}}_n f_\delta(t,\tau)]$$
$$=: \triangle_{11}(t,\tau) + \triangle_{12}(t,\tau) + \triangle_{13}(t,\tau) + \sum_{1\le kj\le n} \langle f - f_\delta, \varphi_{kj}\rangle \varphi'_k(t)\varphi'_j(\tau),$$

where $\triangle_{11}$, $\triangle_{12}$ were defined by (9), (10) and

$$\triangle_{13}(t,\tau) = 4\sum_{k=1}^{n} \sum_{j=\frac{n+1}{k}}^{n} \langle f, \varphi_{kj}\rangle \sideset{}{^*}\sum_{l_1=0}^{k-1} \sqrt{k+1/2}\sqrt{l_1+1/2}\varphi_{l_1}(t)\times$$
$$\times \sideset{}{^*}\sum_{l_2=0}^{j-1} \sqrt{j+1/2}\sqrt{l_2+1/2}\varphi_{l_2}(\tau). \tag{17}$$

Lemma 5. *For arbitrary function* $f \in L^\mu_{2,2}(Q)$, $\mu > 2$, *it holds true*

$$\|f^{(1,1)} - \bar{\mathcal{D}}_n f\|_2 \le c\|f\|_\mu n^{-\mu+2}\ln n.$$

Proof. We need to bound $\|\triangle_{13}\|_2$. Changing the summation order by the indices k, j, l_1, l_2 in (17) we immediately get

$$\triangle_{13}(t,\tau) = \triangle_{131}(t,\tau) + \triangle_{132}(t,\tau) + \triangle_{133}(t,\tau),$$

where

$$\triangle_{131}(t,\tau) = 4\sideset{}{^*}\sum_{l_1=0}^{n-1} \sqrt{l_1+1/2}\varphi_{l_1}(t) \sideset{}{^*}\sum_{l_2=0}^{\frac{n+1}{l_1+1}-1} \sqrt{l_2+1/2}\varphi_{l_2}(\tau)\times$$
$$\times \sum_{k=l_1+1}^{\frac{n+1}{l_2+1}} \sqrt{k+1/2} \sum_{j=\frac{n+1}{k}}^{n} \sqrt{j+1/2}\langle f, \varphi_{k,j}\rangle,$$

$$\triangle_{132}(t,\tau) = 4\sum_{l_1=0}^{n-1}{}^{*}\sqrt{l_1+1/2}\varphi_{l_1}(t)\sum_{l_2=0}^{\frac{n+1}{l_1+1}}{}^{*}\sqrt{l_2+1/2}\varphi_{l_2}(\tau)\times$$

$$\times\sum_{k=\frac{n+1}{l_2}}^{n}\sqrt{k+1/2}\sum_{j=l_2+1}^{n}\sqrt{j+1/2}\langle f,\varphi_{k,j}\rangle,$$

$$\triangle_{133}(t,\tau) = 4\sum_{l_1=0}^{n-1}{}^{*}\sqrt{l_1+1/2}\varphi_{l_1}(t)\sum_{l_2=\frac{n+1}{l_1+1}+1}^{n-1}{}^{*}\sqrt{l_2+1/2}\varphi_{l_2}(\tau)\times$$

$$\times\sum_{k=l_1+1}^{n}\sqrt{k+1/2}\sum_{j=l_2+1}^{n}\sqrt{j+1/2}\langle f,\varphi_{k,j}\rangle.$$

It remains to bound the functions $\triangle_{131}$, $\triangle_{132}$, $\triangle_{133}$ in the metric of $L_2(Q)$. Since

$$\left(\sum_{k=l_1+1}^{\frac{n+1}{l_2+1}}\sqrt{k+1/2}\sum_{j=\frac{n+1}{k}}^{n}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\rangle\right)^2 \le$$

$$\le c\|f\|_\mu^2\sum_{k=l_1+1}^{\frac{n+1}{l_2+1}}\frac{1}{k^{2\mu-1}}\sum_{j=\frac{n+1}{k}}^{n}\frac{1}{j^{2\mu-1}} =$$

$$= c\|f\|_\mu^2 n^{-2\mu+2}\sum_{k=l_1+1}^{\frac{n+1}{l_2+1}}\frac{1}{k} = c\|f\|_\mu^2 n^{-2\mu+2}\ln n,$$

then

$$\|\triangle_{131}\|_2^2 \le 16\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{\frac{n+1}{l_1+1}-1}(l_2+1/2)\times$$

$$\times\left(\sum_{k=l_1+1}^{\frac{n+1}{l_2+1}}\sqrt{k+1/2}\sum_{j=\frac{n+1}{k}}^{n}\sqrt{j+1/2}\,\langle f,\varphi_{k,j}\rangle\right)^2 \le$$

$$\le c\|f\|_\mu^2 n^{-2\mu+2}\ln n\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{\frac{n+1}{l_1+1}-1}(l_2+1/2) =$$

$$= c\|f\|_\mu^2 n^{-2\mu+4}\ln n\sum_{l_1=0}^{n-1}\frac{1}{l_1+1} = c\|f\|_\mu^2 n^{-2\mu+4}\ln^2 n.$$

Further due to

$$\left(\sum_{k=\frac{n+1}{l_2}}^{n}\sqrt{k+1/2}\sum_{j=l_2+1}^{n}\sqrt{j+1/2}\langle f,\varphi_{k,j}\rangle\right)^2 \le$$

$$\leq c\|f\|_\mu^2 \sum_{k=\frac{n+1}{l_2}}^{n} \frac{1}{k^{2\mu-1}} \sum_{j=l_2+1}^{n} \frac{1}{j^{2\mu-1}} =$$

$$= c\|f\|_\mu^2 (l_2+1)^{-2\mu+2} \sum_{k=\frac{n+1}{l_2}}^{n} \frac{1}{k^{2\mu-1}} = c\|f\|_\mu^2 n^{-2\mu+2}$$

we have

$$\|\triangle_{132}\|_2^2 \leq 16 \sum_{l_1=0}^{n-1} (l_1+1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}} (l_2+1/2) \times$$

$$\times \left(\sum_{k=\frac{n+1}{l_2}}^{n} \sqrt{k+1/2} \sum_{j=l_2+1}^{n} \sqrt{j+1/2}\, \langle f, \varphi_{k,j} \rangle \right)^2 \leq$$

$$\leq c\|f\|_\mu^2 n^{-2\mu+2} \sum_{l_1=0}^{n-1} (l_1+1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}} (l_2+1/2) =$$

$$= c\|f\|_\mu^2 n^{-2\mu+4} \sum_{l_1=0}^{n-1} \frac{1}{l_1+1} \leq c\|f\|_\mu^2 n^{-2\mu+4} \ln n.$$

Similarly, we find that

$$\left(\sum_{k=l_1+1}^{n} \sqrt{k+1/2} \sum_{j=l_2+1}^{n} \sqrt{j+1/2}\, \langle f, \varphi_{k,j} \rangle \right)^2 \leq$$

$$\leq c\|f\|_\mu^2 \sum_{k=l_1+1}^{n} \frac{1}{k^{2\mu-1}} \sum_{j=l_2+1}^{n} \frac{1}{j^{2\mu-1}} =$$

$$= c\|f\|_\mu^2 (l_2+1)^{-2\mu+2} \sum_{k=l_1+1}^{n} \frac{1}{k^{2\mu-1}} \leq c\|f\|_\mu^2 [(l_1+1)(l_2+1)]^{-2\mu+2}.$$

Then

$$\|\triangle_{133}\|_2^2 \leq 16 \sum_{l_1=0}^{n-1} (l_1+1/2) \sum_{l_2=\frac{n+1}{l_1+1}+1}^{n-1} (l_2+1/2) \times$$

$$\times \left(\sum_{k=l_1+1}^{n} \sqrt{k+1/2} \sum_{j=l_2+1}^{n} \sqrt{j+1/2}\, \langle f, \varphi_{k,j} \rangle \right)^2 \leq$$

$$\leq c\|f\|_\mu^2 \sum_{l_1=0}^{n-1} \frac{1}{(l_1+1)^{2\mu-3}} \sum_{l_2=\frac{n+1}{l_1+1}+1}^{n-1} \frac{1}{(l_2+1)^{2\mu-3}} =$$

$$= c\|f\|_\mu^2 n^{-2\mu+4} \sum_{l_1=0}^{n-1} \frac{1}{l_1+1} \leq c\|f\|_\mu^2 n^{-2\mu+4} \ln n.$$

Thus we have

$$\|\triangle_{13}\|_2 \le c\|f\|_\mu n^{-\mu+2}\ln n. \tag{18}$$

Combining (18) and (11) we get the assertion of Lemma. □

Lemma 6. *Let the condition (1) be satisfied. Then for arbitrary function $f \in L_2(Q)$ it holds true*

$$\|\bar{\mathcal{D}}_n f - \bar{\mathcal{D}}_n f_\delta\|_2 \le c\delta n^2 \ln n.$$

Proof. Taking into account the decomposition

$$\bar{\mathcal{D}}_n f(t,\tau) - \bar{\mathcal{D}}_n f_\delta(t,\tau) = 4 \sum_{l_1=0}^{n-1}{}^{*} \sum_{l_2=0}^{\frac{n}{l_1+1}-1}{}^{*} \sqrt{l_1+1/2}\sqrt{l_2+1/2}\ \varphi_{l_1}(t)\varphi_{l_2}(\tau)\times$$

$$\times \sum_{k=l_1+1}^{\frac{n}{l_2+1}} \sum_{j=l_2+1}^{\frac{n}{k}} \sqrt{k+1/2}\sqrt{j+1/2}\ \xi_{k,j} \tag{19}$$

we have

$$\|\bar{\mathcal{D}}_n f - \bar{\mathcal{D}}_n f_\delta\|_2^2 \le 16 \sum_{l_1=0}^{n-1} \sum_{l_2=0}^{\frac{n}{l_1+1}-1} \times$$

$$\times (l_1+1/2)(l_2+1/2)\left(\sum_{k=l_1+1}^{\frac{n}{l_2+1}} \sum_{j=l_2+1}^{\frac{n}{k}} \sqrt{k+1/2}\sqrt{j+1/2}\ \xi_{k,j}\right)^2 \le$$

$$\le 16\delta^2 \sum_{l_1=0}^{n-1} \sum_{l_2=0}^{\frac{n}{l_1+1}-1} (l_1+1/2)(l_2+1/2) \sum_{k=l_1+1}^{\frac{n}{l_2+1}} (k+1/2) \sum_{j=l_2+1}^{\frac{n}{k}} (j+1/2) \le$$

$$\le c\delta^2 n^2 \sum_{l_1=0}^{n-1} \sum_{l_2=0}^{\frac{n}{l_1+1}-1} (l_1+1/2)(l_2+1/2) \sum_{k=l_1+1}^{\frac{n}{l_2+1}} \frac{k+1/2}{k^2} \le$$

$$\le c\delta^2 n^2 \ln n \sum_{l_1=0}^{n-1}(l_1+1/2) \sum_{l_2=0}^{\frac{n}{l_1+1}-1} (l_2+1/2) \le c\delta^2 n^4 \ln^2 n.$$

□

The combination of Lemmas 5 and 6 leads to

Theorem 3. *Let $\mu > 2$. Then for arbitrary function $f \in L^\mu_{2,2}(Q)$, $\|f\|_\mu \le 1$, with $n = O(\delta^{-\frac{1}{\mu}})$ it holds true*

$$\|f^{(1,1)} - \bar{\mathcal{D}}_n f_\delta\|_2 \le c\delta^{\frac{\mu-2}{\mu}} \ln\frac{1}{\delta}.$$

Corollary 7. *Under the assumptions of Theorem 3, for achieving the accuracy*

$$O\left(\delta^{\frac{\mu-2}{\mu}} |\ln\delta|\right)$$

the method (16) *requires the following amount of perturbed Fourier-Legendre coefficients*

$$card \asymp \mathrm{n} \ln \mathrm{n} \asymp \delta^{-\frac{1}{\mu}} |\ln \delta|$$

with indices from uniform hyperbolic cross $\{(k, j) : 1 \leq kj \leq n; k, j \in \mathbb{N}\}$.

Remark 4. *A comparison of Corollaries 5 and 7 allows us to conclude that on the class* $L^{\mu}_{2,2}(Q)$ *the modified approach (16) is more effective compared to the standard approach* (6) *in the sense of the accuracy and amount of the Fourier-Legendre coefficients, in the case when the approximation error is measured in the metric of the space* $L_2(Q)$.

Let's estimate the error of the method (16) in the uniform metric.

Lemma 7. *For arbitrary function* $f \in L^{\mu}_{2,2}(Q)$, $\mu > 3$, *it holds true*

$$\|f^{(1,1)} - \bar{\mathcal{D}}_n f\|_C \leq c\|f\|_{\mu} n^{-\mu+3} \ln^{3/2} n.$$

Proof. Let $\mu > 3$. Then we have

$$\|\triangle_{131}\|_C \leq 4 \sum_{l_1=0}^{n-1} (l_1 + 1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}-1} (l_2 + 1/2) \times$$

$$\times \sum_{k=l_1+1}^{\frac{n+1}{l_2+1}} \sum_{j=\frac{n+1}{k}}^{n} \sqrt{k + 1/2}\sqrt{j + 1/2} \frac{(kj)^{\mu}}{(kj)^{\mu}} |\langle f, \varphi_{k,j} \rangle| \leq$$

$$\leq 4\|f\|_{\mu} \sum_{l_1=0}^{n-1} (l_1 + 1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}-1} (l_2 + 1/2) \left(\sum_{k=l_1+1}^{\frac{n+1}{l_2+1}} \frac{k + 1/2}{k^{2\mu}} \sum_{j=\frac{n+1}{k}}^{n} \frac{j + 1/2}{j^{2\mu}} \right)^{1/2} \leq$$

$$c\|f\|_{\mu} n^{-\mu+1} \ln^{1/2} n \sum_{l_1=0}^{n-1} (l_1 + 1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}-1} (l_2 + 1/2) \leq c\|f\|_{\mu} n^{-\mu+3} \ln^{3/2} n.$$

Similarly, we find

$$\|\triangle_{132}\|_c \leq 4 \sum_{l_1=0}^{n-1} (l_1 + 1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}} (l_2 + 1/2) \times$$

$$\times \sum_{k=\frac{n+1}{l_2}}^{n} \sum_{j=l_2+1}^{n} \sqrt{k + 1/2}\sqrt{j + 1/2} \frac{(kj)^{\mu}}{(kj)^{\mu}} |\langle f, \varphi_{k,j} \rangle| \leq$$

$$\leq 4\|f\|_{\mu} \sum_{l_1=0}^{n-1} (l_1 + 1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}} (l_2 + 1/2) \left(\sum_{k=\frac{n+1}{l_2}}^{n} \frac{k + 1/2}{k^{2\mu}} \sum_{j=l_2+1}^{n} \frac{j + 1/2}{j^{2\mu}} \right)^{1/2} \leq$$

$$\leq c\|f\|_{\mu} n^{-\mu+1} \sum_{l_1=0}^{n-1} (l_1 + 1/2) \sum_{l_2=0}^{\frac{n+1}{l_1+1}} (l_2 + 1/2) = c\|f\|_{\mu} n^{-\mu+3} \ln n,$$

$$\|\triangle_{133}\|_C \le 4\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=\frac{n+1}{l_1+1}+1}^{n-1}(l_2+1/2)\times$$

$$\times\sum_{k=l_1+1}^{n}\sum_{j=l_2+1}^{n}\sqrt{k+1/2}\sqrt{j+1/2}\frac{(kj)^\mu}{(kj)^\mu}\,|\langle f,\varphi_{k,j}\rangle| \le$$

$$\le 4\|f\|_\mu\sum_{l_1=0}^{n-1}(l_1{+}1/2)\sum_{l_2=\frac{n+1}{l_1+1}+1}^{n-1}(l_2{+}1/2)\left(\sum_{k=l_1+1}^{n}\frac{k+1/2}{k^{2\mu}}\sum_{j=l_2+1}^{n}\frac{j+1/2}{j^{2\mu}}\right)^{1/2} =$$

$$c\|f\|_\mu\sum_{l_1=0}^{n-1}\frac{1}{(l_1+1)^{\mu-2}}\sum_{l_2=\frac{n+1}{l_1+1}+1}^{n-1}\frac{1}{(l_2+1)^{\mu-2}} \le c\|f\|_\mu n^{-\mu+3}\ln n.$$

Combining obtained above bounds we have

$$\|\triangle_{13}\|_C \le c\|f\|_\mu n^{-\mu+3}\ln^{3/2} n.$$

Taking into account upper bounds for $\|\triangle_{11}\|_C$ and $\|\triangle_{12}\|_C$ from Lemma 3, we get the assertion of Lemma. □

Lemma 8. *Let the condition (1) be satisfied. Then for arbitrary function $f \in L_2(Q)$ it holds true*

$$\|\bar{\mathcal{D}}_n f - \bar{\mathcal{D}}_n f_\delta\|_C \le c\delta n^3 \ln^{3/2} n.$$

Proof. By means of (19), (1) and (3) we have

$$\|\bar{\mathcal{D}}_n f - \bar{\mathcal{D}}_n f_\delta\|_C \le 4\sum_{l_1=0}^{n-1}\sum_{l_2=0}^{\frac{n}{l_1+1}-1}(l_1+1/2)(l_2+1/2)\times$$

$$\times\sum_{k=l_1+1}^{\frac{n}{l_2+1}}\sum_{j=l_2+1}^{\frac{n}{k}}\sqrt{k+1/2}\sqrt{j+1/2}\;\xi_{k,j} \le$$

$$\le 4\delta\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{\frac{n}{l_1+1}-1}(l_2+1/2)\left(\sum_{k=l_1+1}^{\frac{n}{l_2+1}}(k+1/2)\sum_{j=l_2+1}^{\frac{n}{k}}(j+1/2)\right)^{1/2} \le$$

$$\le c\delta n\ln^{1/2} n\sum_{l_1=0}^{n-1}(l_1+1/2)\sum_{l_2=0}^{\frac{n}{l_1+1}-1}(l_2+1/2) \le$$

$$\le c\delta n^3\ln^{1/2} n\sum_{l_1=0}^{n-1}\frac{1}{l_1+1} \le c\delta n^3\ln^{3/2} n.$$

□

Combining Lemmas 7 and 8 we obtain

Theorem 4. *Let $\mu > 3$. Then for arbitrary function $f \in L^{\mu}_{2,2}(Q)$, $\|f\|_\mu \le 1$, with $n = O(\delta^{-\frac{1}{\mu}})$ it holds true*

$$\|f^{(1,1)} - \bar{\mathcal{D}}_n f_\delta\|_C \le c\delta^{\frac{\mu-3}{\mu}} \ln^{3/2}\frac{1}{\delta}.$$

Corollary 8. *Under the assumptions of Theorem 4, for achieving the accuracy*

$$O\left(\delta^{\frac{\mu-3}{\mu}} |\ln\delta|^{3/2}\right)$$

the method (16) *requires the following amount of perturbed Fourier-Legendre coefficients*

$$card \asymp \mathrm{n}\ln\mathrm{n} \asymp \delta^{-\frac{1}{\mu}} |\ln\delta|$$

with indices from uniform hyperbolic cross $\{(k,j) : 1 \le kj \le n; k,j \in \mathbb{N}\}$.

Remark 5. *A comparison of Corollaries 6 and 8 allows to conclude that on the class $L^{\mu}_{2,2}(Q)$ the modified approach (16) is more effective compared to the standard approach* (6) *in the sense of the accuracy and amount of used Fourier-Legendre coefficients, in the case when the approximation error is measured in the metric of the space $C(Q)$.*

Remark 6. *In [5] it was investigated the application of the truncation method to recover the first derivative of the univariate functions, in particular, from the class*

$$L^{\mu}_{2,1}[-1,1] = \{f \in L_2[-1,1] : \sum_{k=0}^{\infty} k^{2\mu} |\langle f, \varphi_k\rangle|^2 \le 1\}.$$

For the method

$$\tilde{D}_n f_\delta(t) = \sum_{k=1}^{n} \langle f_\delta, \varphi_k\rangle \varphi_k(t)$$

in [5] it was proved, that on the class $L^{\mu}_{2,1}[-1,1]$ the following accuracy estimates are guaranteed

$$\|f' - \tilde{D}_n f_\delta\|_2 = O(\delta^{\frac{\mu-2}{\mu}}), \quad \text{if } \mu > 2, \quad n = O(\delta^{-\frac{1}{\mu}}), \tag{20}$$

$$\|f' - \tilde{D}_n f_\delta\|_C = O(\delta^{\frac{\mu-3}{\mu}}), \quad \text{if } \mu > 3, \quad n = O(\delta^{-\frac{1}{\mu}}). \tag{21}$$

Comparing the estimates (20) and (21) with the results of Corollaries 7, 8 respectively, we can to conclude that in the problem of numerical differentiation we managed to move from the univariate functions to the bivariate functions without losing basic accuracy and information costs.

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