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INTEGRAL EQUATION METHOD FOR BOUNDARY VALUE PROBLEMS IN MULTIPLY CONNECTED DOMAINS FOR THE TWO-DIMENSIONAL LAPLACE EQUATION

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Резюме. Розглядаються задачі Діріхле та Неймана для двовимірного рівняння Лапласа в області обмеженій двома гладкими замкнутими контурами. Розв'язок задач подається у вигляді суми потенціалів подвійного шару з невідомими густинами. Досліджено питання існування та єдиності розв'язків поставлених задач у відповідних функціональних просторах. Використовуючи інтегральне подання розв'язків вихідні диференціальні задачі зведені до систем граничних інтегральних та сингулярних інтегро-диференціальних рівнянь. Оскільки отримані системи мають не єдиний розв'язок, запропоновано підхід на основі використання модифікованих систем граничних рівнянь, розв'язки яких є єдиними. Як наслідок ми отримуємо шукані густини інтегрального подання розв'язків задач Діріхле та Неймана, які задовольняють певним інтегральним співвідношенням.

Abstract. We consider Dirichlet and Neumann boundary value problems for the two-dimensional Laplace equation in multiply-connected domain bounded by two smooth closed curves. The solutions of this problems we present as a sum of potentials of double layer with unknown densities. Existence and uniqueness of solutions of the posed problems in appropriate functional spaces is proved. Using integral representation of solutions of the initial boundary value problems we obtain some systems of boundary integral and singular integro-differential equations. Inasmuch the obtained systems have not unique solutions we consider some approach based on modified system of boundary equations which have unique solutions. As a result we get densities of integral representations of the solutions of the Dirichlet and Neumann boundary value problems which satisfies some additional integral conditions.

1. Introduction

Using of the boundary equation method for solving of boundary value problems in many cases gives us opportunity to apply different types of integral representations of solution of initial differential problem. At the same time depending on the type of representation we obtain solutions whose differential properties essentially differ if we consider the jump through the boundary of domain. Also obtained boundary equations have principally different properties depend on integral representation. For instance this equations may have not unique solutions and solutions itself may satisfy some additional conditions. As a result we get systems of boundary equations which have not unique solutions

Key words. Dirichlet and Neumann boundary value problems; double layer potentials; integral and singular integro-differential equations.

and during the numerical solving of this systems we are needed to pose additional integral conditions which on its turn complicates theoretical analysis of convergence.

In [5,7] for solving of the such type equations it was proposed the procedure of using certain modificated equations whose solutions are unique. In [8] it was considered general approach for solving of linear equations with not unique solutions based on some extension of the given operators.

In addition when we construct mathematical models of some physical processes it is necessary to take into account certain transmission conditions. For instance such as a continuity of the solution of boundary value problem itself or his normal derivative in the transition through the boundary of the domain. This imposes some restrictions on the integral representation that on its turn narrows the choice of the boundary data for which the solution exists.

In present paper we consider Dirichlet and Neumann boundary value problems in domain which boundary consists of the two smooth closed curves such that one of them lies inside of another. The solutions of these problems we look for as a sum of the potentials of the double layer over the given curves. Using boundary conditions we can reduce differential problems to the systems of boundary equations and main problem is that these systems have not unique solutions. Thus we try to solve some modificated systems and show the uniqueness and existence of their solutions. As a result we get the choosing densities of the integral representation which gives us the solutions of the initial boundary value problems.

2. Functional spaces and trace operators

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are bounded connected domains. Their boundary curves $\Sigma_1, \Sigma_2 \in C^{1,\alpha}$ and have no self-intersections. $\overline{\Omega}_i = \Omega_i \cup \Sigma_i$, $i = 1, 2$. We suppose that $\overline{\Omega}_2 \subset \Omega_1$, $\mathrm{diam}\Omega_2 \neq 1$ and denote $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, $\Omega_- = \mathbb{R}^2 \setminus \overline{\Omega}_1$. We can define outward pointing unit normal $\vec{n}_x$ and tangent unit vector $\vec{s}_x$ respectively for Ω_1 and Ω_2, $x \in \Sigma_1$ or $x \in \Sigma_2$.

In Ω we consider the Laplace operator

$$Lu = -\Delta u = -\sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2$$

and fundamental solution of L

$$Q(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}, \quad x \neq y, \quad L_x Q(x,y) = \delta(|x-y|).$$

We use the Hilbert spaces $H^1(\Omega)$ and $H^1(\Omega, L)$ of real functions with norms and inner products

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \left\{ |\nabla u|^2 + u^2 \right\} dx,$$

$$(u,v)_{H^1(\Omega)} = \int_\Omega \{ (\nabla u, \nabla v) + uv \} \, dx,$$

$$\|u\|^2_{H^1(\Omega,L)} = \|u\|^2_{H^1(\Omega)} + \|Lu\|^2_{L_2(\Omega)},$$

$$(u,v)_{H^1(\Omega,L)} = (u,v)_{H^1(\Omega)} + (Lu, Lv)_{L_2(\Omega)}.$$

We have the following trace operators in Ω_1, Ω and Ω_- which are continuous and surjective [1, 4]:

$$\gamma_0 = (\gamma_{0,1}^+, \gamma_{0,2}^-) : H^1(\Omega) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2),$$
$$\gamma_{0,1}^- : H^1(\Omega_-) \to H^{1/2}(\Sigma_1), \quad \gamma_{0,2}^+ : H^1(\Omega_2) \to H^{1/2}(\Sigma_2),$$
$$\gamma_1 = (\gamma_{1,1}^+, \gamma_{1,2}^-) : H^1(\Omega, L) \to H^{-1/2}(\Sigma_1) \times H^{-1/2}(\Sigma_2)$$
$$\gamma_{1,1}^- : H^1(\Omega_-, L) \to H^{-1/2}(\Sigma_1), \quad \gamma_{1,2}^+ : H^1(\Omega_2, L) \to H^{-1/2}(\Sigma_2).$$

Here $H^{-1/2}(\Sigma_i) = (H^{1/2}(\Sigma_i))'$, $i = 1, 2$.

We use the first Green's formula in Ω for $u \in H^1(\Omega, L)$ and $v \in H^1(\Omega)$:

$$\int_\Omega (\nabla u, \nabla v)dx = (Lu, v)_{L_2(\Omega)} + \langle \gamma_{1,1}^+ u, \gamma_{0,1} v\rangle - \langle \gamma_{1,2}^- u, \gamma_{0,2}^- v\rangle. \tag{1}$$

Here $\langle \cdot, \cdot \rangle$ are relations of duality between $H^{1/2}(\Sigma_1)$, $H^{-1/2}(\Sigma_1)$ and $H^{1/2}(\Sigma_2)$, $H^{-1/2}(\Sigma_2)$ respectively.

For $\tau_1 \in H^{-1/2}(\Sigma_1)$, $\mu_1 \in H^{1/2}(\Sigma_1)$ we consider the following potentials in $\Omega_1 \cup \Omega_-$:

$$V_1\tau_1(x) = \int_{\Sigma_1} Q(x,y)\tau_1(y)ds_y, \quad W_1\mu_1(x) = \int_{\Sigma_1} \frac{\partial Q(x,y)}{\partial n_y}\mu_1(y)ds_y.$$

Potentials of simple $V_1\tau_1$ and double layers $W_1\mu_1$ satisfy the jump relations which can be written in the next form [1].

Lemma 1. *Let* $\tau_1 \in H^{-1/2}(\Sigma_1)$, $\mu_1 \in H^{1/2}(\Sigma_1)$ *and* $[\gamma_{0,1}] = \gamma_{0,1}^+ - \gamma_{0,1}^-$, $[\gamma_{1,1}] = \gamma_{1,1}^+ - \gamma_{1,1}^-$. *Then:*
1.$[\gamma_{0,1}]V_1\tau_1 = 0, \quad [\gamma_{1,1}]V_1\tau_1 = \tau_1.$
2.$[\gamma_{0,1}]W_1\mu_1 = -\mu_1, \quad [\gamma_{1,1}]W_1\mu_1 = 0.$

If we introduce the operators

$$N_1\tau_1 = \frac{1}{2}(\gamma_{1,1}^+ V_1\tau_1 + \gamma_{1,1}^- V_1\tau_1), \quad M_1\mu_1 = \frac{1}{2}(\gamma_{0,1}^+ W_1\mu_1 + \gamma_{0,1}^- W_1\mu_1),$$

we can rewrite jump relations as

$$\gamma_{1,1}^\pm V_1\tau_1 = \pm\frac{1}{2}\tau_1 + N_1\tau_1, \quad \gamma_{0,1}^\pm W_1\mu_1 = \mp\frac{1}{2}\mu_1 + M_1\mu_1,$$

where

$$M_1\mu_1(x) = \int_{\Sigma_1} \frac{\partial Q(x,y)}{\partial n_y}\mu_1(y)ds_y, \quad x \in \Sigma_1.$$

Let us denote: $H_1 = \gamma_{1,1}^\pm W_1$, $B_1^\pm = \gamma_{1,1}^\pm V_1$, $C_1^\pm = \gamma_{0,1}^\pm W_1$.

If $\tau_1 \in L_2(\Sigma_1)$ then

$$N_1\tau_1(x) = \int_{\Sigma_1} \frac{\partial Q(x,y)}{\partial n_x}\tau_1(y)ds_y, \quad x \in \Sigma_1.$$

In the same way we consider potentials of simple and double layers for the curve Σ_2. Index 1 or 2 will be connected with curve Σ_1 or Σ_2 respectively.

3. Dirichlet boundary value problem

Let us state the following boundary value problem in domain Ω.

Problem D. Find a function $u \in H^1(\Omega)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega,$$

and boundary conditions

$$\gamma_{0,1}^{+} u = g_1, \qquad \gamma_{0,2}^{-} u = g_2. \tag{2}$$

Here $g_i \in H^{1/2}(\Sigma_i)$, $i = 1, 2$, are given.

Since the trace operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2)$ is surjective it's easy to verify that problem D has unique solution for arbitrary $g_i \in H^{1/2}(\Sigma_i)$, $i = 1, 2$.

We look for the solution of the problem D as a sum of potentials of the double layer:

$$u(x) = W_1\mu_1 + W_2\mu_2. \tag{3}$$

Here $\mu_1 = \gamma_{0,1}^{+} u - \gamma_{0,1}^{-} u$, $\mu_2 = \gamma_{0,2}^{+} u - \gamma_{0,2}^{-} u$.

This approach is connected with boundary value problem for stationary heat equation in domain Ω when heat flows through the boundaries Σ_1 and Σ_2 are continuous.

Then the solution of the problem D satisfies the next conditions:

$$\langle \gamma_{1,1}^{+} u, \nu_0 \rangle = 0, \quad \langle \gamma_{1,2}^{-} u, \mu_0 \rangle = 0. \tag{4}$$

Here $C_1^{+}\nu_0 = 0$, $\nu_0(x) = 1$, $x \in \Sigma_1$, $C_2^{-}\mu_0 = 0$, $\mu_0(x) = 1$, $x \in \Sigma_2$.

If we use boundary conditions (2) we obtain the following system of integral equations:

$$\begin{cases} C_1^{+}\mu_1 + W_{2,1}\mu_2 = g_1, \\ W_{1,2}\mu_1 + C_2^{-}\mu_2 = g_2, \end{cases} \tag{5}$$

where $W_{2,1}\mu_2(x) = \gamma_{0,1}^{+} W_2\mu_2(x)$ and $W_{1,2}\mu_1(x) = \gamma_{0,2}^{-} W_1\mu_1(x)$.

The integral representation (3) of the solution u of problem D via the sum of potentials of double layer is connected with the following Dirichlet boundary value problem of transmission type.

Problem DT. Find a function $u \in H^1(\Omega_2) \cup H^1(\Omega) \cup H^1(\Omega_-)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega_2 \cup \Omega \cup \Omega_-,$$

boundary conditions

$$\begin{cases} \gamma_{0,1}^{+} u = g_1, & \gamma_{0,2}^{-} u = g_2, \\ \gamma_{1,1}^{+} u = \gamma_{1,1}^{-} u, & \gamma_{1,2}^{+} u = \gamma_{1,2}^{-} u, \end{cases}$$

and condition at infinity

$$\lim_{|x| \to \infty} u(x) = 0,$$

where $g_i \in H^{1/2}(\Sigma_i)$, $i = 1, 2$.

The problem DT is equivalent to the system (5), i.e. solution of the problem DT has representation (3), where μ_1, μ_2 are solutions of the system (5) and vice versa the function (3) where μ_1, μ_2 are solutions of the system (5) is a solution of the problem DT.

Let us note that function $u = W_2\mu_0$ is a solution of the problem DT with boundary conditions $g_1 = 0$, $g_2 = 0$.

Since the system (5) has not unique solution, i.e. homogeneous system has solution $(0, \mu_0)$, instead of the system (5) we will use the following modificated system:

$$\begin{cases} C_1^+\mu_1 + W_{2,1}\sigma_2 = g_1, \\ W_{1,2}\mu_1 + C_2^-\sigma_2 + (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0 = h_2. \end{cases} \tag{6}$$

We can rewrite the system (6) in the next integral form:

$$\begin{cases} -\frac{1}{2}\mu_1(x) + \int_{\Sigma_1} \frac{\partial Q(x,y)}{\partial n_y}\mu_1(y)ds_y + \\ \qquad + \int_{\Sigma_2} \frac{\partial Q(x,y)}{\partial n_y}\sigma_2(y)ds_y = g_1(x), \quad x \in \Sigma_1, \\ \int_{\Sigma_1} \frac{\partial Q(x,y)}{\partial n_y}\mu_1(y)ds_y + \frac{1}{2}\sigma_2(x) + \\ \qquad + \int_{\Sigma_2} \{\frac{\partial Q(x,y)}{\partial n_y} + 1\}\sigma_2(y)ds_y = h_2(x), \quad x \in \Sigma_2. \end{cases}$$

Theorem 1. *The system (6) has unique solution (μ_1, σ_2) for arbitrary $g_1 \in H^{1/2}(\Sigma_1)$, $h_2 \in H^{1/2}(\Sigma_2)$.*

Proof. Let $g_1 = 0$, $h_2 = 0$ and μ_1, σ_2 are the solutions of the following homogeneous system

$$\begin{cases} C_1^+\mu_1 + W_{2,1}\sigma_2 = 0, \\ W_{1,2}\mu_1 + C_2^-\sigma_2 + (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0 = 0. \end{cases}$$

From [8] we obtain $\sigma_2 = \mu_2 + \frac{\alpha}{c_g}\mu_0$, where $(\mu_2, \mu_0)_{L_2(\Sigma_2)} = 0$, $c_g = \|\mu_0\|^2_{L_2(\Sigma_2)} = |\Sigma_2|$ - the length of the curve Σ_2 and $\alpha = (\sigma_2, \mu_0)_{L_2(\Sigma_2)}$. Then $W_{2,1}\sigma_2 = W_{2,1}\mu_2$ and μ_1, μ_2 are the solutions of the system

$$\begin{cases} C_1^+\mu_1 + W_{2,1}\mu_2 = 0, \\ W_{1,2}\mu_1 + C_2^-\mu_2 = -\alpha\mu_0. \end{cases} \tag{7}$$

Then the function $u = W_1\mu_1 + W_2\mu_2$ is a solution of the problem D with condition $g_1 = 0$ and $g_2 = -\alpha\mu_0$.

From the first Green's formula (1) it follows:

$$\int_\Omega |u(x)|^2 dx = \langle\gamma_{1,1}^+ u, \gamma_{0,1}^+ u\rangle - \langle\gamma_{1,2}^- u, \gamma_{0,2}^- u\rangle =$$
$$= \alpha\langle\gamma_{1,2}^- u, \mu_0\rangle = \alpha\langle\gamma_{1,2}^+ u, \mu_0\rangle = 0.$$

Thus $u(x) = \text{const}$, $x \in \Omega$. Since $\gamma_{0,1}^+ u = 0$ then $u(x) = 0$, $x \in \Omega$. Also using jump relations we have $\gamma_{1,1}^- u = \gamma_{1,1}^+ u = 0$. If $x \in \Omega_-$ then function u is a solution of the Neumann problem for the Laplace equation in Ω_- with boundary condition $\gamma^-_{1,1}u = 0$ and condition at infinity $\lim_{|x|\to\infty} u(x) = 0$. Thus $u(x) = 0$, $x \in \Omega_-$ and $\gamma^-_{0,1}u = 0$. Inasmuch $\mu_1 = \gamma^+_{0,1}u - \gamma^-_{0,1}u$ we have $\mu_1 = 0$. From the system (7) we obtain $C^-_2\mu_2 = -\alpha\mu_0$. Since $(\mu_2, \mu_0)_{L_2(\Sigma_2)} = 0$ from [5, 6] it follows that $\alpha\langle\tau_2, \mu_0\rangle = 0$ where $\gamma^+_{0,2}V_2\tau_2 = 1$. So far as $\langle\tau_2, \mu_0\rangle \neq 0$ [8] we have $\alpha = 0$. Then $C^-_2\mu_2 = 0$ and $\mu_2 = 0$. As a consequence we have $\sigma_2 = 0$.

Now we consider the existence of solution of the system (6). We can rewrite system (6) in the following operator form:

$$\mathbf{C}\vec{\mu} = \begin{pmatrix} C^+_1 & W_{2,1} \\ W_{1,2} & C^-_{2,1} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ h_2 \end{pmatrix},$$

where $\vec{\mu} = (\mu_1, \sigma_2)$, $C^-_{2,1}\sigma_2 = C^-_2 + (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0$.

Operator $\mathbf{C} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2)$ and $\mathbf{C} = \mathbf{A} + \mathbf{B}$ where

$$\mathbf{A} = \begin{pmatrix} C^+_1 & 0 \\ 0 & C^-_{2,1} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & W_{2,1} \\ W_{1,2} & 0 \end{pmatrix}.$$

The operators $C^+_1 : H^{1/2}(\Sigma_1) \to H^{1/2}(\Sigma_1)$ and $C^-_{2,1} : H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_2)$ are isomorphisms [5, 8].

Thus the operator $\mathbf{A} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2)$ is isomorphism.

So far as operators $W_{2,1} : H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_1)$ and $W_{1,2} : H^{1/2}(\Sigma_1) \to H^{1/2}(\Sigma_2)$ are compact [3, 6], then operator $\mathbf{B} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2)$ is compact.

As a consequence we have that operator $\mathbf{C} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2)$ is a Fredholm operator of index zero and the system (6) has unique solution for arbitrary $g_1 \in H^{1/2}(\Sigma_1)$, $h_2 \in H^{1/2}(\Sigma_2)$. □

Theorem 2. *The problem D has unique solution for arbitrary $g_1 \in H^{1/2}(\Sigma_1)$ and for $g_2 = h_2 - (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0$, where σ_2 is solution of the system (6) and $h_2 \in H^{1/2}(\Sigma_2)$ is arbitrary.*

If (μ_1, σ_2) is a solution of the system (6), then the function $u = W_1\mu_1 + W_2\mu_2$, where $\mu_2 = \sigma_2 - \frac{(\sigma_2,\mu_0)_{L_2(\Sigma_2)}}{\|\mu_0\|^2_{L_2(\Sigma_2)}}\mu_0$ and $(\mu_2, \mu_0)_{L_2(\Sigma_2)} = 0$, is a solution of the problem D with boundary conditions $g_1 \in H^{1/2}(\Sigma_1)$ and $g_2 = h_2 - (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0$, $h_2 \in H^{1/2}(\Sigma_2)$.

Proof. Let $g_1 \in H^{1/2}(\Sigma_1)$, $h_2 \in H^{1/2}(\Sigma_2)$ are given and (μ_1, σ_2) is the solution of the system (6). If $\mu_2 = \sigma_2 - \frac{(\sigma_2,\mu_0)_{L_2(\Sigma_2)}}{\|\mu_0\|^2_{L_2(\Sigma_2)}}\mu_0$ then $(\mu_2, \mu_0)_{L_2(\Sigma_2)} = 0$ and the function $u(x) = W_1\mu_1 + W_2\sigma_2 = W_1\mu_1 + W_2\mu_2$ is a unique solution of the problem D with boundary condition $\gamma^+_{0,1}u = g_1$, $\gamma^-_{0,2}u = h_2 - (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0$. □

4. Neumann boundary value problem

Problem N. Find a function $u \in H^1(\Omega)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega,$$
$$\gamma^+_{1,1} u = f_1, \qquad \gamma^-_{1,2} u = f_2. \tag{8}$$

Here $f_i \in H^{-1/2}(\Sigma_i)$, $i = 1, 2$, are given.

It's well known that problem N has solution $u(x) + c$, where c is constant, if f_1 and f_2 satisfy condition $\langle f_1, \nu_0\rangle - \langle f_2, \mu_0\rangle = 0$.

If we look for the solution of the problem N as a sum of potentials of double layer (3), where $\mu_1 = \gamma^+_{0,1}u - \gamma^-_{0,1}u$, $\mu_2 = \gamma^+_{0,2}u - \gamma^-_{0,2}u$, then the solution of the problem N satisfies the conditions (4) or

$$\langle f_1, \nu_0\rangle = 0, \quad \langle f_2, \mu_0\rangle = 0. \tag{9}$$

Using the boundary conditions (8) we obtain the following system of boundary equations:

$$\begin{cases} H_1\mu_1 + H_{2,1}\mu_2 = f_1, \\ H_{1,2}\mu_1 + H_2\mu_2 = f_2, \end{cases} \tag{10}$$

where $H_{2,1}\mu_2(x) = \gamma^+_{1,1}W_2\mu_2(x)$ and $H_{1,2}\mu_1(x) = \gamma^-_{1,2}W_1\mu_1(x)$.

The integral representation of the solution u of the problem N as a sum of potentials of double layer is connected with the following Neumann boundary value problem of transmission type.

Problem NT. Find a function $u \in H^1(\Omega_2) \cup H^1(\Omega) \cup H^1(\Omega_-)$ that satisfies

$$Lu = -\Delta u = 0 \quad \text{in} \quad \Omega_2 \cup \Omega \cup \Omega_-,$$

boundary conditions

$$\begin{cases} \gamma^+_{1,1}u = f_1, \qquad \gamma^-_{1,2}u = f_2, \\ \gamma^+_{1,1}u = \gamma^-_{1,1}u, \qquad \gamma^+_{1,2}u = \gamma^-_{1,2}u, \end{cases}$$

and condition at infinity

$$\lim_{|x|\to\infty} u(x) = 0.$$

The problem NT is equivalent to the system (10), i.e. solution of the problem NT has representation (3), where μ_1, μ_2 are solutions of the system (10) and vice versa the function (3,) where μ_1, μ_2 are solutions of the system (10) is a solution of the problem NT.

Since the system (10) has not unique solution, i.e. homogeneous system has solution (ν_0, μ_0), instead of the system (10) we will use the following modificated system:

$$\begin{cases} H_1\sigma_1 + (\sigma_1, \nu_0)_{L_2(\Sigma_1)}\nu_0 + H_{2,1}\sigma_2 = h_1, \\ H_{1,2}\sigma_1 + H_2\sigma_2 + (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0 = h_2, \end{cases} \tag{11}$$

where

$$h_1 = f_1 + c_1\nu_0, \quad h_2 = f_1 + c_2\mu_0,$$
$$c_1 = \|\nu_0\|^2_{L_2(\Sigma_1)} = |\Sigma_1|, \quad c_2 = \|\mu_0\|^2_{L_2(\Sigma_2)} = |\Sigma_2|.$$

If $\sigma_1 \in H^1(\Sigma_1)$, $\sigma_2 \in H^1(\Sigma_2)$ system (11) has the next integral representation:

$$\begin{cases} -\int_{\Sigma_1}\{\dfrac{\partial Q(x,y)}{\partial s_x}\dfrac{\partial \sigma_1(y)}{\partial s_y} - \sigma_1(y)\}ds_y - \\ \qquad -\int_{\Sigma_2}\dfrac{\partial Q(x,y)}{\partial s_x}\dfrac{\partial \sigma_2(y)}{\partial s_y}ds_y = h_1(x), \quad x \in \Sigma_1, \\ -\int_{\Sigma_1}\dfrac{\partial Q(x,y)}{\partial s_x}\dfrac{\partial \sigma_1(y)}{\partial s_y}ds_y - \\ \qquad -\int_{\Sigma_2}\{\dfrac{\partial Q(x,y)}{\partial s_x}\dfrac{\partial \sigma_2(y)}{\partial s_y} - \sigma_2(y)\}ds_y = h_2(x), \quad x \in \Sigma_2. \end{cases}$$

Theorem 3. *The system (11) has unique solution (σ_1, σ_2) for arbitrary $h_1 \in H^{-1/2}(\Sigma_1)$, $h_2 \in H^{-1/2}(\Sigma_2)$.*

Proof. Let $h_1 = 0$, $h_2 = 0$ and σ_1, σ_2 are the solutions of the following homogeneous system

$$\begin{cases} H_1\sigma_1 + (\sigma_1, \nu_0)_{L_2(\Sigma_1)}\nu_0 + H_{2,1}\sigma_2 = 0, \\ H_{1,2}\sigma_1 + H_2\sigma_2 + (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0 = 0. \end{cases} \tag{12}$$

So far as the function $u = W_1\sigma_1 + W_2\sigma_2$ satisfies condition (4) or $\langle H_1\sigma_1 + H_{2,1}\sigma_2, \nu_0\rangle = 0$ from the first equation of (12) it follows that $(\sigma_1, \nu_0)_{L_2(\Sigma_1)} = 0$. In the same way we have $(\sigma_2, \mu_0)_{L_2(\Sigma_2)} = 0$.

Thus function $u = W_1\sigma_1 + W_2\sigma_2$ is a solution of homogeneous problem N and $u(x) = \text{const}$ in $\Omega_2 \cup \Omega$. Since $\sigma_2 = \gamma_{0,2}^{+}u - \gamma_{0,2}^{-}u$ we have $\sigma_2 = \alpha\mu_0$, $\alpha \in \mathbf{R}$. From $(\sigma_2, \mu_0)_{L_2(\Sigma_2)} = 0$ we obtain $\sigma_2 = 0$. In Ω_- function $u(x) = 0$ because $\gamma_{1,1}^{-}u = 0$ and $\lim_{|x|\to\infty} u(x) = 0$. So far as $\sigma_1 = \gamma_{0,1}^{+}u - \gamma_{0,1}^{-}u$ we have $\sigma_1 = \beta\nu_0$, $\beta \in \mathbf{R}$, and $\sigma_1 = 0$.

Thus system (11) has only trivial solution.

Now we consider the existence of solution of the system (11). We can rewrite system (11) in the following operator form:

$$\mathbf{H}\vec{\sigma} = \begin{pmatrix} H_{1,1} & H_{2,1} \\ H_{1,2} & H_{2,2} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where $\vec{\sigma} = (\sigma_1, \sigma_2)$, $H_{1,1}\sigma_1 = H_1\sigma_1 + (\sigma_1, \nu_0)_{L_2(\Sigma_1)}\nu_0$, $H_{2,2}\sigma_2 = H_2\sigma_2 + (\sigma_2, \mu_0)_{L_2(\Sigma_2)}\mu_0$.

Operator

$$\mathbf{H} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{-1/2}(\Sigma_1) \times H^{-1/2}(\Sigma_2)$$

and

$$\mathbf{H} = \mathbf{D} + \mathbf{F}$$

where

$$\mathbf{D} = \begin{pmatrix} H_{1,1} & 0 \\ 0 & H_{2,2} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & H_{2,1} \\ H_{1,2} & 0 \end{pmatrix}.$$

The operators

$$H_{1,1} : H^{1/2}(\Sigma_1) \to H^{-1/2}(\Sigma_1)$$

and

$$H_{2,2} : H^{1/2}(\Sigma_2) \to H^{-1/2}(\Sigma_2)$$

are isomorphisms [8] .

Thus the operator

$$\mathbf{D} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{-1/2}(\Sigma_1) \times H^{-1/2}(\Sigma_2)$$

is isomorphism.

So far as operators $H_{2,1} : H^{1/2}(\Sigma_2) \to H^{-1/2}(\Sigma_1)$ and $H_{1,2} : H^{1/2}(\Sigma_1) \to H^{-1/2}(\Sigma_2)$ are compact [2, 6] then operator $\mathbf{F} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2)$ is compact.

As a consequence we have that operator $\mathbf{H} : H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \to H^{-1/2}(\Sigma_1) \times H^{-1/2}(\Sigma_2)$ is a Fredholm operator of index zero and the system (11) has unique solution for arbitrary $h_1 \in H^{-1/2}(\Sigma_1)$, $h_2 \in H^{-1/2}(\Sigma_2)$. $\square$

Theorem 4. *If (σ_1, σ_2) is a solution of the system (11), then the function $u = W_1\mu_1 + W_2\mu_2$, where $\mu_1 = \sigma_1 - \nu_0$, $(\mu_1, \nu_0)_{L_2(\Sigma_1)} = 0$ and $\mu_2 = \sigma_2 - \mu_0$, $(\mu_2, \mu_0)_{L_2(\Sigma_2)} = 0$, is a solution of the problem N with boundary conditions $f_1 \in H^{-1/2}(\Sigma_1)$ and $f_2 \in H^{-1/2}(\Sigma_2)$ which satisfy conditions (9).*

Proof. We have $\langle H_1\sigma_1, \nu_0\rangle = 0$ and $\langle H_2\sigma_2, \mu_0\rangle = 0$ for arbitrary $\sigma_1 \in H^{1/2}(\Sigma_1)$, $\sigma_2 \in H^{1/2}(\Sigma_2)$ [6]. Then from the first equation of the system (11) we get:

$$\langle H_1\sigma_1, \nu_0\rangle + (\sigma_1, \nu_0)_{L_2(\Sigma_1)} \|\nu_0\|^2_{L_2(\Sigma_1)} + \langle H_{2,1}\sigma_2, \nu_0\rangle = \langle f_1, \nu_0\rangle + \|\nu_0\|^4_{L_2(\Sigma_1)}. \quad (13)$$

Let us consider functions $w(x) = W_2\sigma_2(x)$ and $v(x) = 1$, $x \in \Omega$. Then $\gamma^+_{0,1}v = \nu_0$, $\gamma^-_{0,2}v = \mu_0$ and from the first Green's formula (1) we obtain:

$$0 = \langle H_{2,1}\sigma_2, \nu_0\rangle - \langle H_2\sigma_2, \mu_0\rangle,$$

or $\langle H_{2,1}\sigma_2, \nu_0\rangle = 0$. Thus from (13) it follows that $(\sigma_1, \nu_0)_{L_2(\Sigma_1)} = \|\nu_0\|^2_{L_2(\Sigma_1)}$ and if $\mu_1 = \sigma_1 - \nu_0$ then $(\mu_1, \nu_0)_{L_2(\Sigma_1)} = 0$.

In the same way we can show that if $\mu_2 = \sigma_2 - \mu_0$ then $(\mu_2, \mu_0)_{L_2(\Sigma_2)} = 0$ and as a consequence we get that function $u = W_1\mu_1 + W_2\mu_2$ satisfies system (11). $\square$

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